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THE HARDY SPACE H^1 ON NON-HOMOGENEOUS SPACES AND ITS APPLICATIONS – A SURVEY

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Key words: non-homogeneous space, Hardy space, RBMO(μ), RBLO(μ), atom, molecule, Calderón-Zygmund operator, fractional integral, Marcinkiewicz integral, commutator.

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Abstract. Let $(\mathcal{X}, d, \mu)$ be a metric measure space satisfying both the upper doubling and the geometrically doubling conditions. In this article, the authors give a survey on the Hardy space H^1 on non-homogeneous spaces and its applications. These results include: the regularized BMO spaces RBMO(μ) and $\widetilde{\mathrm{RBMO}}(\mu)$, the regularized BLO spaces RBLO(μ) and $\widetilde{\mathrm{RBLO}}(\mu)$, the Hardy spaces $H^1(\mu)$ and $\widetilde{H}^1(\mu)$, the behaviour of the Calderón-Zygmund operator and its maximal operator on Hardy spaces and Lebesgue spaces, a weighted norm inequality for the multilinear Calderón-Zygmund operator, the boundedness on Orlicz spaces and the endpoint estimate for the commutator generated by the Calderón-Zygmund operator or the generalized fractional integral with any RBMO(μ) function or any $\widetilde{\mathrm{RBMO}}(\mu)$ function, and equivalent characterizations for the boundedness of the generalized fractional integral or the Marcinkiewicz integral, respectively.

1 Introduction

The real-variable theory of the Hardy space on the D-dimensional Euclidean space $\mathbb{R}^D$ initiated by Stein and Weiss [34] plays an important role in various fields of analysis and partial differential equations; see, for example, [34, 32, 7, 33]. It is well known that the Hardy space is a good substitute of $L^p(\mathbb{R}^D)$ when $p \in (0, 1]$ since some of the singular integrals (for example, the Riesz transform) are bounded on $H^p(\mathbb{R}^D)$, but not on $L^p(\mathbb{R}^D)$ when $p \in (0, 1]$. In 1972, Fefferman and Stein [7] showed that the Hardy space $H^1(\mathbb{R}^D)$ is the predual of the space BMO($\mathbb{R}^D$). Later, Coifman [2] obtained the atomic characterization of $H^p(\mathbb{R})$, which was extended to $D > 1$ by Latter in [21]. In 1974, Coifman [3] introduced the notion of molecules on $\mathbb{R}$ to obtain the characterizations of Fourier transforms of boundary distributions of functions in $H^p(\mathbb{R})$, with $p \in (0, 1]$, which were further applied to establish some Fourier multiplier theorems on $H^p(\mathbb{R})$. Moreover, the atomic and the molecular characterizations enable the extension of the real variable theory of the Hardy spaces on $\mathbb{R}^D$ to general metric measure spaces.

In 1971, Coifman and Weiss [4] introduced the following setting of the space of homogeneous type which generalizes the Euclidean space $\mathbb{R}^D$ with the D-dimensional Lebesgue measure.

Definition 1.1. A metric space $(\mathcal{X}, d)$ equipped with a nonnegative measure μ is called a *space of homogeneous type* if the measure μ satisfies the *doubling condition*: there exists a positive constant $C(\mu)$ such that, for any ball $B(x,r) := \{y \in \mathcal{X} : \ d(x,y) < r\}$ with $x \in \mathcal{X}$ and $r \in (0,\infty)$,

$$\mu(B(x,2r)) \le C(\mu)\mu(B(x,r)). \tag{1.1}$$

Typical examples of spaces of homogeneous type include Euclidean spaces, Euclidean spaces with weighted measure satisfying (1.1), Heisenberg groups, and connected and simply connected nilpotent Lie groups. Since 1970s, there have been a lot of fruitful results on the theory of function spaces and singular integral operators on spaces of homogeneous type; see, for example, [4, 5, 35, 11, 6]. The space of homogeneous type is seen as a natural setting for the study of the theory of Hardy spaces and singular integrals.

In the last two decades, some research indicates that many results in the theory of the classical Hardy spaces and singular integrals are still valid for non-doubling measures; see, for example, [28, 29, 37, 38, 30, 31, 39, 40, 41, 42]. In particular, let μ be a non-negative Radon measure on the Euclidean space $\mathbb{R}^D$ which satisfies the *polynomial growth condition*: there exist positive constants C_0 and $\kappa \in (0, D]$ such that, for all $x \in \mathbb{R}^D$ and $r \in (0,\infty)$,

$$\mu(B(x,r)) \le C_0 r^\kappa, \tag{1.2}$$

where $B(x,r) := \{y \in \mathbb{R}^D : \ |y - x| < r\}$. The measure μ does not necessarily satisfy (1.1). We mention that the analysis with non-doubling measures, especially, the $T(b)$ theorem and the $L^2(\mu)$-boundedness of the Cauchy integral, plays an important role in solving the long open Painlevé's problem by Tolsa in [40]; see also [41, 42].

However, as pointed out by Hytönen [17], $(\mathbb{R}^D, |\cdot|, \mu)$ (or more generally, a metric measure space $(\mathcal{X}, d, \mu)$ with μ satisfying the polynomial growth condition (1.2)) is different from, but not more general than, the space of homogeneous type. Hytönen further introduced the following new class of metric measure spaces, which unifies the spaces of homogeneous type in the sense of Coifman and Weiss and $(\mathbb{R}^D, |\cdot|, \mu)$ with μ satisfying (1.2); see also [1, 20]. In what follows, let $\mathbb{R}_+ := (0,\infty)$.

Definition 1.2. A metric measure space $(\mathcal{X}, d, \mu)$ is said to be *upper doubling* if μ is a Borel measure on $\mathcal{X}$ and there exist a *dominating function* $\lambda : \mathcal{X} \times \mathbb{R}_+ \to \mathbb{R}_+$ and a positive constant $C(\lambda)$, depending on λ, such that, for each $x \in \mathcal{X}$, $r \to \lambda(x,r)$ is non-decreasing and, for all $x \in \mathcal{X}$ and $r \in (0,\infty)$,

$$\mu(B(x,r)) \le \lambda(x,r) \le C(\lambda)\lambda(x,r/2).$$

Remark 1.1. (i) Let $(\mathcal{X}, d, \mu)$ be upper doubling with λ being the dominating function on $\mathcal{X} \times (0, \infty)$ as in Definition 1.2. It was proved in [20] that there exists another dominating function $\widetilde{\lambda}$ such that $\widetilde{\lambda} \le \lambda$, $C(\widetilde{\lambda}) \le C(\lambda)$ and, for all x, $y \in \mathcal{X}$ with $d(x, y) \le r$,

$$\widetilde{\lambda}(x, r) \le C(\widetilde{\lambda})\widetilde{\lambda}(y, r).$$

(ii) It was shown in [36] that the upper doubling condition is equivalent to the *weak growth condition*: there exist a dominating function $\lambda : \mathcal{X} \times \mathbb{R}_+ \to \mathbb{R}_+$ with $r \to \lambda(x, r)$ non-decreasing and positive constants $C(\lambda)$ and ε such that,

(a) for all $r \in (0, \infty)$, $t \in [0, r]$, x, $y \in \mathcal{X}$ and $d(x, y) \in [0, r]$,

$$|\lambda(y, r + t) - \lambda(x, r)| \le C(\lambda) \left[\frac{d(x, y) + t}{r}\right]^{\varepsilon} \lambda(x, r);$$

(b) for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \le \lambda(x, r).$$

Obviously, a space of homogeneous type is a special case of upper doubling spaces, where one can take the dominating function $\lambda(x, r) := \mu(B(x, r))$. Moreover, let μ be a non-negative Radon measure on $\mathbb{R}^D$ which only satisfies the polynomial growth condition (1.2). By taking $\lambda(x, r) := Cr^{\kappa}$ for positive constants C and $\kappa \in (0, D]$, we see that $(\mathbb{R}^D, |\cdot|, \mu)$ is also an upper doubling measure space.

In [17], Hytönen also introduced the following notion of geometrically doubling metric spaces.

Definition 1.3. A metric space $(\mathcal{X}, d)$ is said to be *geometrically doubling* if there exists some $N_0 \in \mathbb{N} := \{1, 2, \dots\}$ such that, for any ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

Remark 1.2. Let $(\mathcal{X}, d)$ be a metric space. In [17], Hytönen showed that the following statements are equivalent:

(i) $(\mathcal{X}, d)$ is geometrically doubling;

(ii) For any $\varepsilon \in (0, 1)$ and ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, \varepsilon r)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0\varepsilon^{-n}$, where here and in what follows, N_0 is as in Definition 1.3 and $n := \log_2 N_0$;

(iii) For every $\varepsilon \in (0, 1)$, any ball $B(x, r) \subset \mathcal{X}$ contains at most $N_0\varepsilon^{-n}$ centers of disjoint balls $\{B(x_i, \varepsilon r)\}_i$;

(iv) There exists $M \in \mathbb{N}$ such that any ball $B(x, r) \subset \mathcal{X}$ contains at most M centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, r/4)\}_{i=1}^M$.

