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Short communications

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HARDY-LITTLEWOOD TYPE THEOREMS

M.I. Dyachenko, E.D. Nursultanov, A.M. Zhantakbayeva

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Key words and phrases: Fourier coefficients, Fourier series, Hardy-Littlewood theorem.

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Abstract. In this paper we state new theorems of Hardy-Littlewood type for functions with general monotone Fourier coefficients. Sharpness of stated results is discussed.

We consider 1-periodic integrable functions f with the Fourier series $\sum\limits_{k\in\mathbb{N}} a_k(f)e^{2\pi ikx}$, where $a_k(f)$ are real numbers. Let us denote by C a constant depending only on the parameters p and α. These constants can be different in different cases.

One of the most interesting problems in the theory of trigonometric series is studying the relationship between the degree of integrability of a function and behavior of its Fourier coefficients. The fundamental result in this direction is the following

Theorem A. (Hardy-Littlewood) *If* $1 < p < \infty$, $f \in L_1[0,1]$, $f \sim \sum\limits_{k\in\mathbb{N}} a_k e^{2\pi ikx}$, *and* $\{a_k\}_{k=1}^{\infty}$ *is monotonically decreasing, then* $f \in L_p[0,1]$ *if and only if*

$$J_p(a) = \left(\sum_{k=1}^{\infty} k^{p-2}|a_k|^p\right)^{1/p} < \infty.$$

In this regard, many authors studied the possibility of weakening of the monotonicity condition on the coefficients a_k, so that the statement of Theorem A still holds.

It is known that the monotonicity condition on the Fourier coefficients cannot be skipped. On the other hand, one can refine the monotonicity condition (see [2] - [6]). In [1], the monotonicity condition was replaced by the general monotonicity condition of the form: for some $c > 1$ and $\alpha \in (0,1]$ for all $k \in \mathbb{N}$

$$\sum_{m=k}^{2k} |\Delta a_m| \leq ck^{\alpha-1} \sum_{m=[\frac{k}{c}]+1}^{\infty} \frac{a_m}{m^{\alpha}}$$

and

$$a_k \geq 0. \tag{1}$$

In this note, we state the following result.

Theorem 1. *Let* $\alpha \in (0, \frac{1}{2}]$, $p > \frac{1}{\alpha}$, $f \in L_1[0,1]$ *and* $f \sim \sum_{k=1}^{\infty} a_k e^{2\pi i k x}$. *If for some* $C > 0$

$$|a_k| \leq C k^{\alpha - 1} \left| \sum_{m=[\frac{k}{2}]+1}^{\infty} \frac{a_m}{m^\alpha} \right|, \quad k \in \mathbb{N}, \tag{2}$$

then $f \in L_p[0,1]$ *if and only if* $J_p(a) < \infty$.

Moreover, there exists $C(p) > 0$ *such that*

$$J_p(a) \leq C(p) \|f\|_{L_p[0,1]}$$

for all $f \in L_1[0,1]$.

Remark 1. In Theorem 1 the case in which the series $\sum\limits_{m=1}^{\infty} \frac{a_m}{m^\alpha}$ diverges is allowed. Note that if this series diverges, then it diverges to $+\infty$ or $-\infty$. In this case (2) is satisfied, and $J_p(f) = +\infty$. Hence in this case $f \notin L_p[0,1]$.

The question on the possibility of extension of Theorem 1 to $p \leq \frac{1}{\alpha}$ arises. We note that if $2 < p < \frac{1}{\alpha}$, the series

$$\sum_{m=1}^{\infty} \frac{a_m}{m^\alpha}$$

converges, and (2) holds, then $|a_k| \leq \frac{C}{k^{1-\alpha}}$. Therefore,

$$J_p(a) \leq C \sum_{k=1}^{\infty} \frac{k^{p-2}}{k^{(1-\alpha)p}} = C \sum_{k=1}^{\infty} k^{\alpha p - 2} < \infty.$$

So $J_p(a) < \infty$ independently of whether $f \in L_p[0,1]$ or not.

