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THE O'NEIL INEQUALITY FOR THE HANKEL CONVOLUTION OPERATOR AND SOME APPLICATIONS

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Abstract. In this paper we prove the O'Neil inequality for the Hankel (Fourier-Bessel) convolution operator and consider some of its applications. By using the O'Neil inequality we study the boundedness of the Riesz-Hankel potential operator $I_{\beta,\alpha}$ associated with the Hankel transform in the Lorentz-Hankel spaces $L_{p,r,\alpha}(0,\infty)$. We establish necessary and sufficient conditions for the boundedness of $I_{\beta,\alpha}$ from the Lorentz-Hankel spaces $L_{p,r,\alpha}(0,\infty)$ to $L_{q,s,\alpha}(0,\infty)$, $1 < p < q < \infty$, $1 \le r \le s \le \infty$. We obtain boundedness conditions in the limiting cases $p = 1$ and $p = (2\alpha+2)/\beta$. Finally, for the limiting case $p = (2\alpha+2)/\beta$ we prove an analogue of the Adams theorem on exponential integrability of $I_{\beta,\alpha}$ in $L_{(2\alpha+2)/\beta,r,\alpha}(0,\infty)$.

1 Introduction

In this section we recall some basic results in harmonic analysis related to the Hankel (Fourier-Bessel) transform. More details can be found in [11]. We first begin with some notation. Assuming that $\alpha > -1/2$ we consider the following space

$$L_{p,\alpha} \equiv L_{p,\alpha}(0,\infty) \equiv L_p((0,\infty),\ x^{2\alpha+1}dx),\ (1 \le p < \infty)$$

of all measurable functions f defined on $(0,\infty)$ for which

$$\|f\|_{L_{p,\alpha}} \equiv \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx\right)^{1/p} < \infty.$$

By $L_{\infty,\alpha}(0,\infty) = L_\infty(0,\infty)$ we denote the space of all essentially bounded measurable functions on $(0,\infty)$.

The Hankel transform appears taking different forms in the literature (see for instance [6, 11]). Here we define the Hankel transform h_α by

$$h_\alpha(f)(x) = \int_0^\infty j_\alpha(xy) f(y) y^{2\alpha+1} dy, \quad x \in (0,\infty),$$

where $j_\alpha(s) = 2^\alpha \Gamma(\alpha+1) s^{-\alpha} J_\alpha(s)$, with J_α being the Bessel function of the first kind and index α.

Definition 1.1. *1) The generalized translation operator T^y, $y \geq 0$, is defined by*

$$T^y f(x) = C_\alpha \int_0^\pi f(\sqrt{x^2 + y^2 - 2xycos\theta})sin^{2\alpha}\theta d\theta,$$

where $C_\alpha = \Gamma(\alpha+1)[\sqrt{\pi}\ \Gamma(\alpha+1/2)]^{-1}$.

2) The Hankel (Fourier-Bessel) convolution operator of two functions f, g on $(0,\infty)$ is defined by

$$(f\#g)(x) = \int_0^\infty T^y f(x)g(y)y^{2\alpha+1}dy, \quad x \in (0,\infty).$$

It is well known that $T^y f$ is the solution of the following differential equation

$$(L_\alpha)_x u = (L_\alpha)_y u, \ u(x,0) = f(x), \ u_y(x,0) = 0.$$

Here $(L_\alpha)_x u = \frac{\partial^2 u}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial u}{\partial x}$.

Definition 1.2. *For $0 < \beta < 2\alpha+2$, the Riesz-Hankel potential operator $I_{\beta,\alpha}$ associated with the Hankel transform is defined by*

$$\begin{aligned} I_{\beta,\alpha} f(x) &= f(x)\#x^{\beta-2\alpha-2} \\ &= \int_0^\infty \frac{1}{y^{2\alpha+2-\beta}} T^y f(x) y^{2\alpha+1} dy \\ &= \int_0^\infty T^y f(x) y^{\beta-1} dy, \quad x \in (0,\infty). \end{aligned}$$

In this paper, we prove the O'Neil inequality for the Hankel convolution, and study boundedness conditions for the Riesz-Hankel potential operator $I_{\beta,\alpha}$ in the Lorentz-Hankel spaces $L_{p,r,\alpha}(0,\infty)$ by using the O'Neil inequality. The paper is organized as follows. In the second section, we prove the O'Neil inequality for the Hankel convolution $f\#g$. We establish necessary and sufficient conditions for the boundedness of $I_{\beta,\alpha}$ from the Lorentz-Hankel spaces $L_{p,r,\alpha}(0,\infty)$ to $L_{q,s,\alpha}(0,\infty)$, $1 < p < q < \infty$, $1 \leq r \leq s \leq \infty$. In the third section, we obtain boundedness conditions in the limiting cases $p = 1$ and $p = (2\alpha+2)/\beta$. After that, for the limiting case $p = (2\alpha+2)/\beta$, we prove an analogue of the Adams theorem on exponential integrability of $I_{\beta,\alpha}$ in $L_{(2\alpha+2)/\beta,r,\alpha}(0,\infty)$.