It is well known that the doubling condition (1.1) implies the geometrically doubling condition; see [4, pp. 67-68]. Conversely, if $(\mathcal{X}, d)$ is a complete geometrically doubling metric space, then there exists a Borel measure μ on $\mathcal{X}$ satisfying the doubling condition (1.1); see [17] and its related references.

In what follows, a metric measure space $(\mathcal{X}, d, \mu)$ satisfying both the upper doubling and geometrically doubling conditions is simply called a *non-homogeneous space*. A natural and interesting question is how to generalize the classical theory of function spaces and singular integrals to the non-homogeneous context. This question has been answered from many aspects concerning the atomic Hardy space $H^1(\mu)$, the regularized bounded mean oscillation space $\mathrm{RBMO}(\mu)$, the Calderón-Zygmund operator, the fractional integral and the Marcinkiewicz integral. However, there are still many open problems such as the characterization of $H^1(\mu)$ in terms of the maximal function and the Littlewood-Paley function, and the theory of the Hardy space $H^p(\mu)$ for $p \in (0, 1)$; see [44] for the details and some other unsolved questions.

One of main motivations for developing harmonic analysis on the non-homogeneous space is the natural existence of some important singular integral operators on some non-homogeneous spaces, such as Bergman-type singular operators from several complex variables (see [43, 19]).

The purpose of this article is to give a survey about the Hardy space $H^1(\mu)$ over a non-homogeneous space and its applications. The main results that we review include: the regularized BMO spaces $\mathrm{RBMO}(\mu)$ and $\widetilde{\mathrm{RBMO}}(\mu)$, the regularized BLO spaces $\mathrm{RBLO}(\mu)$ and $\widetilde{\mathrm{RBLO}}(\mu)$, the Hardy spaces $H^1(\mu)$ and $\widetilde{H}^1(\mu)$, the behaviour of the Calderón-Zygmund operator and its maximal operator on Hardy spaces and Lebesgue spaces, a weighted norm inequality for the multilinear Calderón-Zygmund operator, the boundedness on Orlicz spaces and the endpoint estimate for the commutator generated by a Calderón-Zygmund operator or the generalized fractional integral with any $\mathrm{RBMO}(\mu)$ function or any $\widetilde{\mathrm{RBMO}}(\mu)$ function, and equivalent characterizations of the boundedness of the generalized fractional integral, the Marcinkiewicz integral respectively.

This survey is organized as follows.

In Section 2, we begin with some basic facts on non-homogeneous spaces $(\mathcal{X}, d, \mu)$. Then we review the results on the regularized BMO space $\mathrm{RBMO}(\mu)$ and its variant, $\widetilde{\mathrm{RBMO}}(\mu)$, including the John-Nirenberg inequality and the John-Strömberg sharp maximal characterization of $\mathrm{RBMO}(\mu)$ or $\widetilde{\mathrm{RBMO}}(\mu)$, and the relation between $\mathrm{RBMO}(\mu)$ and $\widetilde{\mathrm{RBMO}}(\mu)$. Some related topics on the space of type BLO are also included in this section and, in particular, we show that, if $(\mathcal{X}, d, \mu)$ is a space of homogeneous type and $\mu(\mathcal{X}) = \infty$, then $\mathrm{RBLO}(\mu) = \mathrm{BLO}(\mu)$. At the end of Section 2, we review the results on the atomic Hardy spaces $H^1(\mu)$ and $\widetilde{H}^1(\mu)$. In particular, we discuss the molecular characterizations of $H^1(\mu)$ and $\widetilde{H}^1(\mu)$, the duality between $H^1(\mu)$ and $\mathrm{RBMO}(\mu)$, or between $\widetilde{H}^1(\mu)$ and $\widetilde{\mathrm{RBMO}}(\mu)$, and the relation between $H^1(\mu)$ and $\widetilde{H}^1(\mu)$.

Section 3 is mainly devoted to the boundedness of the Calderón-Zygmund operator and its maximal operator on Hardy spaces and Lebesgue spaces. We first discuss the Calderón-Zygmund decomposition and two interpolation theorems in the non-homogeneous space. Then we review the results of the equivalent boundedness of T on $L^p(\mu)$ for all $p \in (1, \infty)$ and the boundedness of T on $\widetilde{H}^1(\mu)$. We also review the boundedness of the maximal Calderón-Zygmund operator $T^{\#}$ on $L^p(\mu)$ for all $p \in (1, \infty)$ and its endpoint estimate. We further survey the weighted norm inequality for the multilinear Calderón-Zygmund operator and the boundedness on Orlicz spaces, especially, on $L^p(\mu)$ with $p \in (1, \infty)$, and the weak type endpoint estimate of the commutator generated by the Calderón-Zygmund operator with any $\mathrm{RBMO}(\mu)$ function or any $\widetilde{\mathrm{RBMO}}(\mu)$ function.

In Section 4, we first present some equivalent characterizations of the boundedness of the generalized fractional integral, then review the boundedness of the commutator generated by the generalized fractional integral with any $\mathrm{RBMO}(\mu)$ function or any $\widetilde{\mathrm{RBMO}}(\mu)$ function on Orlicz spaces, especially, on $L^p(\mu)$ with $p \in (1, \infty)$, and its weak type endpoint estimate. At the end of Section 4, we review the results on the equivalent boundedness of the Marcinkiewicz integral and its endpoint estimates.

Finally, we make some conventions on notation. Throughout the whole paper, C stands for a *positive constant* which is independent of the main parameters, but it may vary from line to line. Further, we use $C(\rho, \alpha, \dots)$ to denote a positive constant depending on the parameters ρ, α, ... For any ball B, its center and radius are denoted, respectively, by c_B and r_B. Moreover, for any ball $B := B(c_B, r_B)$ and $\rho \in (0, \infty)$,

$$\rho B := B(c_B, \rho r_B).$$

Furthermore, for any subset E of $\mathcal{X}$, we use χ_E to denote its *characteristic function*.

2 The Hardy space H^1 on non-homogeneous spaces

In this section, we review the results on $\mathrm{RBMO}(\mu)$ and $\widetilde{\mathrm{RBMO}}(\mu)$, the Hardy spaces $H^1(\mu)$ and $\widetilde{H}^1(\mu)$, and $\mathrm{RBLO}(\mu)$. To this end, we first recall some geometrical properties of the non-homogeneous space. Then we state some results of $\mathrm{RBMO}(\mu)$, $\widetilde{\mathrm{RBMO}}(\mu)$ and the regularized BLO spaces $\mathrm{RBLO}(\mu)$ and $\widetilde{\mathrm{RBLO}}(\mu)$. We further discuss the results on $H^1(\mu)$ and $\widetilde{H}^1(\mu)$, including the duality between $H^1(\mu)$ and $\mathrm{RBMO}(\mu)$, or between $\widetilde{H}^1(\mu)$ and $\widetilde{\mathrm{RBMO}}(\mu)$, the molecular characterizations of $H^1(\mu)$ and $\widetilde{H}^1(\mu)$, and the relation between $H^1(\mu)$ and $\widetilde{H}^1(\mu)$.

2.1 Doubling balls and coefficients $K_{B,S}$ and $\widetilde{K}^{(\alpha)}_{B,S}$

In this subsection, we recall some necessary notions and notation, and we also state some known basic facts and fundamental results on non-homogeneous spaces.

Though the doubling condition on the measure μ is not assumed uniformly for all balls in the non-homogeneous space $(\mathcal{X}, d, \mu)$, there still exist many balls which have the following (η, β)-doubling property.

Definition 2.1. Let η, $\beta \in (1, \infty)$. A ball $B \subset \mathcal{X}$ is said to be (η, β)-*doubling* if

$$\mu(\eta B) \le \beta \mu(B).$$

The upper doubling condition ensures the abundance of large doubling balls. On the other hand, for β big enough, there exist many small (η, β)-doubling balls under the assumption of the geometrically doubling condition. To be precise, Hytönen [17] obtained the following properties; see [17, Lemmas 3.2 and 3.3].

Lemma 2.1. *Let $(\mathcal{X}, d, \mu)$ be upper doubling, η, $\beta \in (1,\infty)$ and $\beta > [C(\lambda)]^{\log_2 \eta} =: \eta^{\nu}$. Then, for every ball $B \subset \mathcal{X}$, there exists $j \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ such that $\eta^j B$ is (η, β)-doubling.*

Lemma 2.2. *Let $\eta \in (1,\infty)$, $(\mathcal{X}, d)$ be geometrically doubling and $\beta > \eta^n$, where n is as in Remark 1.2(ii). If μ is a Borel measure on $\mathcal{X}$ which is finite on bounded sets, then, for almost every $x \in \mathcal{X}$, there exist arbitrarily small (η, β)-doubling balls centered at x. Indeed, their radii may be chosen to be the form $\eta^{-j} r$ for $j \in \mathbb{N}$ and for any preassigned number $r \in (0,\infty)$.*

In what follows, for any $\eta \in (1,\infty)$ and ball B, the *smallest (η, β_η)-doubling ball of the form $\eta^j B$ with $j \in \mathbb{N}$* is denoted by $\widetilde{B}^{\eta}$, where

$$\beta_\eta := (\max\{\eta^{3n}, \eta^{3\nu}\}) + 30^n + 30^{\nu} = \eta^{3(\max\{n,\nu\})} + 30^n + 30^{\nu}. \tag{2.1}$$

The following coefficient $K_{B,S}$ for all balls $B \subset S$ was introduced in [17] as an analogue of the coefficient $K_{Q,R}$ from Tolsa [37]; see also [38, 39].

