**EURASIAN MATHEMATICAL JOURNAL**
ISSN 2077-9879
Volume 4, Number 3 (2013), 120 – 126

# BEST POLYNOMIAL APPROXIMATIONS AND WIDTHS OF CERTAIN CLASSES OF FUNCTIONS IN THE SPACE $L_2$

**G.A. Yusupov**

Communicated by T.V. Tararykova

**Key words:** best polynomial approximations, generalized modulus of continuity, extremal characteristics, widths.

**AMS Mathematics Subject Classification:** 42A10.

**Abstract.** In the paper exact values of the $n$-widths are found for the class of differentiable periodic functions in the space $L_2[0, 2\pi]$, satisfying the condition

$$\left(\int_0^t \tau\, \Omega_m^{2/m}(f^{(r)}, \tau) d\tau\right)^{m/2} \leq \Phi(t),$$

where $0 < t \leq \pi/n,\ m, n, r \in \mathbb{N}$, $\Omega_m(f^{(r)}, \tau)$ is the generalized modulus of continuity of order $m$ of the derivative $f^{(r)} \in L_2[0, 2\pi]$, and $\Phi(t)$, $0 \leq t < \infty$ is a continuous non-decreasing function, such that $\Phi(0) = 0$ and $\Phi(t) > 0$ for $t > 0$.

## 1 Preliminaries and definitions

Denote by $L_2 := L_2[0, 2\pi]$ the space of all real-valued $2\pi$-periodic Lebesgue measurable functions $f$ with finite norm

$$\|f\| \overset{def}{=} \|f\|_{L_2} = \left(\frac{1}{\pi}\int_0^{2\pi} |f(x)|^2 dx\right)^{1/2} < \infty.$$

Consider its subspace

$$\Im_{n-1} = \left\{T_{n-1}(x) : T_{n-1}(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{n-1} (\alpha_k \cos kx + \beta_k \sin kx)\right\}$$

of all trigonometric polynomials of order $\leq n-1$. It is well known that for an arbitrary function $f \in L_2$ with the formal Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)\,,$$

its best approximation in the metrics of $L_2$ by the subspace $\Im_{n-1}$ is given by the following formula

$$E_n(f) \stackrel{def}{=} \inf\{\|f-T_{n-1}\|: \ T_{n-1}\in\Im_{n-1}\}$$
$$= \|f-S_{n-1}(f)\| = \left\{\sum_{k=n}^{\infty}(a_k^2+b_k^2)\right\}^{1/2}, \tag{1.1}$$

where

$$S_{n-1}(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n-1}(a_k\cos kx + b_k\sin kx)$$

is the partial sum of order $n-1$ of the Fourier series of the function $f$.

Denote by $L_2^{(r)}$, $r\in\mathbb{N}$ the set of all functions $f\in L_2$, whose derivatives $f^{(r-1)}$ are absolutely continuous and $f^{(r)}\in L_2$.

Let

$$\omega_m(f;t)_2 \stackrel{def}{=} \sup\{\|\triangle_h^m f(\cdot)\|: \ |h|\le t\} \tag{1.2}$$

be the modulus of continuity of of order $m$ of the function $f\in L_2$, where

$$\triangle_h^m f(x) = \sum_{k=0}^{m}(-1)^k\binom{m}{k}f(x+(m-k)h)$$

is the difference of order $m$ of the function $f$ with step $h$.

Among extremal problems of the theory of approximations of functions $f\in L_2$ one of the most important is the problem of finding sharp constants in inequalities of Jackson-Stechkin type

$$E_n(f)\le\chi\cdot n^{-r}\omega_m(f^{(r)},\tau/n), \quad f\in L_2^{(r)}, \quad \tau>0, \quad n\in\mathbb{N}. \tag{1.3}$$

Aiming at optimizing the constant $\chi$ in this inequality various smoothness characteristics were introduced (see, for example, [3], [10], [13], [1], [5], [4], [12], [2], [8] and references therein). Application of such smoothness characteristics of $2\pi$-periodic functions as, for example, the trigonometric modulus of continuity in [2] and $\mathcal{K}$-functionals in [12], allowed obtaining new significant results related to further studies of inequalities of Jackson-Stechkin type (1.3).