2 O'Neil inequality for the Hankel convolutions and the boundedness of the Riesz-Hankel potential in the Lorentz-Hankel spaces

In this section, we prove the O'Neil inequality for the Hankel convolution, and study boundedness conditions for the Riesz-Hankel potential operator $I_{\beta,\alpha}$ in the Lorentz-Hankel spaces $L_{p,r,\alpha}(0,\infty)$ by using the O'Neil inequality.

Let $f : (0,\infty) \to \mathbb{R}$ be a measurable function and for any measurable set E, $|E|_\alpha = \int_E x^{2\alpha+1}dx$. We define α-rearrangement of f in decreasing order by

$$f^*_\alpha(t) = \inf\{s > 0 \ : \ f_{*,\alpha}(s) \leq t\}, \quad \forall t \in (0,\infty),$$

where $f_{*,\alpha}(s)$ denotes the α-distribution function of f given by

$$f_{*,\alpha}(s) = |\{x \in (0,\infty) : |f(x)| > s\}|_\alpha.$$

For the rearrangement of f the following properties hold (see [12]).

1) If $f \in L_{p,\alpha}(0,\infty)$, $1 \le p < \infty$, then

$$\begin{aligned} \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx\right)^{1/p} &= \left(p\int_0^\infty s^{p-1} f_{*,\alpha}(s) ds\right)^{1/p} \\ &= \left(\int_0^\infty (f_\alpha^*(t))^p dt\right)^{1/p} \end{aligned} \tag{2.1}$$

2)

$$\int_0^t f_\alpha^*(s) ds = t f_\alpha^*(t) + \int_{f_\alpha^*(t)}^\infty f_{*,\alpha}(s) ds \tag{2.2}$$

We denote by $WL_{p,\alpha}(0,\infty)$ the weak $L_{p,\alpha}$ space of all measurable functions f on $(0,\infty)$ with finite norm

$$\|f\|_{WL_{p,\alpha}} = \sup_{t>0} t^{1/p} f_\alpha^*(t), \quad 1 \le p < \infty.$$

The function $f_\alpha^{**} : (0,\infty) \to [0,\infty]$ is defined as $f_\alpha^{**}(t) = \frac{1}{t}\int_0^t f_\alpha^*(s) ds$. It is clear that for the function f_α^{**} the subadditivity property is satisfied.

Definition 2.1. *[2, 3] If $0 < p, q < \infty$, then the Lorentz-Hankel space $L_{p,q,\alpha}(0,\infty) = L_{p,q}((0,\infty), x^{2\alpha+1}dx)$ is the set of all measurable functions f on $(0,\infty)$ with finite quasi-norm*

$$\|f\|_{p,q,\alpha} \equiv \|f\|_{L_{p,q,\alpha}} = \left(\int_0^\infty \left(t^{1/p} f_\alpha^*(t)\right)^q \frac{dt}{t}\right)^{1/q}.$$

If $0 < p \le \infty$, $q = \infty$, then $L_{p,\infty,\alpha}(0,\infty) = WL_{p,\alpha}(0,\infty)$.

If $1 \le q \le p$ or $p = q = \infty$, then the functional $\|f\|_{p,q,\alpha}$ is a norm. If $p = q = \infty$, then the space $L_{\infty,\infty,\alpha}(0,\infty)$ is denoted by $L_\infty(0,\infty)$.

We need the following lemma to prove the O'Neil inequality for the rearrangements of the Hankel convolutions associated with the Hankel transform.

Lemma 2.1. *Let f and g be measurable functions on $(0,\infty)$ such that $\sup\{f(x) : x \in (0,\infty)\} \le \lambda$ and f vanishes outside of a measurable set E with $|E|_\alpha = \tau$. Then, for all $t > 0$,*

$$(f\#g)_\alpha^{**}(t) \le \lambda\tau \min\{g_\alpha^{**}(\tau), g_\alpha^{**}(t)\}. \tag{2.3}$$