Definition 2.2. For any two balls $B \subset S$, let

$$K_{B,S} := 1 + \int_{2S\setminus B} \frac{1}{\lambda(c_B, d(x, c_B))}\, d\mu(x),$$

where c_B is the center of the ball B.

The coefficient $K_{B,S}$ measures how close the ball B is to the ball S geometrically. Here, we state some useful properties of $K_{B,S}$ established in [17, 20].

Proposition 2.1. (i) *For all balls $B \subset R \subset S$,*

$$K_{B,R} \le K_{B,S}.$$

(ii) *For any $\rho \in [1,\infty)$, there exists a positive constant $C(\rho)$, depending on ρ, such that, for all balls $B \subset S$ with $r_S \le \rho r_B$,*

$$K_{B,S} \le C(\rho).$$

(iii) *For any $\alpha \in (1,\infty)$, there exists a positive constant $C(\alpha)$, depending on α, such that, for all balls B,*

$$K_{B,\widetilde{B}^{\alpha}} \le C(\alpha).$$

(iv) *There exists a positive constant c such that, for all balls $B \subset R \subset S$,*

$$K_{B,S} \le K_{B,R} + cK_{R,S}.$$

In particular, if B and R are concentric, then $c = 1$.

(v) *There exists a positive constant $\widetilde{c}$ such that, for all balls $B \subset R \subset S$,*

$$K_{R,S} \le \widetilde{c} K_{B,S};$$

moreover, if B and R are concentric, then

$$K_{R,S} \le K_{B,S}.$$

The following coefficient $\widetilde{K}^{(\alpha)}_{B,S}$, introduced by Fu, Yang and Yuan [9], is the discrete version of $K_{B,S}$, which was introduced by Tolsa [37] when $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, \mu)$ with μ as in (1.2), and by Bui and Duong [1] in a general non-homogeneous space with $\alpha := 6$.

Definition 2.3. For any two balls $B \subset S$ and $\alpha \in (1, \infty)$, let

$$\widetilde{K}^{(\alpha)}_{B,S} := 1 + \sum_{k=1}^{N^{(\alpha)}_{B,S}} \frac{\mu(\alpha^k B)}{\lambda(c_B, \alpha^k r_B)},$$

where c_B denotes the center of the ball B, r_B and r_S respectively denote the *radii* of B and S, and $N^{(\alpha)}_{B,S}$ is the *smallest integer* satisfying $\alpha^{N^{(\alpha)}_{B,S}} r_B \geq r_S$.

When $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, \mu)$ with μ as in (1.2), it is easy to see that, for any $\alpha \in (1, \infty)$,

$$K_{B,S} \sim \widetilde{K}^{(\alpha)}_{B,S}; \tag{2.2}$$

see [37]. For a general non-homogeneous space $(\mathcal{X}, d, \mu)$, obviously, $K_{B,S} \lesssim \widetilde{K}^{(\alpha)}_{B,S}$ for any $\alpha \in (1, \infty)$ and all balls $B \subset S$ of $\mathcal{X}$. On the other hand, for a given $\alpha \in (1, \infty)$, in general, (2.2) is not true. Nevertheless, the coefficient $\widetilde{K}^{(\alpha)}_{B,S}$ also has the following useful properties similar to those of $K_{B,S}$ (see [8, 9]). Observe that the coefficient $\widetilde{K}^{(\alpha)}_{B,S}$ preserves most of the properties of the coefficient $K_{B,S}$ from (i) through (v) in Proposition 2.1 except those when the balls B and R are concentric in (iv) and (v) of Proposition 2.1.

Proposition 2.2. *Let* $\alpha \in [0, 1)$.

(i) *For all balls* $B \subset R \subset S$,

$$\widetilde{K}^{(\alpha)}_{B,R} \leq 2\widetilde{K}^{(\alpha)}_{B,S}.$$

(ii) *For any* $\rho \in [1, \infty)$, *there exists a positive constant* $C(\rho)$, *depending only on* ρ, *such that, for all balls* $B \subset S$ *with* $r_S \leq \rho r_B$,

$$\widetilde{K}^{(\alpha)}_{B,S} \leq C(\rho).$$

(iii) *There exists a positive constant* $C(\alpha)$, *depending on* α, *such that, for all balls* B,

$$\widetilde{K}^{(\alpha)}_{B,\widetilde{B}^{\alpha}} \leq C(\alpha).$$

(iv) *There exists a positive constant* c, *depending on* $C(\lambda)$ *and* α, *such that, for all balls* $B \subset R \subset S$,

$$\widetilde{K}^{(\alpha)}_{B,S} \leq \widetilde{K}^{(\alpha)}_{B,R} + c\widetilde{K}^{(\alpha)}_{R,S}.$$

(v) *There exists a positive constant* $\widetilde{c}$, *depending on* $C(\lambda)$ *and* α, *such that, for all balls* $B \subset R \subset S$,

$$\widetilde{K}^{(\alpha)}_{R,S} \leq \widetilde{c}\widetilde{K}^{(\alpha)}_{B,S}.$$

Remark 2.1. We remark that, among the results related to $K_{B,S}$ or $\widetilde{K}^{(\alpha)}_{B,S}$ we review below, all conclusions, from Theorems 2.2, 2.3, 2.5, 2.6, 2.7, 3.1, 3.2, 3.3, 3.4, 3.12, 3.14, 3.15, 3.16, 4.2 and 4.3 below, hold true for both $K_{B,S}$ and $\widetilde{K}^{(\alpha)}_{B,S}$. On the other hand, Theorems 3.6, 3.7, 4.1, 4.4 and 4.5 only hold true for $K_{B,S}$, while Theorem 3.8 is only established for $\widetilde{K}^{(\alpha)}_{B,S}$. To be precise, let $B \subset S$ be two balls with $\mu(2S\backslash B) = 0$. Then we see that

$$K_{B,S} = 1, \tag{2.3}$$

which plays an important role in the proofs of Theorems 3.6, 3.7, 4.1, 4.4 and 4.5. However, (2.3) is not true for $\widetilde{K}^{(\alpha)}_{B,S}$ and it is unclear whether, for two balls $B \subset S$ with $\mu(2S\backslash B) = 0$, there exists a positive constant C, independent of B and S, such that $\widetilde{K}^{(\alpha)}_{B,S} \le C$ or not. Thus, it is unknown whether the conclusions of Theorems 3.6, 3.7, 4.1, 4.4 and 4.5 are true or not for $\widetilde{K}^{(\alpha)}_{B,S}$; see Remarks 3.2, 3.3, 4.2, 4.4 and 4.5(iv) below. Also, the discrete form $\widetilde{K}^{(\alpha)}_{B,S}$ plays an important role in the proof of Theorem 3.8. It is still unknown in general whether the conclusion of Theorem 3.8 holds true for $K_{B,S}$ or not; see the statement below Corollary 3.1.

At the end of this subsection, we discuss the sufficient condition for (2.2). To this end, we first recall the following notion of the weak reverse doubling condition for the dominating function λ from [10].

Definition 2.4. The dominating function λ as in Definition 1.2 is said to satisfy the *weak reverse doubling condition* if, for all $r \in (0, 2\,\mathrm{diam}\,(\mathcal{X}))$ and $a \in [1, 2\,\mathrm{diam}\,(\mathcal{X})/r)$, there exists a number $C(a) \in [1,\infty)$, depending only on a and $\mathcal{X}$, such that, for all $x \in \mathcal{X}$,

$$\lambda(x, ar) \ge C(a)\lambda(x,r) \quad \text{and} \quad \sum_{k=1}^{\vartheta_a} \frac{1}{C(a^k)} < \infty,$$

where ϑ_a is the *smallest integer* such that $\vartheta_a > \log_a(2\,\mathrm{diam}\,(\mathcal{X})/r)$ if $\mathrm{diam}\,(\mathcal{X}) < \infty$ and $\vartheta_a := \infty$ if $\mathrm{diam}\,(\mathcal{X}) = \infty$.

Remark 2.2. It was shown in [10, Example 3.2] that there exists a large class of spaces with dominating functions satisfying the weak reverse doubling condition. To be precise, let $(\mathcal{X}, d, \mu)$ be a connected metric measure space with a doubling measure μ. The minimal dominating function $F_\beta(x,r)$, defined in [36] by setting, for all $x \in \mathcal{X}$ and $r \in (0,\infty)$,

$$F_\beta(x,r) := \beta \int_1^\infty \frac{\mu(B(x,\, sr))}{s^{\beta+1}}\,ds$$

satisfies Definition 1.2 and the weak reverse doubling condition with $C(a) \sim a^m$ for some $m \in (0,n]$.

