Short communications

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 4, Number 3 (2013), 127 – 131

AN INEQUALITY FOR THE WEIGHTED HARDY OPERATOR FOR $0 < p < 1$

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Communicated by E.D. Nursultanov

Key words: Hardy-type inequality, weighted L_p-spaces with $0 < p < 1$, weighted Hardy operator.

AMS Mathematics Subject Classification: 35J20, 35J25.

Abstract. A Hardy-type inequality for $0 < p < 1$ with sharp constant is established in [7], [4]. The aim of this work is to extend this inequality for the weighted Hardy operator.

1 Introduction

Let w denote a weight function on $(0, \infty)$, i.e. a positive measurable function on $(0, \infty)$. For $0 < p < \infty$ the weighted space $L_{p,w}(0, \infty)$ is the space of all real valued functions with finite quasi-norm

$$\|f\|_{L_{p,w}(0,\infty)} = \Big(\int_0^\infty |f(x)|^p \; w(x) \; dx \Big)^{\frac{1}{p}}.$$

The weighted Hardy operator H_w is defined by

$$(H_w f)(r) = \frac{1}{W(r)} \int_0^r f(x)\, w(x)\, dx.$$

where $0 < W(r) := \displaystyle\int_0^r w(x)\, dx < \infty$ for all $r > 0$. Note that for $w(x) \equiv 1$ the operator H_w is the usual Hardy operator $(Hf)(x) = \frac{1}{x} \int_0^x f(t)\, dt$.

In [7], [4], in particular, the following statement was proved.

Theorem 1.1. *Let* $0 < p < 1$, $\alpha < 1 - \frac{1}{p}$ *and* $M > 0$. *Moreover, let* f *be a non-negative measurable function on* $(0, \infty)$ *such that for all* $x > 0$

$$f(x) \leq \frac{M}{x}\Big(\int_0^x f^p(y)\ y^{p-1}\ dy\Big)^{\frac{1}{p}}. \tag{1.1}$$

Then

$$\|r^{\alpha}(Hf)(r)\|_{L_p(0,\infty)} \leq N \||x|^{\alpha} f(x)\|_{L_p(0,\infty)} \tag{1.2}$$

where

$$N = M^{1-p}\Big(1-\alpha-\frac{1}{p}\Big)^{-\frac{1}{p}} p^{1-\frac{1}{p}}. \tag{1.3}$$

The constant N is sharp.

Remark 1. If f is a non-negative non-increasing function on $(0,\infty)$, then inequality (1.1) is satisfied with $M = p^{\frac{1}{p}}$, hence for such functions inequality (1.2) takes the form

$$\|r^{\alpha}(Hf)(r)\|_{L_p(0,\infty)} \leq \Big(1-\alpha-\frac{1}{p}\Big)^{-\frac{1}{p}} \||x|^{\alpha} f(x)\|_{L_p(0,\infty)}.$$

The factor $\big(1-\alpha-\frac{1}{p}\big)^{-\frac{1}{p}}$ in this inequality is sharp. This inequality was earlier proved in [5, p. 90], [3].

2 Main results

Lemma 2.1. *Let $0 < p < 1$, $c > 0$, w be a weight function on $(0,\infty)$ such that*

$$w(x) \leq c w(y) \quad for \quad 0 < y < x < \infty\ , \tag{2.1}$$

then for all $0 < x < r < \infty$

$$\Big(\int_0^x w(y)\ y^{p-1}\ dy\Big)^{1-\frac{1}{p}} \leq B\ x^{p-1} \frac{1}{r w(r)^{\frac{1}{p}}} \int_0^r w(t)\ dt, \tag{2.2}$$

where $B = c^{\frac{2}{p}-1}\ p^{\frac{1}{p}-1}$.