Proof. Without loss of generality we can assume that the functions f and g are non-negative. For $a > 0$, define

$$g_a(x) = \begin{cases} g(x), & \text{if} \quad g(x) \le a \\ a, & \text{if} \quad g(x) > a \end{cases}$$

and let

$$g^a(x) = g(x) - g_a(x).$$

Then we can write

$$f\#g = f\#g_a + f\#g^a.$$

If $s > a$, then $g^a_{*,\alpha}(s) = g_{*,\alpha}(s)$. If $s \leq a$, then we have

$$\begin{aligned} g^a_{*,\alpha}(s) &= \int_{\{y: g^a(y) > s\}} y^{2\alpha+1} dy \\ &= \int_{\{y: s < g^a(y) \leq a\}} y^{2\alpha+1} dy = 0 \end{aligned}$$

and setting $a = g^*_\alpha(t)$ we have

$$\begin{aligned} (f\#g^a)^{**}_\alpha(t) &\leq \sup_{(0,\infty)} |(f\#g^a)(y)| \\ &\leq \sup_E f(y) ||g^a||_{L_{1,\alpha}} \\ &\leq \lambda \int_a^\infty g^a_{*,\alpha}(s) ds \\ &\leq \lambda \tau a \leq \lambda \tau g^{**}_\alpha(t). \end{aligned}$$

The last inequality follows by equality (2.2) and thus the first inequality of the lemma is established.

To prove the second inequality, set $a = g^*_\alpha(\tau)$ to obtain

$$\begin{aligned} (f\#g)^{**}_\alpha(t) &= \frac{1}{t} \sup_{|A|_\alpha = t} \int_A |(f\#g)(y)| y^{2\alpha+1} dy \\ &\leq \sup_{(0,\infty)} |(f\#g)(y)| \\ &\leq \sup_{(0,\infty)} |(f\#g_a)(y)| + \sup_{(0,\infty)} |(f\#g^a)(y)| \\ &\leq \lambda \tau g^*_\alpha(t) + \lambda \int_{g^*_\alpha(\tau)}^\infty g_{*,\alpha}(s) ds \\ &\leq \lambda \tau \left[g^*_\alpha(\tau) + \frac{1}{\tau} \int_{g^*_\alpha(\tau)}^\infty g_{*,\alpha}(s) ds \right] \\ &= \lambda \tau g^{**}_\alpha(\tau) \end{aligned}$$

by equation (2.2). □

In the following theorem we show that the O'Neil inequality holds for the rearrangements of the Hankel convolution. The methods of the proof used here are close to those in [12].

Theorem 2.1. *(The O'Neil inequality for the rearrangements of the Hankel convolutions) Let f and g be measurable functions, then for any $t > 0$*

$$(f\#g)^{**}_\alpha(t) \leq t f^{**}_\alpha(t) g^{**}_\alpha(t) + \int_t^\infty f^*_\alpha(u) g^*_\alpha(u) du. \tag{2.4}$$

Proof. Fix $t > 0$ and select a doubly infinite sequence $\{y_i\}$ whose indices range from $-\infty$ to ∞ such that

$$y_0 = f_\alpha^*(t),\ y_i \le y_{i+1},\ \lim_{i\to\infty} y_i = \infty,\ and\ \lim_{i\to-\infty} y_i = 0.$$

Let

$$f(z) = \sum_{i=-\infty}^{\infty} f_i(z),$$

where

$$f_i(z) = \begin{cases} 0, & \text{if} \quad |f(z)| \le y_{i-1}; \\ f(z) - y_{i-1} sgn f(z), & \text{if} \quad y_{i-1} < |f(z)| \le y_i; \\ y_i - y_{i-1} sgn f(z), & \text{if} \quad y_i < |f(z)|. \end{cases}$$

Clearly, the series converges absolutely and therefore,

$$\begin{aligned} f\#g &= \left(\sum_{i=-\infty}^{\infty} f_i\right)\#g \\ &= \left(\sum_{i=-\infty}^{0} f_i\right)\#g + \left(\sum_{i=1}^{\infty} f_i\right)\#g \\ &= h_1 + h_2 \end{aligned}$$

with

$$(f\#g)_\alpha^{**}(t) \le (h_1)_\alpha^{**}(t) + (h_2)_\alpha^{**}(t).$$

To evaluate $(h_2)_\alpha^{**}(t)$ we use inequality (2.3) with $E_i \equiv \{z : |f(z)| > y_{i-1}\} = E$ and $\lambda = y_i - y_{i-1}$ to obtain

$$\begin{aligned} (h_2)_\alpha^{**}(t) &\le \sum_{i=1}^{\infty} (y_i - y_{i-1}) f_{*,\alpha}(y_{i-1}) g_\alpha^{**}(t) \\ &= g_\alpha^{**}(t) \sum_{i=1}^{\infty} f_{*,\alpha}(y_{i-1})(y_i - y_{i-1}). \end{aligned}$$

The series on the right is the infinite Riemann sum for the integral

$$\int_{f_\alpha^*(t)}^{\infty} f_{*,\alpha}(y)dy,$$

and provides an arbitrarily close approximation with an appropriate choice of the sequence $\{y_i\}$. Therefore,

$$(h_2)_\alpha^{**}(t) \le g_\alpha^{**}(t) \int_{f_\alpha^*(t)}^{\infty} f_{*,\alpha}(y)dy. \tag{2.5}$$

By inequality (2.3),

$$(h_1)_\alpha^{**}(t) \le \sum_{i=1}^{\infty} (y_i - y_{i-1}) f_{*,\alpha}(y_{i-1}) g_\alpha^{**}(f_{*,\alpha}(y_{i-1})).$$

The sum on the right is the infinite Riemann sum tending (with proper choice of y_i) to the integral,

$$\int_0^{f_\alpha^*(t)} f_{*,\alpha}(y) g_\alpha^{**}(f_{*,\alpha}(y))dy.$$

It is not hard to see that the integral can be evaluated by making the substitution $y = f_\alpha^*(u)$ and then integrating by parts.