The following sufficient condition of $K_{B,S} \sim K^{(\alpha)}_{B,S}$ in terms of the weak reverse doubling condition was obtained in [10].

Theorem 2.1. *Let $\alpha \in (1,\infty)$. If the dominating function λ satisfies the weak reverse doubling condition, then, for any two balls $B \subset S$,*

$$K_{B,S} \sim \widetilde{K}^{(\alpha)}_{B,S}.$$

On the other hand, it was shown in [10, Example 3.6] that the weak reverse doubling condition is necessary to guarantee $K_{B,S} \sim \widetilde{K}^{(\alpha)}_{B,S}$ for all balls $B \subset S$ in the sense that, there exists a large class of spaces which do not satisfy the weak reverse doubling condition and $K_{B,S} \nsim \widetilde{K}^{(\alpha)}_{B,S}$ for some balls $B \subset S$ and $\alpha \in (1,\infty)$.

2.2 The regularized BMO spaces $\mathrm{RBMO}(\mu)$ and $\widetilde{\mathrm{RBMO}}(\mu)$

This subsection is devoted to some results of $\mathrm{RBMO}(\mu)$ and $\widetilde{\mathrm{RBMO}}(\mu)$. To be precise, we first state the John-Nirenberg inequality and an equivalent characterization of $\mathrm{RBMO}(\mu)$. Then we discuss the relation between $\mathrm{RBMO}(\mu)$ and $\widetilde{\mathrm{RBMO}}(\mu)$.

Now we recall the following notion of the space $\mathrm{RBMO}_\gamma(\mu)$ from [17, 20].

Definition 2.5. Let $\rho \in (1,\infty)$ and $\gamma \in [1,\infty)$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the *space* $\mathrm{RBMO}_\gamma(\mu)$ if there exist a positive constant C and, for any ball $B \subset \mathcal{X}$, a number f_B such that

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - f_B| \, d\mu(x) \le C \tag{2.4}$$

and, for any two balls B and B_1 such that $B \subset B_1$,

$$|f_B - f_{B_1}| \le C(K_{B,B_1})^\gamma. \tag{2.5}$$

The infimum of the positive constants C satisfying both (2.4) and (2.5) is defined to be the $\mathrm{RBMO}_\gamma(\mu)$ *norm* of f and denoted by $\|f\|_{\mathrm{RBMO}_\gamma(\mu)}$.

The space $\mathrm{RBMO}_\gamma(\mu)$ was proved to be independent of the choices of $\rho \in (1,\infty)$ in [17, Lemma 4.6] and $\gamma \in [1,\infty)$ in [20, Proposition 2.5]. Moreover, similar to the case that $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, \mu)$, several equivalent characterizations of $\mathrm{RBMO}(\mu)$ were established in [20]. In what follows, we *denote* $\mathrm{RBMO}_\gamma(\mu)$ *simply by* $\mathrm{RBMO}(\mu)$.

Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type in the sense of Coifman and Weiss [4, 5] with μ as in (1.1) and $\mathrm{BMO}(\mu)$ as in [5]. Then it is easy to see that $\mathrm{RBMO}(\mu) \subset \mathrm{BMO}(\mu)$. Moreover, if $\mu(\mathcal{X}) = \infty$, by [17, Proposition 4.7] (which is false when $\mu(\mathcal{X}) < \infty$), we know that $\mathrm{BMO}(\mu) = \mathrm{RBMO}(\mu)$. However, for a general doubling measure μ with $\mu(\mathcal{X}) < \infty$, it may happen that

$$\mathrm{RBMO}(\mu) \subsetneqq \mathrm{BMO}(\mu);$$

see [37, p. 106, Example 2.13]. Nevertheless, $\mathrm{RBMO}(\mu)$ is still seen as a suitable substitute for $\mathrm{BMO}(\mathbb{R}^D)$ since many properties fulfilled by $\mathrm{BMO}(\mathbb{R}^D)$ are still satisfied by $\mathrm{RBMO}(\mu)$.

The following theorem is a version of the John-Nirenberg inequality for $\mathrm{RBMO}(\mu)$ obtained by Hytönen [17, Proposition 6.1]; see [37] for the case when

$$(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, \mu)$$

with μ as in (1.2).

Theorem 2.2. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous space. Then, for every $\rho \in (1, \infty)$, there exists a positive constant c such that, for all $f \in \mathrm{RBMO}(\mu)$, balls B_0 and $t \in (0, \infty)$,*

$$\mu\left(\{x \in B_0 : \ |f(x) - f_{B_0}| > t\}\right) \le 2\mu(\rho B_0) e^{-ct/\|f\|_{\mathrm{RBMO}(\mu)}},$$

where f_{B_0} is as in Definition 2.5 with B replaced by B_0.

The following result stated in [17, Corollary 6.3] is a straightforward consequence of Theorem 2.2.

Corollary 2.1. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous space. Then, for every $\rho \in (1, \infty)$ and $p \in [1, \infty)$, there exists a positive constant C such that, for all $f \in \mathrm{RBMO}(\mu)$ and balls B,*

$$\left[\frac{1}{\mu(\rho B)} \int_B |f(x) - f_B|^p \, d\mu(x)\right]^{1/p} \le C\|f\|_{\mathrm{RBMO}(\mu)},$$

where f_B is as in Definition 2.5.

Remark 2.3. As a consequence of Corollary 2.1, together with the Hölder inequality, we obtain an equivalent definition of $\mathrm{RBMO}(\mu)$ by replacing (2.4) with

$$\left[\frac{1}{\mu(\rho B)} \int_B |f(x) - f_B|^p \, d\mu(x)\right]^{1/p} \le C,$$

where f_B and C are as in Definition 2.5.

In [15], Hu, Meng and Yang established an equivalent characterization of $\mathrm{RBMO}(\mu)$ in terms of the John-Strömberg sharp maximal function. To be precise, let f be a μ-measurable function. If f is real-valued, then, for all balls B with $\mu(B) \neq 0$, the *median value* of f on the ball B, denoted by $m_f(B)$, is defined to be one of the numbers such that

$$\mu(\{x \in B : \ f(x) > m_f(B)\}) \le \mu(B)/2$$

and

$$\mu(\{x \in B : \ f(x) < m_f(B)\}) \le \mu(B)/2.$$

For all balls B with $\mu(B) = 0$, let $m_f(B) := 0$. If f is complex-valued, we take

$$m_f(B) := m_{\mathrm{Re} f}(B) + i m_{\mathrm{Im} f}(B),$$

where Ref and Imf denote the *real part* and the *imaginary part* of f, respectively.

Let $s \in (0, 1)$ and $\rho \in (1, \infty)$. For any fixed ball B and μ-measurable function f, define $m^{\rho}_{0,\, s;\, B}(f)$ by setting

$$m^{\rho}_{0,\, s;\, B}(f) := \inf\{t \in (0, \infty) : \ \mu(\{y \in B : \ |f(y)| > t\}) < s\mu(\rho B)\}$$

when $\mu(B) > 0$, and

$$m^{\rho}_{0,\, s;\, B}(f) := 0$$

when $\mu(B)=0$. For any μ-measurable function f, the *John-Strömberg sharp maximal function* $M_{0,s}^{\rho,\#}(f)$ is defined by setting, for all $x\in\mathcal{X}$,

$$M_{0,s}^{\rho,\#}(f)(x):=\sup_{B\ni x} m_{0,s;B}^{\rho}\left(f-m_f\left(\widetilde{B}^{6\rho^2}\right)\right)+\sup_{\substack{x\in B\subset S,\\ B,S\ (6\rho^2,\beta_{6\rho^2})-\text{doubling}}}\frac{|m_f(B)-m_f(S)|}{K_{B,S}}.$$

Using $M_{0,s}^{\rho,\#}$, we define a version of RBMO(μ) as follows.

Definition 2.6. Let $s\in(0,1)$ and $\rho\in(1,\infty)$. A μ-measurable function f is said to belong to the *space* $\mathrm{RBMO}_{0,s}(\mu)$ if $M_{0,s}^{\rho,\#}(f)\in L^\infty(\mu)$. Moreover, $\|M_{0,s}^{\rho,\#}(f)\|_{L^\infty(\mu)}$ is defined to be the $\mathrm{RBMO}_{0,s}(\mu)$ *norm* of f and denoted by $\|f\|_{\mathrm{RBMO}_{0,s}(\mu)}$.

Now we state the following equivalent characterization of RBMO(μ) in terms of $M_{0,s}^{\rho,\#}$ from [15].

Theorem 2.3. *Let $\rho\in(1,\infty)$ and $s\in(0,\beta_{6\rho^2}^{-2}/4)$. Then the space* RBMO$(\mu)$ *and* $\mathrm{RBMO}_{0,s}(\mu)$ *coincide with equivalent norms.*

Let φ be a strictly increasing and nonnegative continuous function on $[0,\infty)$ such that

$$\lim_{t\to\infty}\varphi(t)=\infty. \tag{2.6}$$

Then, by Theorem 2.3, the following conclusion holds true.