Proof. By (2.1) for $0 < y < x < r$ we have

$$c\, w(y)\ y^{p-1} \geq w(x)\ y^{p-1}.$$

Integrating over $(0,x)$ we get

$$\int_0^x w(y) y^{p-1}\, dx \geq c^{-1}\ w(x)\ \Big(\int_0^x y^{p-1}\ dy\Big) = c^{-1}\ w(x)\ \frac{x^p}{p} \geq c^{-2}\ w(r)\ \frac{x^p}{p}\ .$$

Since $1-\frac{1}{p} < 0$

$$\Big(\int_0^x w(y)\ y^{p-1}\ dy\Big)^{1-\frac{1}{p}} \leq c^{\frac{2}{p}-2}\ \frac{w(r)}{w^{\frac{1}{p}}(r)}\ \Big(\frac{x^p}{p}\Big)^{1-\frac{1}{p}}.$$

Since for $0<t<r$ we have $w(r)\le c\,w(t)$, it follows by integration over $(0,r)$ that

$$w(r)\le \frac{c}{r}\int_0^r w(t)\,dt,$$

consequently

$$\begin{aligned}\Big(\int_0^x w(y)\ y^{p-1}\ dy\Big)^{1-\frac{1}{p}} &\le \frac{c^{\frac{2}{p}-1}}{rw(r)^{\frac{1}{p}}}\int_0^r w(t)\ dt\ \Big(\frac{x^p}{p}\Big)^{1-\frac{1}{p}}\\ &= c^{\frac{2}{p}-1}\ p^{\frac{1}{p}-1}\ x^{p-1}\frac{1}{rw(r)^{\frac{1}{p}}}\int_0^r w(t)\ dt.\end{aligned}$$

□

Lemma 2.2. *Let $0<p<1$, $c>0$, $A>0$, w be a weight function on $(0,\infty)$ satisfying condition (2.1). If f is a non-negative measurable function on $(0,\infty)$ such that for almost all $0<x<\infty$*

$$f(x)\le A\Big(\int_0^x w(y)\ y^{p-1}\ dy\Big)^{-\frac{1}{p}}\Big(\int_0^x f^p(y)w(y)\ y^{p-1}\ dy\Big)^{\frac{1}{p}}, \tag{2.3}$$

Then for all $r>0$

$$(H_w f)(r)\le \frac{C}{r\ w^{\frac{1}{p}}(r)}\Big(\int_0^r f^p(y)w(y)\ y^{p-1}\ dy\Big)^{\frac{1}{p}}, \tag{2.4}$$

where $C=pA^{1-p}\ B$.

Proof. Indeed, by (2.3) it follows that

$$f^{1-p}(x)\le A^{1-p}\Big(\int_0^x w(y)\ y^{p-1}\ dy\Big)^{-\frac{1-p}{p}}\Big(\int_0^x f^p(y)w(y)\ y^{p-1}\ dy\Big)^{\frac{1-p}{p}},$$

hence

$$\begin{aligned}f(x)w(x)&\le A^{1-p}\Big(\int_0^x w(y)\ y^{p-1}\ dy\Big)^{1-\frac{1}{p}} f^p(x)w(x)\Big(\int_0^x f^p(y)w(y)\ y^{p-1}\ dy\Big)^{\frac{1}{p}-1}\\ &= pA^{1-p}\Big(\int_0^x w(y)\ y^{p-1}\ dy\Big)^{1-\frac{1}{p}} x^{1-p}\Big[\Big(\int_0^x f^p(y)w(y)\ y^{p-1}\ dy\Big)^{\frac{1}{p}}\Big]'.\end{aligned}$$

Hence by (2.2) for $0<x<r$

$$f(x)w(x)\le pA^{1-p}\ B\frac{1}{rw(r)^{\frac{1}{p}}}\int_0^r w(t)\ dt\Big[\Big(\int_0^x f^p(y)w(y)\ y^{p-1}\ dy\Big)^{\frac{1}{p}}\Big]'$$