Therefore, we have

$$\begin{aligned}(h_1)_\alpha^{**}(t) &\le \int_0^{f_\alpha^*(t)} f_{*,\alpha}(y) g_\alpha^{**}(f_{*,\alpha}(y))dy \\ &= -\int_t^\infty u g_\alpha^{**}(u) df_\alpha^*(u) \\ &= -u g_\alpha^{**}(u) f_\alpha^*(u)|_t^\infty + \int_t^\infty f_\alpha^*(u) g_\alpha^*(u)du \\ &\le t g_\alpha^{**}(t) f_\alpha^*(t) + \int_t^\infty f_\alpha^*(u) g_\alpha^*(u)du\end{aligned} \tag{2.6}$$

Thus by (2.6), (2.5), and (2.2),

$$\begin{aligned}(h_1)_\alpha^{**}(t) + (h_2)_\alpha^{**}(t) &\le g_\alpha^{**}(t)\left[t f_\alpha^*(t) + \int_{f_\alpha^*(t)}^\infty f_{*,\alpha}(y)dy\right] + \int_t^\infty f_\alpha^*(u) g_\alpha^*(u)du \\ &\le t f_\alpha^{**}(t) g_\alpha^{**}(t) + \int_t^\infty f_\alpha^*(u) g_\alpha^*(u)du.\end{aligned}$$

□

By Theorem 2.1 we get the following result.

Theorem 2.2. *If $g \in WL_{r,\alpha}([0,\infty))$, $1 < r < \infty$, then*

$$\begin{aligned}(f\#g)_\alpha^*(t) &\le (f\#g)_\alpha^{**}(t) \\ &\le \|g\|_{WL_{r,\alpha}}\left(r' t^{-1/r}\int_0^t f_\alpha^*(s)ds + \int_t^\infty s^{-1/r} f_\alpha^*(s)ds\right).\end{aligned} \tag{2.7}$$

Proof. Since $g \in WL_{r,\alpha}([0,\infty))$, we have $g_\alpha^*(t) \le \|g\|_{WL_{r,\alpha}} t^{-1/r}$, $g_\alpha^{**}(t) \le r'\|g\|_{WL_{r,\alpha}} t^{-1/r}$. Taking into account inequality (2.4) we get inequality (2.7). □

Next we give a pointwise rearrengement estimate of the Riesz-Hankel potential operator $I_{\beta,\alpha}$.

Corollary 2.1. *If $g(x) = g_{\beta,\alpha}(x) = x^{\beta-2\alpha-2}$, $0 < \beta < 2\alpha+2$, and $r = \frac{2\alpha+2}{2\alpha+2-\beta}$, then by* (2.7)

$$\begin{aligned}&(I_{\beta,\alpha}f)_\alpha^*(t) \le (I_{\beta,\alpha}f)_\alpha^{**}(t) \\ &\le \|g_{\beta,\alpha}\|_{WL_{\frac{2\alpha+2}{2\alpha+2-\beta},\alpha}}\left(\left(\frac{2\alpha+2}{\beta}\right) t^{\frac{\beta}{2\alpha+2}-1}\int_0^t f_\alpha^*(s)ds + \int_t^\infty s^{\frac{\beta}{2\alpha+2}-1} f_\alpha^*(s)ds\right).\end{aligned}$$

In Theorem 2.3 we obtain necessary and sufficient conditions for the boundedness of $I_{\beta,\alpha}$ from the Lorentz-Hankel spaces $L_{p,r,\alpha}(0,\infty)$ to $L_{q,s,\alpha}(0,\infty)$. The methods of the proof of this theorem are close to those in [4]. We omit the proof.

Theorem 2.3. *Let $0<\beta<2\alpha+2$. Then, if $1<p<(2\alpha+2)/\beta$, $1\le r\le s\le\infty$, then the condition $1/p-1/q=\beta/(2\alpha+2)$ is necessary and sufficient for the boundedness of $I_{\beta,\alpha}$ from $L_{p,r,\alpha}(0,\infty)$ to $L_{q,s,\alpha}(0,\infty)$.*

3 Boundedness of the Riesz-Hankel potential in the limiting cases

In this section, we obtain the boundedness conditions for $I_{\beta,\alpha}$ in the limiting cases $p=1$ and $p=(2\alpha+2)/\beta$.