When studying certain extremal problems of the theory of approximations of functions $f\in L_2$ sometimes it is more convenient to use the following characteristics equivalent to (1.2)

$$\Omega_m(f;t)_2 = \left\{\frac{1}{t^m}\int_0^t\cdots\int_0^t\|\Delta_{\overline{h}}^m f(\cdot)\|^2 dh_1\cdots dh_m\right\}^{1/2}, \quad t>0,$$

where $\overline{h}=(h_1,\ldots,h_m)$, $\Delta_{\overline{h}}^m=\Delta_{h_1}^1\circ\cdots\circ\Delta_{h_m}^1$, $\Delta_{h_j}^1 f = f(\cdot+h_j)-f(\cdot)$, $j=\overline{1,m}$ (see, for example, [7], [8], [10]). The following statement holds.

**Theorem 1.1.** *Let $m,n,r\in\mathbb{N}, r\ge m$. Then the following equality is valid*

$$\mathcal{M}_{m,n,r} = \sup_{\substack{f\in L_2^{(r)}\\ f^{(r)}\ne const}} \frac{n^{r-m}E_n(f)}{\left(\int\limits_0^{\pi/n}\tau\,\Omega_m^{2/m}(f^{(r)};\,\tau)d\tau\right)^{m/2}} = \frac{1}{(\pi^2-4)^{m/2}}. \tag{1.4}$$

*Proof.* If $f \in L_2$ and

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

is its Fourier series, then direct calculation shows that

$$\Omega_m^2(f^{(r)};\tau) = 2^m \sum_{k=1}^{\infty} k^{2r} \varrho_k^2 \left(1 - \frac{\sin k\tau}{k\tau}\right)^m, \tag{1.5}$$

where $\varrho_k^2 = a_k^2 + b_k^2,\ k \in \mathbb{N}$. By Hölder's inequality for sums and inequalities (1.1) and (1.5) we get

$$E_n^2(f) - \sum_{k=n}^{\infty} \frac{\sin k\tau}{k\tau} \varrho_k^2 = \sum_{k=n}^{\infty} \varrho_k^2 \left(1 - \frac{\sin k\tau}{k\tau}\right) = \sum_{k=n}^{\infty} \varrho_k^{2(1-1/m)} \varrho_k^{2/m} \left(1 - \frac{\sin k\tau}{k\tau}\right)$$

$$\leq \left(\sum_{k=n}^{\infty} \varrho_k^2\right)^{1-1/m} \left(\sum_{k=n}^{\infty} \varrho_k^2 \left(1 - \frac{\sin k\tau}{k\tau}\right)^m\right)^{1/m}$$

$$\leq \left(E_n^2(f)\right)^{1-1/m} 2^{-1} n^{-2r/m} \Omega_m^{2/m}(f^{(r)},\tau).$$

Therefore

$$E_n^2(f) \leq \left(E_n^2(f)\right)^{1-1/m} 2^{-1} n^{-2r/m}\, \Omega_m^{2/m}(f^{(r)},\tau) + \sum_{k=n}^{\infty} \frac{\sin k\tau}{k\tau} \varrho_k^2.$$

Multiplying both sides of this inequality by $\tau > 0$ and integrating over $\tau \in [0, \pi/n]$, we get

$$\frac{1}{2}\left(\frac{\pi}{n}\right)^2 E_n^2(f) \leq \left(E_n^2(f)\right)^{1-1/m} \cdot \frac{1}{2} \cdot n^{-2r/m} \left(\int_0^{\pi/n} \tau\, \Omega_m^{2/m}(f^{(r)};\tau) d\tau\right) + \frac{2}{n^2} \sum_{k=n}^{\infty} \varrho_k^2.$$

Hence

$$\left(\frac{\pi^2 - 4}{n^2}\right) E_n^2(f) \leq \left(E_n^2(f)\right)^{1-1/m} n^{-2r/m} \left(\int_0^{\pi/n} \tau\, \Omega_m^{2/m}(f^{(r)};\tau) d\tau\right)$$

and

$$E_n^2(f) \leq (\pi^2 - 4)^{-m/2} n^{-r+m} \left(\int_0^{\pi/n} \tau \Omega_m^{2/m}(f^{(r)},\tau) d\tau\right)^{m/2}. \tag{1.6}$$

