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APPLICATIONS OF SOBOLEV LATTICE CUBATURE FORMULAS

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Abstract. In this note we discuss new theoretical results on Sobolev lattice formulas and applications to programmes for multi-dimensional approximate integrating and solving integral equations.

S.L. Sobolev created the algorithm of lattice cubature formulas with boundary layer for approximate integration of functions defined on multi-dimensional domains. He derived the asymptotic optimality property of this algorithm for integrands in the $L_2^{(m)}$ space [9]. Research was continued for the W_p^m spaces with $p \in (1, \infty)$ and fractional $m > n/p$. It resulted in creating the theory of asymptotically optimal formulas with bounded boundary layers (BBL-formulas):

$$K_h f = h^n \sum_{\mathrm{dist}(hk,\Omega) \leqslant Lh} c_k(h) f(hk), \qquad c_k(h) \equiv 1 \ \text{ if } \ \mathrm{dist}(hk, \mathbb{R}^n \backslash \Omega) > Lh.$$

We implimented algorithms of BBL-formulas in parallel programmes for calculating integrals over multi-dimensional domains of arbitrary shapes and for solving integral equations.

The first program “CubaInt" is for approximate integration in domains with dimensions from 2 to 10 (see [4]). It was designed by means of C++ programming language with the library of parallel functions MPI. The program has high efficiency of parallelism (about 0.8-0.9) and high precision. It was tested on the supercomputer MVS-100k of the Joint Supercomputer Center, Russian Academy of Sciences, Moscow.

We also created a version of this programme for calculations on systems with the accelerators on graphics cards (the technology NVIDIA CUDA, see [5]).

A programme was also written for solving integral equations of the form

$$u(x) - \int_\Omega K(x,y) u(y) dy = f(x), \quad x \in \Omega \subset \mathbb{R}^2.$$

Here Ω is a two-dimensional bounded closed domain with smooth boundary, $K \in C^M(\Omega \times \Omega)$ and $f \in C^M(\Omega)$. It was assumed that $||K||_{C(\Omega\times\Omega)} < 1$.

This parallel programme was based on C++/MPI technology (see [8], [2]).

Our algorithms and programmes were named “conditionally unsaturated” because respective cubature formulas are asymptotically optimal for any $m < M$.

Below we give some results confirming the sharpness of our algorithms.

Theorem 1. ([6]) *Let $1 < p_1 < p_2 < \infty$ and $\frac{n}{p_1} < m_1 < m_2$. Then the family $\{K_h\}$ is asymptotically optimal in every space of the family*

$$\{W_p^m(\Omega)\}_{\substack{m\in(m_1,m_2)\\ p\in(p_1,p_2)}},$$

if and only if it is optimal in order in every of these spaces.

Idea of the proof. We relied on estimates of differences between norms of error functionals with equal orders of optimality $O(h^m)$. It was found that the difference had the order $o(h^m)$ that implied the asymptotic optimality of the error functionals.

□

Theorem 2.([7]) *Let χ_Ω be the characteristic function of a bounded domain Ω with smooth boundary.*

Let $c_k(h) \equiv C(x,h)$, $x = hk$, and $C(x,h)$ be equal to zero if dist$(x,\Omega) \geqslant h^\gamma$, be equal to 1 if dist$(x,\mathbb{R}^n\backslash\Omega) \geqslant h^\gamma$, and be equal to

$$\int\limits_{\mathbb{R}^n} \chi_\Omega(y) \prod_{j=1}^{n} \frac{\sin\left(2\pi(x_j - y_j)/h^\delta\right)}{\pi(x_j - y_j)} dy$$

if dist$(x,\partial\Omega) \leqslant h^\gamma$, with some $0 < \delta < \gamma < \frac{1}{2}$.

Then the lattice cubature formula is asymptotically optimal in all spaces W_2^m with $m > \frac{n}{2}$.

This formula is unsaturated in the sense of K.I. Babenko [1].

Idea of the proof. We derive the optimal formula and then simplify it to the above preserving the asymptotic optimality.

□

Plans. At the moment we are facing the following problems.

It is necessary to write our programmes for different platforms, including OpenMP, MATLAB, Maple, and other languages.

We also need to make programmes based on our unsaturated algorithms.

Certainly we need to estimate, theoretically and by computational experiments, remainder terms in the asymptotic formulas.

The programmes for solving integral equations should be extended to solving equations with less restrictive conditions on the smoothness of kernels of integral operators.

Problems of application of elastic and plastic deformations need calculations of integrals on domains arising in computing processes [3]. Our model problem is calculating integrals of smooth functions u on domains where they are positive, $\int u_+(x)dx$.

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References

[1] K.I. Babenko, *Osnovy chislennogo analiza,* 2nd edition. Izhevsk, 2002, 848 p. (in Russian)

[2] E.L. Bannikova, *The programme of numerical solution of Fredholm integral equations "IntUr",* Certificate of registration 10418 of 15.04.2008 in the Sectoral fund of algorithms and programmes.

[3] T.I. Buriev. *Razrabotka algoritmov vyichisleniya integralov v mnogomernyih oblastyah s zaranee neizvestnyimi granitsami*, Sbornik "Voprosyi matematicheskogo analiza", Krasnoyarsk: IPTs KGU, 6 (2003), 51–66. *(in Russian)*

[4] D.Y. Rakhmatullin, *The programme "CubaInt",* Certificate of registration 2007614331 of 10.10.2007 in EVM programme registry.

[5] D.Y. Rakhmatullin, L.E. Yakhina, *The program for approximate integrating on multidimensional domains with NVIDIA CUDA technology,* Certificate of registration 18601 of 17.10.2012.

[6] M.D. Ramazanov, *To the L_p-theory of Sobolev formulas*, Siberian advances in mathematics, 9 (1999), no. 1, 99-125.

[7] M.D. Ramazanov, *Asymptotically optimal unsaturated lattice cubature formulae with bounded boundary layer,* Sbornik: Mathematics. 204 (2013), no. 7, 1003-1027.

[8] M.D. Ramazanov, D.Y. Rakhmatullin, E.L. Bannikova, *The cubature formulas of S.L. Sobolev: evolution of the theory and applications,* Eurasian Math. J. 1 (2010), no. 1, 123-136.

[9] S.L. Sobolev, *The theory of cubature formulas (Mathematics and its applications),* 1st edition, Springer, 1997, ISBN 0792346319.

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