Integrating over $(0,r)$, we obtain

$$\begin{aligned}&\int_0^r f(x)\ w(x)\ dx\\ &\le pA^{1-p}\ B\frac{1}{rw(r)^{\frac{1}{p}}}\int_0^r w(t)\ dt\Big[((\int_0^r f^p(y)w(y)\ y^{p-1}\ dy\Big)^{\frac{1}{p}}\Big].\end{aligned}$$

Hence inequality (2.4) follows. □

Remark 2. *For non-negative measurable functions equality in (6) holds if and only if*

$$f(x) = K\Big(\int_0^x \omega(y)y^{p-1}dy\Big)^{\frac{A^p-1}{p}}, K \geq 0,$$

for almost all $x \in (0,\infty)$. To verify it suffices to rewrite (6) with the equality sign in the form

$$f^p(x)\int_0^x \omega(y)y^{p-1}dy = A\int_0^x f^p(y)\omega(y)y^{p-1}dy,$$

differentiate and solve the resulting differential equation.

Theorem 2.1. *Let $0 < p < 1$, $c > 0$, $A > 0$, w be a weight function on $(0,\infty)$ satisfying condition (2.1), and $\alpha < 1 - \frac{1}{p}$. If f is a non-negative measurable function on $(0,\infty)$ satisfying (2.3), then*

$$\| r^\alpha (H_w f)(r) \|_{L_{p,w}(0,\infty)} \leq D \, \| x^\alpha \; f(x)\|_{L_{p,w}(0,\infty)}, \tag{2.5}$$

where

$$D = A^{1-p}c^{\frac{2}{p}-1}\Big(1-\alpha-\frac{1}{p}\Big)^{-\frac{1}{p}}. \tag{2.6}$$

Proof. Indeed

$$\| r^\alpha (H_w f)(r) \|_{L_{p,w}(0,+\infty)} = \Big(\int_0^\infty r^{\alpha p}\; (H_w f)^p(r)w(r)\; dr\Big)^{\frac{1}{p}}$$

and by (2.4) we get

$$\| r^\alpha (H_w f)(r) \|_{L_{p,w}(0,+\infty)} \leq \Big(\int_0^\infty r^{\alpha p}\; C^p \frac{1}{r^p\; w(r)}\; \Big(\int_0^r f^p(y)\; w(y)\; y^{p-1}\; dy\Big)\; w(r)\; dr\Big)^{\frac{1}{p}}$$

$$= C\; \Big(\int_0^\infty r^{\alpha p - p}\; \Big(\int_0^r f^p(y)\; w(y)\; y^{p-1}\; dy\Big)\; dr\Big)^{\frac{1}{p}}$$

$$= C\; \Big(\int_0^\infty \Big(\int_y^\infty r^{\alpha p - p}\; dr\Big)\; f^p(y)\; w(y)\; y^{p-1}\; dy\Big)^{\frac{1}{p}}.$$

Since $\alpha p - p + 1 < 0$ we get

$$\| r^\alpha (H_w f)(r) \|_{L_{p,w}(0,+\infty)} \leq C\; \Big(\int_0^\infty \frac{y^{\alpha p - p + 1}}{\alpha p - p + 1}\; f^p(y)\; w(y)\; y^{p-1}\; dy\Big)^{\frac{1}{p}}$$

$$= \frac{C}{(-\alpha p + p - 1)^{\frac{1}{p}}}\; \Big(\int_0^\infty y^{\alpha p}\; f^p(y)\; w(y)\; dy\Big)^{\frac{1}{p}}$$

$$= \frac{C}{(-\alpha p + p - 1)^{\frac{1}{p}}}\; \| x^\alpha\; f(x)\|_{L_{p,w}(0,\infty)}.$$

□

Remark 3. *If f is a non-increasing function on $(0,\infty)$, then (2.3) holds with $A = 1$.*

Remark 4. *If in Theorem 2.1 $w \equiv 1$, then inequality (2.3) takes form (1.1) with $M = Ap^{\frac{1}{p}}$. Also $c = 1$, hence $D = N$.*

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Received: 7.08.2013