Rewriting this inequality in the form

$$\frac{n^{r-m} E_n(f)}{\left(\int_0^{\pi/n} \tau\, \Omega_m^{2/m}(f^{(r)};\tau) d\tau\right)^{m/2}} \leq (\pi^2 - 4)^{-m/2},$$

and taking into account the definition of the quantity $\mathcal{M}_{m,n,r}$ in the left-hand side of equality (1.4) we get the following estimate above

$$\mathcal{M}_{m,n,r} \le (\pi^2 - 4)^{-m/2}. \tag{1.7}$$

For obtaining an estimate below consider the extremal function $f_0(x) = \sin nx \in L_2$. Since $E_n(f_0) = 1,\ \Omega_m(f_0^{(r)};\tau) = 2^{m/2} n^r \left(1 - \dfrac{\sin n\tau}{n\tau}\right)^{m/2}$,

$$\left(\int\limits_0^{\pi/n} \tau\, \Omega_m^{2/m}(f_0^{(r)};\tau) d\tau\right)^{m/2} = n^{r-m}(\pi^2-4)^{m/2},$$

we get

$$\mathcal{M}_{m,n,r} \ge \frac{n^{r-m} E_n(f_0)}{\left(\int\limits_0^{\pi/n} \tau\, \Omega_m^{2/m}(f_0^{(r)};\tau) d\tau\right)^{m/2}} = (\pi^2-4)^{-m/2}. \tag{1.8}$$

Combining inequalities (1.7) and (1.8) we obtain equality (1.4). □

## 2 Main results

Denote by $b_n(\mathfrak{M}, L_2)$, $d^n(\mathfrak{M}, L_2)$, $d_n(\mathfrak{M}, L_2)$, $\delta_n(\mathfrak{M}, L_2)$, and $\Pi_n(\mathfrak{M}, L_2)$ Bernshtein, Gelfand, Kolmogorov, linear, and projective $n$-widths respectively of a a centrally symmetric compact $\mathfrak{M}$ in the space $L_2$ (see the definitions, for example, in [12],[6]). These $n$-widths satisfy the following relations

$$b_n(\mathfrak{M}, L_2) \le d^n(\mathfrak{M}, L_2) \le d_n(\mathfrak{M}, L_2) = \delta_n(\mathfrak{M}, L_2) = \Pi_n(\mathfrak{M}, L_2). \tag{2.1}$$

Let also $E_n(\mathfrak{M}) := \sup\{E_n(f) : f \in \mathfrak{M}\}$.

Moreover, let $\Phi(t)$, $t \ge 0$ be an arbitrary continuous non-decreasing function, such that $\Phi(0) = 0$ and $\Phi(t) > 0$ for $t > 0$. By $W_m^{(r)}(\Phi), m, r \in \mathbb{N}$ we denote the class of all functions $f \in L_2^{(r)}$, for which for all $0 < t \le \pi/n$ the following inequality

$$\left(\int\limits_0^{t} \tau\, \Omega_m^{2/m}(f^{(r)},\tau) d\tau\right)^{m/2} \le \Phi(t)$$

is satisfied.

**Theorem 2.1.** *For all $m, n, r \in \mathbb{N}, r \ge m$*

$$\begin{aligned} p_{2n}\left(W_m^{(r)}(\Phi); L_2\right) &= p_{2n-1}\left(W_m^{(r)}(\Phi); L_2\right) \\ &= E_n\left(W_m^{(r)}(\Phi)\right) = (\pi^2-4)^{-m/2}\, n^{-r+m}\, \Phi(\pi/n), \end{aligned} \tag{2.2}$$

*where $p_n(\cdot)$ is any of the $n$-widths $b_n(\cdot)$, $d^n(\cdot)$, $d_n(\cdot)$, $\delta_n(\cdot)$, and $\Pi_n(\cdot)$.*