Corollary 2.2. *Let $\rho\in(1,\infty)$ and φ be a strictly increasing and nonnegative continuous function on $[0,\infty)$ satisfying (2.6). If $f\in L^1_{\mathrm{loc}}(\mu)$ and there exists a positive constant C such that, for all balls $B\subset\mathcal{X}$,*

$$\frac{1}{\mu(\rho B)}\int_B\varphi\left(\left|f(x)-m_f(\widetilde{B})\right|\right)\,d\mu(x)\le C$$

and that, for all $(6\rho^2,\beta_{6\rho^2})$-doubling balls $B\subset S$,

$$|m_f(B)-m_f(S)|\le CK_{B,S},$$

then $f\in$ RBMO(μ).

Notice that a typical example of φ satisfying Corollary 2.2 is $\varphi(r):=r^p$ for all $r\in[0,\infty)$ and $p\in(0,\infty)$. Thus, Remark 2.3 also holds true for $p\in(0,1)$; see [15].

If we replace the coefficient $K_{B,S}$ with its discrete version $\widetilde{K}_{B,S}^{(\rho)}$ for $\rho\in(1,\infty)$, we have the following version of the BMO-type space RBMO(μ).

Definition 2.7. Let $\rho\in(1,\infty)$ and $\gamma\in[1,\infty)$. A function $f\in L^1_{\mathrm{loc}}(\mu)$ is said to be in the *space* $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$, if it satisfies (2.4) and (2.5) with K_{B,B_1} replaced by $\widetilde{K}_{B,B_1}^{(\rho)}$.

The infimum of the corresponding positive constants C satisfying both (2.4) and (2.5) with K_{B,B_1} replaced by $\widetilde{K}_{B,B_1}^{(\rho)}$ is defined to be the $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ *norm* of f and denoted by $\|f\|_{\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)}$.

As is shown in [10], the space $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ is independent of the choices of $\rho \in (1,\infty)$ and $\gamma \in [1,\infty)$. In what follows, we *denote* $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ *simply by* $\widetilde{\mathrm{RBMO}}(\mu)$.

By [10, Remark 2.5], when $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, \mu)$ with μ as in (1.2), by (2.2), we see that $\widetilde{\mathrm{RBMO}}(\mu)$ becomes the regularized BMO space $\mathrm{RBMO}(\mu)$ introduced in [37] for $\gamma = 1$ and in [14] for $\gamma > 1$. Moreover, for $\rho \in (1,\infty)$ and $\gamma \in [1,\infty)$,

$$\mathrm{RBMO}(\mu) \subset \widetilde{\mathrm{RBMO}}(\mu).$$

However, it is still unclear whether we always have $\mathrm{RBMO}(\mu) = \widetilde{\mathrm{RBMO}}(\mu)$ or not. Nevertheless, as an application of Theorem 2.1, we obtain the following result from [10].

Corollary 2.3. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous space with the dominating function satisfying the weak reverse doubling condition. Then $\widetilde{\mathrm{RBMO}}(\mu) = \mathrm{RBMO}(\mu)$ with equivalent norms.*

Remark 2.4. (i) If $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, \mu)$ with μ as in (1.2) and $\lambda(x, r) := C_0 r^\kappa$ for all $x \in \mathbb{R}^D$ and $r \in [0,\infty)$, where $\kappa \in (0, D]$, it was proved in [10, Remark 3.5] that the weak reverse condition holds true automatically in this case and hence the conclusion of Corollary 2.3 is true in this case.

(ii) If $(\mathcal{X}, d, \mu)$ is a space of homogeneous type in the sense of Coifman and Weiss with μ as in (1.1), and $\mathcal{X}$ connected, it was proved in [10, Remark 3.5] that $\lambda(x, r) := \mu(B(x, r))$ for all $x \in \mathcal{X}$ and $r \in [0, 2\,\mathrm{diam}\,(\mathcal{X}))$ satisfies the weak reverse condition and hence the conclusion of Corollary 2.3 is also true in this case.

(iii) However, if $(\mathcal{X}, d, \mu)$ is a non-homogeneous space without the dominating function satisfying the weak reverse doubling condition, then it is still unclear whether $\widetilde{\mathrm{RBMO}}(\mu) = \mathrm{RBMO}(\mu)$ or not.

2.3 The regularized BLO spaces $\mathrm{RBLO}(\mu)$ and $\widetilde{\mathrm{RBLO}}(\mu)$

In this subsection, we first review the regularized BLO space $\mathrm{RBLO}(\mu)$ introduced in [22]. We begin with the notion of $\mathrm{RBLO}(\mu)$ as follows.

Definition 2.8. Let ε, $\rho \in (1,\infty)$, and β_ρ be as in (2.1). A real-valued function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the *space* $\mathrm{RBLO}(\mu)$ if there exists a non-negative constant C such that, for all balls B,

$$\frac{1}{\mu(\varepsilon B)} \int_B \left[f(y) - \operatorname*{ess\,inf}_{\widetilde{B}^\rho} f \right] d\mu(y) \le C \tag{2.7}$$

and that, for all (ρ, β_ρ)-doubling balls $B \subset S$,

$$\operatorname*{ess\,inf}_B f - \operatorname*{ess\,inf}_S f \le C K_{B,S}. \tag{2.8}$$

Moreover, the $\mathrm{RBLO}(\mu)$-*norm* of f is defined to be the minimal constant C as in (2.7) and (2.8) and denoted by $\|f\|_{\mathrm{RBLO}(\mu)}$.

Remark 2.5. By [22, Propositions 2.1 and 2.2], we see that $\mathrm{RBLO}(\mu)$ is independent of the choices of the constants ε, $\rho \in (1, \infty)$. Moreover, Lin and Yang established several equivalent characterizations of $\mathrm{RBLO}(\mu)$ and showed that $\mathrm{RBLO}(\mu) \subset \mathrm{RBMO}(\mu)$ in [22].

Recall that, on a space of homogeneous type, $(\mathcal{X}, d, \mu)$, a real-valued function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the *space* $\mathrm{BLO}(\mu)$, if there exists a non-negative constant C such that, for all balls B,

$$\frac{1}{\mu(B)} \int_B \left[f(y) - \operatorname*{ess\,inf}_B f \right] \, d\mu(y) \leq C. \tag{2.9}$$

Moreover, the $\mathrm{BLO}(\mu)$*-norm* of f is defined to be the minimal constant C as in (2.9) and denoted by $\|f\|_{\mathrm{BLO}(\mu)}$.

Now we discuss the relation between $\mathrm{RBLO}(\mu)$ and $\mathrm{BLO}(\mu)$ when μ is as in (1.1). We first show a positive result.

Proposition 2.3. *Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type, with $\mu(\mathcal{X}) = \infty$, and $\lambda(x, r) := \mu(B(x, r))$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$. Then $\mathrm{RBLO}(\mu) = \mathrm{BLO}(\mu)$ with equivalent norms.*

Proof. It is obvious that $\mathrm{RBLO}(\mu) \subset \mathrm{BLO}(\mu)$ and, for all $f \in \mathrm{RBLO}(\mu)$,

$$\|f\|_{\mathrm{BLO}(\mu)} \leq \|f\|_{\mathrm{RBLO}(\mu)}.$$

Conversely, let $f \in \mathrm{BLO}(\mu)$. By the facts that $\mathrm{BLO}(\mu) \subset \mathrm{BMO}(\mu)$ and, for all $f \in \mathrm{BLO}(\mu)$,

$$\|f\|_{\mathrm{BMO}(\mu)} \lesssim \|f\|_{\mathrm{BLO}(\mu)},$$

where $\mathrm{BMO}(\mu)$ is defined as in [5], and the fact that $\mathrm{RBMO}(\mu) = \mathrm{BMO}(\mu)$ when $\mu(\mathcal{X}) = \infty$, we see that, for all balls $B \subset S$,

$$\begin{aligned}
\operatorname*{ess\,inf}_B f - \operatorname*{ess\,inf}_S f &\leq \operatorname*{ess\,inf}_B f - m_B(f) + |m_B(f) - f_B| + |f_B - f_S| \\
&\quad + |f_S - m_S(f)| + m_S(f) - \operatorname*{ess\,inf}_S f \\
&\leq \frac{1}{\mu(B)} \int_B |f(y) - f_B| \, d\mu(y) + |f_B - f_S| \\
&\quad + \frac{1}{\mu(S)} \int_S |f(y) - f_S| \, d\mu(y) + \frac{1}{\mu(S)} \int_S \left[f(y) - \operatorname*{ess\,inf}_S f \right] \, d\mu(y) \\
&\lesssim \|f\|_{\mathrm{BLO}(\mu)} + K_{B,S} \|f\|_{\mathrm{BMO}(\mu)} \lesssim K_{B,S} \|f\|_{\mathrm{BLO}(\mu)},
\end{aligned}$$

where, for all balls B, f_B is as in Definition 2.5 and

$$m_B(f) := \frac{1}{\mu(B)} \int_B f(x) \, d\mu(x).$$

This shows that (2.8) holds true. We then obtain $\mathrm{BLO}(\mu) \subset \mathrm{RBLO}(\mu)$ and, for all $f \in \mathrm{BLO}(\mu)$,

$$\|f\|_{\mathrm{RBLO}(\mu)} \lesssim \|f\|_{\mathrm{BLO}(\mu)}.$$

This finishes the proof of Proposition 2.3. □

We further borrow an example from [37, Example 2.13] to show that, when μ is as in (1.1) with $\mu(\mathcal{X})<\infty$, it may happen that $\mathrm{RBLO}(\mu) \subsetneqq \mathrm{BLO}(\mu)$.