Theorem 3.1. *Let $0<\beta<2\alpha+2$. Then the condition $1-1/q=\beta/(2\alpha+2)$ is necessary and sufficient for the boundedness of $I_{\beta,\alpha}$ from $L_{1,\alpha}(0,\infty)$ to $WL_{q,\alpha}(0,\infty)$.*

Proof. Sufficiency: Let $1-\frac{1}{q}=\frac{\beta}{2\alpha+2}$ and $f\in L_{1,\alpha}(0,\infty)$. By using inequality (2.4) we get

$$\begin{aligned}
\|I_{\beta,\alpha}f\|_{WL_{q,\alpha}} &= \sup_{t>0} t^{1/q}\,(I_{\beta,\alpha}f)^*_\alpha(t)\\
&\le (2\alpha+2)^{\frac{\beta}{2\alpha+2}-1}\sup_{t>0} t^{1/q}\left(\left(\frac{2\alpha+2}{\beta}\right)t^{\frac{\beta}{2\alpha+2}-1}\int_0^t f^*_\alpha(s)ds+\int_t^\infty s^{\frac{\beta}{2\alpha+2}-1}f^*_\alpha(s)ds\right)\\
&= (2\alpha+2)^{\frac{\beta}{2\alpha+2}-1}\left(\frac{2\alpha+2}{\beta}\right)\sup_{t>0}\int_0^t f^*_\alpha(s)ds\\
&+ (2\alpha+2)^{\frac{\beta}{2\alpha+2}-1}\sup_{t>0} t^{1/q}\int_t^\infty s^{-1/q}f^*_\alpha(s)ds\\
&\le (2\alpha+2)^{\frac{\beta}{2\alpha+2}-1}\left(1+\frac{2\alpha+2}{\beta}\right)\|f^*_\alpha\|_{L_1(0,\infty)}\\
&= (2\alpha+2)^{\frac{\beta}{2\alpha+2}-1}\left(1+\frac{2\alpha+2}{\beta}\right)\|f\|_{L_{1,\alpha}}.
\end{aligned}$$

Necessity: Suppose that the operator $I_{\beta,\alpha}$ is bounded from $L_{1,\alpha}(0,\infty)$ to $WL_{q,\alpha}(0,\infty)$, i.e., the following inequality is valid

$$\|I_{\beta,\alpha}f\|_{WL_{q,\alpha}}\le C\|f\|_{L_{1,\alpha}},$$

where C is independent of f. Define $f_t(x)=:f(tx)$ for $t>0$. It is easy to show that $\|f_t\|_{L_{1,\alpha}}=t^{-(2\alpha+2)}\,\|f\|_{L_{1,\alpha}}$ and

$$\|I_{\beta,\alpha}f_t\|_{WL_{q,\alpha}}=t^{-\beta-(2\alpha+2)/q}\,\|I_{\beta,\alpha}f\|_{WL_{q,\alpha}}.$$

Then we have
$(I_{\beta,\alpha}f_t)_{*,\alpha}(\tau)=t^{-(2\alpha+2)}(I_{\beta,\alpha}f)_{*,\alpha}(t^\beta\tau)$, $\|I_{\beta,\alpha}f_t\|_{WL_{q,\alpha}}=t^{-\beta-\frac{2\alpha+2}{q}}\,\|I_{\beta,\alpha}f\|_{WL_{q,\alpha}}$, and

$$\begin{aligned}
\|I_{\beta,\alpha}f\|_{WL_{q,\alpha}} &= t^{\beta+\frac{2\alpha+2}{q}}\|I_{\beta,\alpha}f_t\|_{WL_{q,\alpha}}\\
&\le Ct^{\beta+\frac{2\alpha+2}{q}}\|f_t\|_{L_{1,\alpha}}=Ct^{\beta+\frac{2\alpha+2}{q}-(2\alpha+2)}\|f\|_{L_{1,\alpha}}.
\end{aligned}$$

If $1 < \frac{1}{q} + \frac{\beta}{2\alpha+2}$, then for all $f \in L_{1,\alpha}(0,\infty)$ we have $\|I_{\beta,\alpha} f\|_{WL_{q,\alpha}} = 0$ as $t \to 0$.