Therefore, the following result is of interest.

Theorem 2. *Let* $\alpha \in (0, \frac{1}{2})$, $2 < p < 1/\alpha$, *then for every* $\delta > 0$ *there exists a function* f *such that the series*

$$\sum_{m=1}^{\infty} \frac{a_m}{m^\alpha}$$

converges,

$$|a_k| \leq \frac{4}{k^{1-\alpha}} \left| \sum_{m=k}^{\infty} \frac{a_m}{m^\alpha} \right|, \quad k \in \mathbb{N},$$

and $\|f\|_{L_p} < \delta$, *but* $J_p(a) \geq 1$.

Next we can state the condition on the coefficients, which allow us to obtain a criterion in the case of $\alpha \in (1/2, 1)$.

Theorem 3. *Let* $\alpha \in (1/2, 1)$, $1/\alpha < p < \infty$, $f \in L_1[0,1]$ *and* $f \sim \sum_{k=1}^{\infty} a_k e^{2\pi i k x}$. *If for some* $C > 0$

$$\sum_{k=2^{m-1}}^{2^m-1} |\Delta a_k| \le C \frac{1}{2^{(1-\alpha)m}} \left| \sum_{k=2^m}^{\infty} \frac{a_k}{k^\alpha} \right|, \quad m \in \mathbb{N}, \tag{3}$$

then $f \in L_p[0,1]$ *if and only if the following series converges*

$$\left(\sum_{m=0}^{\infty} \left(2^{m(\frac{1}{p'}+\alpha-1)} \left| \sum_{k=2^m}^{\infty} \frac{a_k}{k^\alpha} \right| \right)^p \right)^{\frac{1}{p}}. \tag{4}$$

In Theorem 3 the case in which the series $\sum\limits_{m=1}^{\infty} \frac{a_m}{m^\alpha}$ diverges is allowed. See Remark 1.

The following example shows that Theorem 3 is not true for $1 < p \le \frac{1}{\alpha}$.

Indeed, let us consider the function

$$f(x) = \sum_{k=2}^{\infty} a_k e^{2\pi i k x}, \text{ where } a_k = \frac{1}{k^{\frac{1}{p'}} \ln k}.$$

Then for all $m \in \mathbb{N}$ even a stronger condition than (3) holds

$$\sum_{k=2^{m-1}}^{2^m-1} |\Delta a_k| \le C \frac{1}{2^{(1-\alpha)m}} \left| \sum_{k=2^m}^{2^m-1} \frac{a_k}{k^\alpha} \right|.$$

In this case $J_p(a) = \left(\sum\limits_{k=2}^{\infty} \frac{1}{k(\ln k)^p} \right)^{\frac{1}{p}} < \infty$. Hence by Theorem A, $f \in L_p[0,1]$. On the other hand, since

$$\sum_{k=2^m}^{\infty} \frac{a_k}{k^\alpha} = \infty,$$

condition (3) holds and series (4) diverges.

Note that in the paper [1] there was proved a statement similar to Theorem 3, but with the additional condition (1) of coefficients a_k.

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Mikhail Ivanovich Dyachenko
Faculty of Mechanics and Mathematics
M.V. Lomonosov Moscow State University
Moscow 119992, Russia
E-mail: dyach@mail.ru

Yerlan Dautbekovich Nursultanov
Kazakh branch of M.V. Lomonosov Moscow State University
2 Mirzoyan St, 010008 Astana, Kazakhstan
E-mail: er-nurs@yandex.ru

Ayagoz Melisovna Zhantakbayeva
L.N. Gumilyov Eurasian National University
2 Mirzoyan St, 010008 Astana, Kazakhstan
E-mail: ayagoz.zhantakbayeva@yandex.ru

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Евразийский национальный университет имени Л.Н. Гумилева,
главный корпус, каб. 355
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