Example 1. Let $(\mathcal{X}, d, \mu) := (\mathbb{R}^2, |\cdot|, \mu)$, with μ the 2-dimensional Lebesgue measure restricted to the unit ball $B(0,1)$, and $\lambda(x,r) := r$ for all $x\in\mathbb{R}^2$ and $r\in(0,\infty)$. This measure is doubling and $\mu(\mathbb{R}^2)<\infty$. Now we claim that $\mathrm{RBLO}(\mu)=L^\infty(\mu)/\mathbb{C}$, $L^\infty(\mu)$ modulo constant functions, with equivalent norms. Indeed, by an argument similar to that used in [37, Example 2.13], we see that $\mathrm{RBMO}(\mu)=L^\infty(\mu)/\mathbb{C}$ with equivalent norms. Then, from the fact that

$$L^\infty(\mu)/\mathbb{C}\subset \mathrm{RBLO}(\mu)\subset \mathrm{RBMO}(\mu),$$

the claim follows. On the other hand, it is not difficult to see that

$$-(\log|x|)\chi_{\{x\in\mathbb{R}^2:\ 0<|x|<1\}}(x)\in \mathrm{BLO}(\mu)\backslash(L^\infty(\mu)/\mathbb{C});$$

see Zhou [45]. We further conclude that, in this case,

$$\mathrm{RBLO}(\mu)=L^\infty(\mu)/\mathbb{C}\subsetneqq \mathrm{BLO}(\mu),$$

which completes the proof of Example 1.

Now we state a result from [22] about the boundedness, from $\mathrm{RBMO}(\mu)$ into $\mathrm{RBLO}(\mu)$, of the natural maximal operator $\mathcal{N}$ defined by setting, for all $f\in L^1_{\rm loc}(\mu)$ and $x\in\mathcal{X}$,

$$\mathcal{N}(f)(x):=\sup_{B\ni x,\,B\ (6,\beta_6)-\text{doubling}}\frac{1}{\mu(B)}\int_B f(y)\,d\mu(y).$$

Theorem 2.4. *Let $f\in\mathrm{RBMO}(\mu)$. Then $\mathcal{N}(f)$ is either infinite everywhere or finite almost everywhere and, in the latter case, there exists a positive constant C, independent of f, such that*

$$\|\mathcal{N}(f)\|_{\mathrm{RBLO}(\mu)}\le C\|f\|_{\mathrm{RBMO}(\mu)}.$$

Then we review the equivalent characterization of $\mathrm{RBLO}(\mu)$ from [22] in terms of $\mathcal{N}$.

Theorem 2.5. *A locally integrable function f belongs to $\mathrm{RBLO}(\mu)$ if and only if there exist $h\in L^\infty(\mu)$ and $g\in\mathrm{RBMO}(\mu)$ with $\mathcal{N}(g)$ finite μ-almost everywhere such that*

$$f=\mathcal{N}(g)+h. \tag{2.10}$$

Furthermore,

$$\|f\|_{\mathrm{RBLO}(\mu)}\sim\inf\{\|g\|_{\mathrm{RBMO}(\mu)}+\|h\|_{L^\infty(\mu)}\},$$

where the infimum is taken over all representations of f as in (2.10) *and the equivalent positive constants are independent of f.*

If we replace the coefficient $K_{B,S}$ with its discrete version $\widetilde{K}^{(\rho)}_{B,S}$ for $\rho\in(1,\infty)$, we have the following version of the BLO-type space.

Definition 2.9. Let ε, $\rho \in (1, \infty)$. A real-valued function $f \in L^1_{\rm loc}(\mu)$ is said to be in the *space* $\widetilde{\rm RBLO}(\mu)$ if it satisfies (2.7) and (2.8) with $K_{B,S}$ replaced by $\widetilde{K}^{(\rho)}_{B,S}$. Moreover, the $\widetilde{\rm RBLO}(\mu)$*-norm* of f is defined to be the minimal constant C as in (2.7) and (2.8) with $K_{B,S}$ replaced by $\widetilde{K}^{(\rho)}_{B,S}$ and denoted by $\|f\|_{\widetilde{\rm RBLO}(\mu)}$.

Remark 2.6. (i) Arguing as the proofs of [22, Propositions 2.1 and 2.2], we conclude that $\widetilde{\rm RBLO}(\mu)$ is independent of the choices of the constants ε, $\rho \in (1, \infty)$. Moreover, it is easy to see that the equivalent characterizations of ${\rm RBLO}(\mu)$ in [22] also hold true for $\widetilde{\rm RBLO}(\mu)$, and

$$\widetilde{\rm RBLO}(\mu) \subset \widetilde{\rm RBMO}(\mu).$$

(ii) We point out that the conclusions of Proposition 2.3 and Example 1 are also true for $\widetilde{\rm RBLO}(\mu)$.

(iii) It is easy to see that the conclusions of Theorems 2.4 and 2.5 also hold true with ${\rm RBLO}(\mu)$ replaced by $\widetilde{\rm RBLO}(\mu)$ and ${\rm RBMO}(\mu)$ by $\widetilde{\rm RBMO}(\mu)$.

(iv) For any ρ, $\varepsilon \in (1, \infty)$, it is obvious that ${\rm RBLO}(\mu) \subset \widetilde{\rm RBLO}(\mu)$.

As an application of Theorem 2.1, we obtain the following result.

Corollary 2.4. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous space with the dominating function satisfying the weak reverse doubling condition. Then $\widetilde{\rm RBLO}(\mu) = {\rm RBLO}(\mu)$ with equivalent norms.*

Remark 2.7. (i) If $(\mathcal{X}, d, \mu)$ is as in Remark 2.4(i), by the same reason as therein, we see that the conclusion of Corollary 2.4 is true in this case.

(ii) If $(\mathcal{X}, d, \mu)$ is as in Remark 2.4(ii), by the same reason as therein, we know that the conclusion of Corollary 2.4 is also true in this case.

(iii) However, if $(\mathcal{X}, d, \mu)$ is a non-homogeneous space without the dominating function satisfying the weak reverse doubling condition, then it is still unclear whether $\widetilde{\rm RBLO}(\mu) = {\rm RBLO}(\mu)$ or not.

2.4 The Hardy spaces $H^{1,p}_{\rm atb}(\mu)$ and $\widetilde{H}^{1,p}_{\rm atb}(\mu)$

In this subsection, we discuss the duality between $H^{1,p}_{\rm atb}(\mu)$ and ${\rm RBMO}(\mu)$, or between $\widetilde{H}^{1,p}_{\rm atb}(\mu)$ and $\widetilde{\rm RBMO}(\mu)$. The molecular characterizations of $\widetilde{H}^{1,p}_{\rm atb}(\mu)$ and $H^{1,p}_{\rm atb}(\mu)$ are also included in this subsection. To begin with, we recall the notion of the atomic Hardy space (see, for example, [20]).

Definition 2.10. Let $\rho \in (1, \infty)$, $\gamma \in [1, \infty)$ and $p \in (1, \infty]$. A function $b \in L^1(\mu)$ is called a $(p, \gamma)_\lambda$*-atomic block* if

(i) there exists a ball B such that ${\rm supp}\,(b) \subset B$;

(ii)

$$\int_{\mathcal{X}} b(x)\, d\mu(x) = 0;$$

(iii) for any $j \in \{1, 2\}$, there exist a function a_j supported on ball $B_j \subset B$ and a number $\lambda_j \in \mathbb{C}$ such that

$$b = \lambda_1 a_1 + \lambda_2 a_2 \tag{2.11}$$

and

$$\|a_j\|_{L^p(\mu)} \le [\mu(\rho B_j)]^{1/p-1}(K_{B_j,B})^{-\gamma}.$$

Moreover, let

$$|b|_{H^{1,\,p,\,\gamma}_{\rm atb,\,\rho}(\mu)} := |\lambda_1| + |\lambda_2|. \tag{2.12}$$

A function $f \in L^1(\mu)$ is said to belong to the *atomic Hardy space* $H^{1,\,p,\,\gamma}_{\rm atb,\,\rho}(\mu)$ if there exist $(p,\gamma)_\lambda$-atomic blocks $\{b_i\}_{i=1}^\infty$ such that $f = \sum_{i=1}^\infty b_i$ in $L^1(\mu)$ and

$$\sum_{i=1}^\infty |b_i|_{H^{1,\,p,\,\gamma}_{\rm atb,\,\rho}(\mu)} < \infty.$$

The $H^{1,\,p,\,\gamma}_{\rm atb,\,\rho}(\mu)$ *norm* of f is defined by

$$\|f\|_{H^{1,\,p,\,\gamma}_{\rm atb,\,\rho}(\mu)} := \inf\left\{\sum_{i=1}^\infty |b_i|_{H^{1,\,p,\,\gamma}_{\rm atb,\,\rho}(\mu)}\right\},$$

where the infimum is taken over all the possible decompositions of f as above.