If $1 > \frac{1}{q} + \frac{\beta}{2\alpha+2}$, then for all $f \in L_{1,\alpha}(0,\infty)$ we have $\|I_{\beta,\alpha} f\|_{WL_{q,\alpha}} = 0$ as $t \to \infty$. Therefore we get the equality $1 = \frac{1}{q} + \frac{\beta}{2\alpha+2}$ and the proof of the theorem is completed. □

Theorem 3.2. *Let $0 < \beta < 2\alpha + 2$, and $f \in L_{\frac{2\alpha+2}{\beta},1,\alpha}(0,\infty)$, then $I_{\beta,\alpha} f \in L_{\infty,\alpha}(0,\infty)$ and*

$$\|I_{\beta,\alpha} f\|_{L_{\infty,\alpha}} \le (2\alpha+2)^{\frac{\beta}{2\alpha+2}-1}\left(1 + \frac{2\alpha+2}{\beta}\right)\|f\|_{L_{\frac{2\alpha+2}{\beta},1,\alpha}}.$$

Proof. Let $p = \frac{2\alpha+2}{\beta}$, $r = 1$, $q = s = \infty$, and $f \in L_{\frac{2\alpha+2}{\beta},1,\alpha}(0,\infty)$. By using inequality (2.4) we have

$$\begin{aligned}
\|I_{\beta,\alpha} f\|_{L_{\infty,\alpha}} &= \sup_{t>0} \left(I_{\beta,\alpha} f\right)^*_\alpha (t) \\
&\le (2\alpha+2)^{\frac{\beta}{2\alpha+2}-1} \sup_{t>0}\left(\left(\frac{2\alpha+2}{\beta}\right) t^{\frac{\beta}{2\alpha+2}-1}\int_0^t f^*_\alpha(s)ds + \int_t^\infty s^{\frac{\beta}{2\alpha+2}-1} f^*_\alpha(s)ds\right) \\
&\le (2\alpha+2)^{\frac{\beta}{2\alpha+2}-1}\left(1 + \frac{2\alpha+2}{\beta}\right)\int_0^\infty s^{\frac{\beta}{2\alpha+2}-1} f^*_\alpha(s)ds \\
&= (2\alpha+2)^{\frac{\beta}{2\alpha+2}-1}\left(1 + \frac{2\alpha+2}{\beta}\right)\|f\|_{L_{\frac{2\alpha+2}{\beta},1,\alpha}}.
\end{aligned}$$

□

In the limiting case $p = (2\alpha+2)/\beta$ the boundedness of the Riesz-Hankel potential operator $I_{\beta,\alpha}$ in $L_{(2\alpha+2)/\beta,r,\alpha}(0,\infty)$ for $r \neq 1$ does not hold. However, the following theorem can be regarded as a substitute of the boundedness for $I_{\beta,\alpha}$ in this case. This theorem is an analogue of the Adams theorem given in [1] on exponential integrability for the Riesz potential of order β $(0 < \beta < n)$.

We need the following lemma to prove Theorem 3.3.

Lemma 3.1. *[1] Let $a(s,t)$ be a nonnegative measurable function on $(-\infty,+\infty) \times [0,+\infty)$ such that $0 < s < t$*

$$a(s,t) \le 1, \quad a.e. \text{ if } \quad 0 < s < t,$$

$$\operatorname*{ess\,sup}_{t>0}\left(\int_{-\infty}^0 + \int_t^\infty a(s,t)^{p'} ds\right)^{\frac{1}{p'}} = b < \infty. \tag{3.1}$$

Then there is a constant $C_0 = C_0(p,b)$, such that for $\phi \ge 0$ with

$$\int_{-\infty}^\infty \phi(s)^p ds \le 1, \tag{3.2}$$

we have

$$\int_0^\infty e^{-F(t)} dt \le C_0,$$

where

$$F(t) = t - \left(\int_{-\infty}^\infty a(s,t)\phi(s)ds\right)^{p'}.$$

Theorem 3.3. *Let* $0<\beta<2\alpha+2$, $r\in(1,\infty]$ *and* $f\in L_{\frac{2\alpha+2}{\beta},r,\alpha}(0,\infty)$.

i) If $r\in(1,\infty)$, *then there exists a constant* $C=C(\alpha,\beta,l,r)$ *such that*

$$\int_{B(0,l)}\exp\Big(K_0\Big|\frac{I_{\beta,\alpha}f(x)}{\|f\|_{L_{(2\alpha+2)/\beta,r,\alpha}}}\Big|\Big)^{r'}x^{2\alpha+1}dx\le C,$$

where $K_0=|B(0,1)|_\alpha^{\frac{\beta}{2\alpha+2}-1}=(2\alpha+2)^{1-\frac{\beta}{2\alpha+2}}$.

ii) If $r=\infty$, *then for every* $M<K_0$ *there exists a constant* $C=C(\alpha,\beta,l,M)$ *such that*

$$\int_{B(0,l)}\exp\Big(M\Big|\frac{I_{\beta,\alpha}f(x)}{\|f\|_{L_{(2\alpha+2)/\beta,\infty,\alpha}}}\Big|\Big)x^{2\alpha+1}dx\le C.$$