*Proof.* Applying the definition of the classes $W_m^{(r)}(\Phi)$ and taking into account relations (2.1), by inequality (1.6) we get the following estimate above

$$p_{2n}\left(W_m^{(r)}(\Phi);L_2\right)\le p_{2n-1}\left(W_m^{(r)}(\Phi);L_2\right)\le d_{2n-1}\left(W_m^{(r)}(\Phi);L_2\right)$$
$$\le E_n\left(W_m^{(r)}(\Phi)\right)\le(\pi^2-4)^{-m/2}\,n^{-r+m}\,\Phi(\pi/n). \tag{2.3}$$

For obtaining an estimate below for the *Bernshtein* width $b_{2n}\left(W_m^{(r)}(\Phi);L_2\right)$ we consider the $(n+1)$-dimensional ball of polynomials

$$S_{n+1}=\left\{T_n(x)\in\Im_n:\ \|T_n\|\le(\pi^2-4)^{-m/2}\,n^{-r+m}\,\Phi(\pi/n)\right\}$$

and claim that $S_{n+1}\subset W_m^{(r)}(\Phi)$. To verify this it is required to prove that for an arbitrary polynomial $T_n\in S_{n+1}$ the following inequality

$$\left(\int\limits_0^{\pi/n}\tau\,\Omega_m^{2/m}(T_n^{(r)},\tau)d\tau\right)^{m/2}\le\Phi\left(\frac{\pi}{n}\right)$$

holds.

Let

$$T_n(x)=\frac{\alpha_0}{2}+\sum_{k=1}^{n}\left(\alpha_k\cos kx+\beta_k\sin kx\right),\quad \varrho_k^{*2}=\alpha_k^2+\beta_k^2$$

be an arbitrary polynomial, belonging to the ball $S_{n+1}$. Then, since

$$(1-\sin k\tau/(k\tau))\le(1-\sin n\tau/(n\tau))$$

for all $1\le k\le n$, we get

$$\Omega_m^2(T_n^{(r)};\tau)=2^m\sum_{k=1}^{n}k^{2r}\varrho_k^2\left(1-\frac{\sin k\tau}{k\tau}\right)^m$$
$$\le 2^m n^{2r}\left(1-\frac{\sin n\tau}{n\tau}\right)^m\sum_{k=1}^{n}\varrho^{*2}_k=2^m n^{2r}\left(1-\frac{\sin n\tau}{n\tau}\right)^m\|T_n\|^2.$$

Hence

$$\left(\int\limits_0^{\pi/n}\tau\,\Omega_m^{2/m}(T_n^{(r)},\tau)d\tau\right)^{m/2}\le 2^{m/2}\left(\int_0^{\pi/n}\left(\tau-\frac{\sin n\tau}{n}\right)d\tau\right)^{m/2}n^r\,\|T_n\|$$
$$=(\pi^2-4)^{m/2}\,n^{r-m}\ \|T_n\|\le\Phi(\pi/n).$$

Therefore the inclusion $S_{n+1}\subset W_m^{(r)}(\Phi)$ is proved. From the Bernstein $n$-width [12] we obtain the lower bound

$$b_{2n}\left(W_m^{(r)}(\Phi);L_2\right)\ge b_{2n}\left(S_{n+1};L_2\right)=(\pi^2-4)^{-m/2}\,n^{-r+m}\,\Phi(\pi/n). \tag{2.4}$$

Taking into account relations (2.1) and combining inequalities (2.3) and (2.4) we arrive at the statement of the theorem. □

**Corollary 2.1.** *Under the assumptions of Theorem 2.1*

$$\sup\left\{|a_n(f)|:\ f(x)\in W_m^{(r)}(\Phi)\right\}=\sup\left\{|b_n(f)|:\ f(x)\in W_m^{(r)}(\Phi)\right\}$$

$$=(\pi^2-4)^{-m/2}\,n^{-r+m}\,\Phi(\pi/n), \tag{2.5}$$

*where $a_n(f)$ and $b_n(f)$ are cosine and sine Fourier coefficients of the function $f$.*