Remark 2.8. (i) It was proved in [20] that, for each $p \in (1,\infty]$, the atomic Hardy space $H^{1,\,p,\,\gamma}_{\rm atb,\,\rho}(\mu)$ is independent of the choice of $\rho \in (1,\infty)$. Thus, we *denote* $H^{1,\,p,\,\gamma}_{\rm atb,\,\rho}(\mu)$ *simply by* $H^{1,\,p,\,\gamma}_{\rm atb}(\mu)$.

(ii) When $\gamma = 1$, we *denote* $H^{1,\,p,\,\gamma}_{\rm atb}(\mu)$ *simply by* $H^{1,\,p}_{\rm atb}(\mu)$. Let us denote by $\widehat{H}^{1,\,p}_{\rm atb}(\mu)$ temporarily the *atomic Hardy space*, defined by $(p,1)_\lambda$-atomic blocks, introduced by Bui and Duong [1]. Recall that, in the definition of $(p,1)_\lambda$-atomic blocks in [1], instead of (2.11) and (2.12), it requires that

$$b = \sum_{j=1}^\infty \lambda_j a_j \quad \text{and} \quad |b|_{\widehat{H}^{1,\,p}_{\rm atb}(\mu)} := \sum_{j=1}^\infty |\lambda_j|.$$

It was proved, in [27, Remark 1.3(ii)], that the atomic Hardy space $\widehat{H}^{1,\,p}_{\rm atb}(\mu)$ and $H^{1,\,p}_{\rm atb}(\mu)$ coincide with equivalent norms.

The duality between $\mathrm{RBMO}_\gamma(\mu)$ and $H^{1,\,p,\,\gamma}_{\rm atb,\,\rho}(\mu)$ is the following result obtained in [20]; see also [1] for $\gamma = 1$.

Theorem 2.6. (i) *Let* $\gamma \in [1,\infty)$ *and* $p \in (1,\infty)$. *Then the spaces* $H^{1,\,p,\,\gamma}_{\rm atb}(\mu)$ *and* $H^{1,\,\infty,\,\gamma}_{\rm atb}(\mu)$ *coincide with equivalent norms and*

$$\left[H^{1,\,\infty,\,\gamma}_{\rm atb}(\mu)\right]^* = \mathrm{RBMO}_\gamma(\mu).$$

(ii) *Let* $\gamma \in (1,\infty)$ *and* $p \in (1,\infty]$. *Then the spaces* $H^{1,\,p,\,\gamma}_{\rm atb}(\mu)$ *and* $H^{1,\,p}_{\rm atb}(\mu)$ *coincide with equivalent norms.*

Remark 2.9. (i) By Theorem 2.6(ii), we see that, for each $p \in (1,\infty]$, the atomic Hardy space $H_{\rm atb}^{1,p,\gamma}(\mu)$ is independent of the choice of $\gamma \in [1,\infty)$. Thus, in what follows, we *denote* $H_{\rm atb}^{1,p,\gamma}(\mu)$ *simply by* $H_{\rm atb}^{1,p}(\mu)$.

(ii) By Theorem 2.6 and Remark 2.8(i), we *denote the atomic Hardy space* $H_{\rm atb}^{1,p}(\mu)$ *simply by* $H^1(\mu)$.

Replacing the coefficient $K_{B,S}$ by its discrete version $\widetilde{K}_{B,S}^{(\rho)}$ as in Definition 2.3, we present the following notion of the atomic Hardy space $\widetilde{H}_{\rm atb}^{1,p}(\mu)$ from [10].

Definition 2.11. Let $\rho \in (1,\infty)$, $p \in (1,\infty]$ and $\gamma \in [1,\infty)$. A function $b \in L^1(\mu)$ is called a $(p,\gamma,\rho)_\lambda$-*atomic block* if b satisfies (i)-(iii) in Definition 2.10 with $K_{B_j,B}$ replaced by $\widetilde{K}_{B_j,B}^{(\rho)}$. A function $f \in L^1(\mu)$ is said to belong to the *atomic Hardy space* $\widetilde{H}_{\rm atb,\,\rho}^{1,p,\gamma}(\mu)$ if there exist $(p,\gamma,\rho)_\lambda$-atomic blocks $\{b_i\}_{i=1}^\infty$ such that $f = \sum_{i=1}^\infty b_i$ in $L^1(\mu)$ and

$$\sum_{i=1}^\infty |b_i|_{\widetilde{H}_{\rm atb,\,\rho}^{1,p,\gamma}(\mu)} < \infty.$$

The $\widetilde{H}_{\rm atb,\,\rho}^{1,p,\gamma}(\mu)$ *norm* of f is defined by

$$\|f\|_{\widetilde{H}_{\rm atb,\,\rho}^{1,p,\gamma}(\mu)} := \inf\left\{\sum_{i=1}^\infty |b_i|_{\widetilde{H}_{\rm atb,\,\rho}^{1,p,\gamma}(\mu)}\right\},$$

where the infimum is taken over all the possible decompositions of f as above.

Remark 2.10. (i) When $(\mathcal{X},d,\mu) := (\mathbb{R}^D,|\cdot|,\mu)$ with μ as in (1.2), by (2.2), we see that $\widetilde{H}_{\rm atb,\,\rho}^{1,p,\gamma}(\mu)$ becomes the atomic Hardy space $H_{\rm atb,\,\rho}^{1,p,\gamma}(\mu)$ in [37] for $\gamma = 1$ and in [14] for $\gamma > 1$. Obviously, for $\rho \in (1,\infty)$, $p \in (1,\infty]$ and $\gamma \in [1,\infty)$, we always have

$$\widetilde{H}_{\rm atb,\,\rho}^{1,p,\gamma}(\mu) \subset H_{\rm atb,\,\rho}^{1,p,\gamma}(\mu).$$

(ii) It was shown in [10] that, for each $p \in (1,\infty]$, the atomic Hardy space $\widetilde{H}_{\rm atb,\,\rho}^{1,p,\gamma}(\mu)$ is independent of the choices of ρ and γ and that, for all $p \in (1,\infty)$, the spaces $\widetilde{H}_{\rm atb,\,\rho}^{1,p,\gamma}(\mu)$ and $\widetilde{H}_{\rm atb,\,\rho}^{1,\infty,\gamma}(\mu)$ coincide with equivalent norms. Thus, in what follows, we *denote* $\widetilde{H}_{\rm atb,\,\rho}^{1,p,\gamma}(\mu)$ *simply by* $\widetilde{H}_{\rm atb}^{1,p}(\mu)$ *or* $\widetilde{H}^1(\mu)$. Moreover, it was shown in [10] that $[\widetilde{H}_{\rm atb}^{1,p}(\mu)]^* = \widetilde{\rm RBMO}(\mu)$ for all $p \in (1,\infty]$.

(iii) Let $(\mathcal{X},d,\mu)$ be a space of homogeneous type with μ as in (1.1) and $H^{1,p}(\mu)$, the atomic Hardy space as in [5] with $p \in (1,\infty]$. Then, it is easy to see that

$$H^{1,p}(\mu) \subset \widetilde{H}_{\rm atb}^{1,p}(\mu) \subset H_{\rm atb}^{1,p}(\mu).$$

Moreover, if $\mu(\mathcal{X}) = \infty$, by [17, Proposition 4.7] and [20, Proposition 3.5], together with [25, Lemma 2.12], we know that

$$H^{1,p}(\mu) = \widetilde{H}_{\rm atb}^{1,p}(\mu) = H_{\rm atb}^{1,p}(\mu).$$

However, for general doubling measure μ with $\mu(\mathcal{X}) < \infty$, it may happen that

$$H^{1,p}(\mu) \subsetneqq \widetilde{H}_{\rm atb}^{1,p}(\mu) \subset H_{\rm atb}^{1,p}(\mu);$$

see [39, p. 317, lines 15 to 16] and [37, p. 125, Example 5.6].

The boundedness of Calderón-Zygmund operators on $H^{1,p}_{\mathrm{atb}}(\mu)$ is still unknown. However, it turns out that Calderón-Zygmund operators are bounded on $\widetilde{H}^{1,p}_{\mathrm{atb}}(\mu)$, which was proved via the molecular characterization of $\widetilde{H}^{1,p}_{\mathrm{atb}}(\mu)$ in [10]. We first state the notion of the molecular Hardy space.