Proof. i) First, assume that $\|f\|_{L_{(2\alpha+2)/\beta,r,\alpha}(0,\infty)}=1$. By Corollary 2.1, by using the O'Neil inequality for the rearrangement of a convolution, we have

$$\begin{aligned}(I_{\beta,\alpha}f)^*_\alpha(t)&\le(I_{\beta,\alpha}f)^{**}_\alpha(t)\\&\le(2\alpha+2)^{\frac{\beta}{2\alpha+2}-1}\Big(\Big(\frac{2\alpha+2}{\beta}\Big)t^{\frac{\beta}{2\alpha+2}-1}\int_0^t f^*_\alpha(s)ds+\int_t^{|B|}s^{\frac{\beta}{2\alpha+2}-1}f^*_\alpha(s)ds\Big)\\&=\frac{1}{K_0}\Big(\Big(\frac{2\alpha+2}{\beta}\Big)t^{\frac{\beta}{2\alpha+2}-1}\int_0^t f^*_\alpha(s)ds+\int_t^{|B|}f^*_\alpha(s)s^{\frac{\beta}{2\alpha+2}-1}ds\Big)\end{aligned}$$

where $|B|=|B(0,l)|_\alpha=\frac{l^{2\alpha+2}}{2\alpha+2}$. Hence, by an appropriate change of variables, we obtain

$$\begin{aligned}K_0(I_{\beta,\alpha}f)^*_\alpha(|B|e^{-\tau})&\le\Big(\frac{2\alpha+2}{\beta}\Big)(|B|e^{-\tau})^{\frac{\beta}{2\alpha+2}-1}\int_\tau^\infty f^*_\alpha(|B|e^{-\sigma})(|B|e^{-\sigma})d\sigma\\&+\int_0^\tau f^*_\alpha(|B|e^{-\sigma})(|B|e^{-\sigma})^{\frac{\beta}{2\alpha+2}-1}(|B|e^{-\sigma})d\sigma=\int_0^\infty a(\sigma,\tau)\phi(\sigma)d\sigma,\end{aligned}\tag{3.3}$$

where

$$\phi(\sigma)=f^*_\alpha(|B|e^{-\sigma})(|B|e^{-\sigma})^{\frac{\beta}{2\alpha+2}}\qquad if\quad\sigma>0,\tag{3.4}$$

and

$$a(\sigma,\tau)=\begin{cases}1, & 0\le\sigma<\tau<\infty,\\ \left(\frac{2\alpha+2}{\beta}\right)(|B|e^{-\tau})^{\frac{\beta}{2\alpha+2}-1}(|B|e^{-\sigma})^{1-\frac{\beta}{2\alpha+2}}, & 0\le\tau<\sigma<\infty.\end{cases}\tag{3.5}$$

Lemma 3.1 comes into play at this stage. Assumption (3.1) is obviously satisfied if a is given by (3.5). As far as (3.2) is concerned we have

$$\begin{aligned}\int_\tau^\infty a(\sigma,\tau)^{r'}d\sigma&=\Big(\frac{2\alpha+2}{\beta}\Big)^{r'}(|B|e^{-\tau})^{r'(\frac{\beta}{2\alpha+2}-1)}\int_\tau^\infty(|B|e^{-\sigma})^{r'(1-\frac{\beta}{2\alpha+2})}d\sigma\\&=\Big(\frac{2\alpha+2}{\beta}\Big)^{r'}\frac{1}{r'(1-\frac{\beta}{2\alpha+2})+1},\quad for\ \tau>0,\end{aligned}$$

whence $\sup\limits_{\tau>0}\int_\tau^\infty a(\sigma,\tau)^{r'}d\sigma<\infty$. By (3.4) for $r\in(1,\infty)$,

$$\begin{aligned}\|\phi\|^r_{L_{r(0,\infty)}}&=\int_0^\infty\Big(f^*_\alpha(|B|e^{-\sigma})(|B|e^{-\sigma})^{\frac{\beta}{2\alpha+2}}\Big)^r d\sigma\\&=\int_0^{|B|}\Big(f^*_\alpha(t)t^{\frac{\beta}{2\alpha+2}}\Big)^r\frac{dt}{t}=\|f\|^r_{L_{\frac{2\alpha+2}{\beta},r,\alpha}(B(0,l))}\le 1.\end{aligned}$$

Thus, by (3.3) and Theorem 2.1

$$\begin{aligned}\int_{B(0,l)}exp(K_0|I_{\beta,\alpha}f(x)|)^{r'}x^{2\alpha+1}dx&=\int_0^{|B|}exp(K_0(I_{\beta,\alpha}f)^*_\alpha(t))^{r'}dt\\&=|B|\int_0^\infty exp[(K_0(I_{\beta,\alpha}f)^*_\alpha(|B|e^{-\tau}))^{r'}-\tau]d\tau\\&\le|B|\int_0^\infty exp\left[\left(\int_0^\infty a(\sigma,\tau)\phi(\sigma)d\sigma\right)^{r'}-\tau\right]d\tau\\&=|B|\int_0^\infty e^{-F(\tau)}d\tau\le C\end{aligned}\tag{3.6}$$

for some constant $C=C(\alpha,\beta,l,r)$, where $\|f\|_{L_{(2\alpha+2)/\beta,r,\alpha}}=1$.