*Proof.* Let us prove equality (2.5) for the cosine Fourier coefficients $a_n(f)$. Keeping in mind that

$$a_n(f)=\frac{1}{\pi}\int\limits_0^{2\pi} f(x)\cos nx dx=\frac{1}{\pi}\int\limits_0^{2\pi}[f(x)-S_{n-1}(f;x)]\cos nx dx, \tag{2.6}$$

by applying the Cauchy-Schwartz inequality to the integral in the right-hand side of (2.6) and taking into account formulas (1.1) и (2.3), we obtain the following estimate above

$$\sup\left\{\,|a_n(f)|:\ f(x)\in W_m^{(r)}(\Phi)\right\}\le E_n\left(W_m^{(r)}(\Phi)\right)$$

$$\le(\pi^2-4)^{-m/2}\,n^{-r+m}\,\Phi(\pi/n). \tag{2.7}$$

In order to get an estimate below consider the function

$$f_1(x)=(\pi^2-4)^{-m/2}\,n^{-r+m}\,\Phi(\pi/n)\cos nx.$$

Clearly $f_1(x)\in W_m^{(r)}(\Phi)$. Hence

$$\sup\left\{\,|a_n(f)|:\ f(x)\in W_m^{(r)}(\Phi)\right\}\ge|a_n(f_1)|=(\pi^2-4)^{-m/2}\,n^{-r+m}\,\Phi(\pi/n). \tag{2.8}$$

The statement follows by combining inequalities (2.7) and (2.8). □

## Acknowledgments

The author thanks the reviewer for useful advices and comments which were used in the paper.

## References

[1] A.G. Babenko, *On the sharp constant in Jackson's inequality in* $L^2$, Matem. Notes, 39 (1986) no. 5, 651 - 664 (in Russian).

[2] A.G. Babenko, N.I. Chernykh, V.T. Shevaldin, *Jackson-Stechkin's inequalities in* $L_2$ *with the trigonometric modulus of continuity*, Matem. Notes, 65 (1999), no. 6, 928 - 932 (in Russian).

[3] N.I. Chernykh, *On the best approximation of periodic functions by trigonometric polynomials in* $L_2$, Matem. Notes, 2 (1967), no. 5, 513 - 522 (in Russian).

[4] V.I. Ivanov, O.I. Smirnov, *Jackson's constants and Young's constant in the spaces* $L_p$, Tula: Tula State University, 1995 (in Russian).

[5] A.A. Ligun, *Sharp Jackson type inequalities for periodic functions in the space* $L_2$, Matem. Notes, 43(1988), no. 6, 757 - 769 (in Russian).

[6] A. Pinkus, *n-Widths in approximation theory*, Berlin: Springer-Verlag, Heidelberg, New York, Tokyo, 1985.

[7] K.V. Runovski, *On approximation by families of linear polynomial operators in the spaces* $L_p, 0 < p < 1$, Matem. Sbornik, 185 (1994), no. 8, 81 - 102 (in Russian).

[8] K.V. Runovski, H.-J. Schmeisser, *Methods of trigonometric approximation and generalized smoothness. I*, Eurasian Math. J., 2 (2011), no. 3, 98-124.

[9] M.Sh. Shabozov, *Widths of certain classes of periodic differentiable functions in the space* $L_2[0, 2\pi]$, Matem. Notes, 87 (2010), no. 4, 616 - 623 (in Russian).

[10] E.A. Storozhenko, V.G. Krotov, P. Osvald, *Direct and inverse theorems of Jackson type in the spaces* $L_p, 0 < p < 1$, Matem. Sbornik, 98(140) (1975), no. 3(11), 395 - 415 (in Russian).

[11] L.V. Taikov, *Inequalities containing best approximations and modulus of continuity of functions in* $L_2$, Matem. Notes, 20 (1976), no. 3, 433 - 438 (in Russian).

[12] V.M. Tikhomirov, *Certain topics of the theory of approximations,* Moscow, Moscow State University, 1976 (in Russian).

[13] S.B. Vakarchuk, $\mathcal{K}$*-functionals and sharp values of the n-widths of certain classes in* $L_2$, Matem. Notes , 66 (1999), no. 4, 494 - 499 (in Russian).

[14] V.A. Yudin, *Diofant approximations in extremal problems*, Doklady Ac. Sci. USSR, 251 (1980), no. 1, 54 - 57 (in Russian).

Gulzorkhon Amirshoevich Yusupov
Tajik National University
734025, Tajikistan, Dushanbe, Rudaki Av. 17
E-mail: G_7777@mail.ruE-mail

Received: 4.10.2011
Revised version: 15.06.2012