Now consider the general case.

If $\|f\|_{L_{(2\alpha+2)/\beta,r,\alpha}}\ne1$, then we denote $g=f/\|f\|_{L_{(2\alpha+2)/\beta,r,\alpha}}$. Thus $I_{\beta,\alpha}g(x)=I_{\beta,\alpha}f(x)/\|f\|_{L_{(2\alpha+2)/\beta,r,\alpha}}$ and $\|g\|_{L_{(2\alpha+2)/\beta,r,\alpha}}=1$. By (3.6), it follows that

$$\int_{B(0,l)}\exp\Big(K_0\Big|\frac{I_{\beta,\alpha}f(x)}{\|f\|_{L_{(2\alpha+2)/\beta,r,\alpha}}}\Big|\Big)^{r'}x^{2\alpha+1}dx\le C.$$

ii) First, assume that $\|f\|_{L_{(2\alpha+2)/\beta,\infty,\alpha}}=1$, then it can be easily seen that

$$f^*_\alpha(t)\le t^{-\frac{\beta}{2\alpha+2}}\qquad\text{for } t\in(0,|B|).\tag{3.7}$$

By (3.3) and (3.7), we infer that

$$\begin{aligned}(I_{\beta,\alpha}f)^*_\alpha(t)&\le\frac{1}{K_0}\Big[\Big(\frac{2\alpha+2}{\beta}\Big)t^{\frac{\beta}{2\alpha+2}-1}\int_0^t f^*_\alpha(s)ds+\int_t^\infty f^*_\alpha(s)s^{\frac{\beta}{2\alpha+2}-1}ds\Big]\\&\le\frac{1}{K_0}\Big[\Big(\frac{2\alpha+2}{\beta}\Big)t^{\frac{\beta}{2\alpha+2}-1}\int_0^t s^{-\frac{\beta}{2\alpha+2}}ds+\int_t^{|B|}s^{-1}ds\Big]\\&=\frac{1}{K_0}\Big[\frac{(2\alpha+2)^2}{\beta(2\alpha+2-\beta)}+log\frac{|B|}{t}\Big],\quad t\in(0,|B|).\end{aligned}$$

Thus a constant $C=C(\alpha,\beta)$ exists such that

$$\begin{aligned}\int_{B(0,l)}exp(M|I_{\beta,\alpha}f(x)|)x^{2\alpha+1}dx&=\int_0^{|B|}exp(M(I_{\beta,\alpha}f)^*_\alpha(t))dt\\&\le\int_0^{|B|}exp\Big(M\Big[C+\frac{1}{K_0}log\frac{|B|}{t}\Big]\Big)dt<\infty\end{aligned}\tag{3.8}$$

for every $M < K_0$.

Now consider the general case.

If $\|f\|_{L_{(2\alpha+2)/\beta,\infty,\alpha}} \neq 1$, then we denote $g = f/\|f\|_{L_{(2\alpha+2)/\beta,\infty,\alpha}}$. Thus $I_{\beta,\alpha}g(x) = I_{\beta,\alpha}f(x)/\|f\|_{L_{(2\alpha+2)/\beta,\infty,\alpha}}$ and $\|g\|_{L_{(2\alpha+2)/\beta,\infty,\alpha}} = 1$. By (3.8) it follows that

$$\int_{B(0,l)} \exp\Big(M\Big|\frac{I_{\beta,\alpha}f(x)}{\|f\|_{L_{(2\alpha+2)/\beta,\infty,\alpha}}}\Big|\Big)x^{2\alpha+1}dx < \infty.$$

□

Corollary 3.1. *Let $0 < \beta < 2\alpha + 2$, then there is a constant $C = C(\beta, \alpha, l)$ depending only on β, α and l such that for all $f \in L_{(2\alpha+2)/\beta,\alpha}(B(0,l))$*

$$\int_{B(0,l)} \exp\Big((2\alpha+2)\left|\frac{I_{\beta,\alpha}f(x)}{\|f\|_{L_{(2\alpha+2)/\beta,\alpha}}}\right|^{(2\alpha+2)/(2\alpha+2-\beta)}\Big)x^{2\alpha+1}dx \le C.$$

Corollary 3.1 was proved in [5] for the Lebesgue spaces $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ associated with the Laplace-Bessel differential operator for the Riesz potential.

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