Definition 2.12. Let $\rho \in (1,\infty)$, $p \in (1,\infty]$, $\gamma \in [1,\infty)$ and $\varepsilon \in (0,\infty)$. A function $b \in L^1(\mu)$ is called a $(p,\gamma,\varepsilon,\rho)_\lambda$-*molecular block* if

(i)

$$\int_{\mathcal{X}} b(x)\,d\mu(x) = 0;$$

(ii) there exist some ball B and some constants $\widetilde{M}$, $M \in \mathbb{N}$ such that, for all $k \in \mathbb{Z}_+$ and $j \in \{0,\ldots,M_k\}$ with $M_k := \widetilde{M}$ if $k = 0$ and $M_k := M$ if $k \in \mathbb{N}$, there exist functions $m_{k,j}$ supported on some balls $B_{k,j} \subset U_k(B)$ for all $k \in \mathbb{Z}_+$, where $U_0(B) := \rho^2 B$ and $U_k(B) := \rho^{k+2}B \backslash \rho^{k-2}B$ with $k \in \mathbb{N}$, and $\lambda_{k,j} \in \mathbb{C}$ such that

$$b = \sum_{k=0}^{\infty}\sum_{j=1}^{M_k} \lambda_{k,j} m_{k,j},$$

$$\|m_{k,j}\|_{L^p(\mu)} \le \rho^{-k\varepsilon}\left[\mu(\rho B_{k,j})\right]^{1/p-1}\left[\widetilde{K}^{(\rho)}_{B_{k,j},\,\rho^{k+2}B}\right]^{-\gamma}$$

and

$$|b|_{\widetilde{H}^{1,p,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)} := \sum_{k=0}^{\infty}\sum_{j=1}^{M_k} |\lambda_{k,j}| < \infty.$$

A function $f \in L^1(\mu)$ is said to belong to the *molecular Hardy space* $\widetilde{H}^{1,p,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ if there exist $(p,\gamma,\varepsilon,\rho)_\lambda$-molecular blocks $\{b_i\}_{i=1}^\infty$ such that $f = \sum_{i=1}^\infty b_i$ in $L^1(\mu)$ and

$$\sum_{i=1}^{\infty} |b_i|_{\widetilde{H}^{1,p,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)} < \infty.$$

The $\widetilde{H}^{1,p,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ *norm* of f is defined by

$$\|f\|_{\widetilde{H}^{1,p,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)} := \inf\left\{\sum_{i=1}^{\infty} |b_i|_{\widetilde{H}^{1,p,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)}\right\},$$

where the infimum is taken over all the possible decompositions of f as above.

Now we present the molecular characterization of the atomic Hardy space $\widetilde{H}^{1,p,\gamma}_{\mathrm{atb},\rho}(\mu)$ obtained in [10].

Theorem 2.7. *Let $\rho \in (1,\infty)$, $p \in (1,\infty]$, $\gamma \in [1,\infty)$ and $\varepsilon \in (0,\infty)$. Then $\widetilde{H}^{1,p,\gamma}_{\mathrm{atb},\rho}(\mu) = \widetilde{H}^{1,p,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ with equivalent norms.*

Remark 2.11. (i) It was shown in [10] that the conclusion of Theorem 2.7 is still valid with $\widetilde{K}^{(\rho)}_{B,S}$ in Definitions 2.10 and 2.11 replaced by $K_{B,S}$ in Definition 2.2.

(ii) We point out that there exists no equivalent characterization for $H^{1,p}_{\mathrm{atb}}(\mu)$ or $\widetilde{H}^{1,p}_{\mathrm{atb}}(\mu)$ by the maximal function or the Littlewood-Paley function in the present setting.

Now we discuss the relation between $H^{1,p}_{\rm atb}(\mu)$ and $\widetilde{H}^{1,p}_{\rm atb}(\mu)$. As an application of Theorem 2.1, we obtain the following result from [10].

Corollary 2.5. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous space with the dominating function satisfying the weak reverse doubling condition. Then*

$$\widetilde{H}^{1,p}_{\rm atb}(\mu) = \widetilde{H}^{1,p}_{\rm mb}(\mu) = H^{1,p}_{\rm atb}(\mu)$$

with equivalent norms.

Remark 2.12. (i) If $(\mathcal{X}, d, \mu)$ is as in Remark 2.4(i), by the same reason as therein, we know that the conclusion of Corollary 2.5 is true in this case.

(ii) If $(\mathcal{X}, d, \mu)$ is as in Remark 2.4(ii), by the same reason as therein, we find that the conclusion of Corollary 2.5 is also true in this case.

(iii) However, if $(\mathcal{X}, d, \mu)$ is a non-homogeneous space without the dominating function satisfying the weak reverse doubling condition, then it is still unclear whether $\widetilde{H}^{1,p}_{\rm atb}(\mu) = H^{1,p}_{\rm atb}(\mu)$ or not.

3 Calderón-Zygmund operators

In this section, we mainly discuss the boundedness of the Calderón-Zygmund operator on the Hardy space as well as the Lebesgue space. We also consider the boundedness of the maximal Calderón-Zygmund operator, the multilinear Calderón-Zygmund operator and the multilinear commutator generated by the Calderón-Zygmund operator on various function spaces.

To this end, we first introduce the notion of the Calderón-Zygmund operator in [19] (see also [18]).

Definition 3.1. A function $K \in L^1_{\rm loc}((\mathcal{X}\times\mathcal{X})\backslash\{(x,x):\ x\in\mathcal{X}\})$ is called a *Calderón-Zygmund kernel* if there exists a positive constant $C(K)$, depending on K, such that,

(i) for all $x,\ y\in\mathcal{X}$ with $x\neq y$,

$$|K(x,y)| \le C(K)\frac{1}{\lambda(x,d(x,y))}; \qquad (3.1)$$

(ii) there exist positive constants $\delta\in(0,1]$ and $c(K)$, depending on K, such that, for all $x,\ \widetilde{x},\ y\in\mathcal{X}$ with $d(x,y)\ge c(K)d(x,\widetilde{x})$,

$$|K(x,y)-K(\widetilde{x},y)| + |K(y,x)-K(y,\widetilde{x})| \le C(K)\frac{[d(x,\widetilde{x})]^\delta}{[d(x,y)]^\delta\lambda(x,d(x,y))}. \qquad (3.2)$$

A linear operator T is called a *Calderón-Zygmund operator* with kernel K satisfying (3.1) and (3.2) if, for all $f\in L^\infty_b(\mu)$, the *space of all $L^\infty(\mu)$ functions with bounded support*, and $x\notin \operatorname{supp}(f)$,

$$Tf(x) := \int_{\mathcal{X}} K(x,y)f(y)\,d\mu(y). \qquad (3.3)$$

Moreover, let $\mathscr{M}(\mathcal{X})$ be the *space of all complex-valued Borel measures on* $\mathcal{X}$. The maximal operator $T^{\#}$ associated with T is defined as follows. For every $f \in L^\infty_b(\mu)$, $\nu \in \mathscr{M}(\mathcal{X})$ and $x \in \mathcal{X}$, let

$$T^{\#}f(x) := \sup_{r\in(0,\infty)} |T_r f(x)| \quad \text{and} \quad T^{\#}\nu(x) := \sup_{r\in(0,\infty)} |T_r \nu(x)|, \tag{3.4}$$

where, for every $r \in (0,\infty)$,

$$T_r f(x) := \int_{\{y\in\mathcal{X}:\ d(x,y)>r\}} K(x,y)f(y)\,d\mu(y)$$

and

$$T_r \nu(x) := \int_{\{y\in\mathcal{X}:\ d(x,y)>r\}} K(x,y)\,d\nu(y).$$

Remark 3.1. (i) We remark that a new example of operators with kernel satisfying (3.1) and (3.2) is the so-called Bergman-type operator appearing in [43]; see also [19] for an explanation.

(ii) We note that the Tb theorem for the Calderón-Zygmund operators was established by Hytönen and Martikainen [19] via the construction of the random system of dyadic cubes, and later Tan and Li [36] obtained the $T1$ theorem by establishing the Littlewood-Paley theory.

3.1 Boundedness of Calderón-Zygmund operators

This subsection is devoted to the boundedness of the Calderón-Zygmund operator on $H^{1,p}_{\rm atb}(\mu)$ as well as other function spaces. To this end, we first state the following Calderón-Zygmund decomposition theorem obtained by Bui and Duong [1, Theorem 6.3] (see [37, Lemma 7.3] for the case $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, \mu)$ with μ as in (1.2)). Let γ_0 be a fixed positive constant satisfying that $\gamma_0 > (216)^{\max\{\nu,n\}}$, where ν is as in Lemma 2.1 and $n := \log_2 N_0$ as in Remark 1.2(ii).

Theorem 3.1. *Let $p \in [1, \infty)$, $f \in L^p(\mu)$ and $t \in (0, \infty)$ ($t > \gamma_0^{1/p}\|f\|_{L^p(\mu)}/[\mu(\mathcal{X})]^{1/p}$ when $\mu(\mathcal{X}) < \infty$). Then*

(i) *there exists a family of finite overlapping balls, $\{6B_j\}_j$, such that $\{B_j\}_j$ is pairwise disjoint,*

$$\frac{1}{\mu(36B_j)}\int_{B_j} |f(x)|^p\,d\mu(x) > \frac{t^p}{\gamma_0} \text{ for all } j,$$

$$\frac{1}{\mu(36\eta B_j)}\int_{\eta B_j} |f(x)|^p\,d\mu(x) \le \frac{t^p}{\gamma_0} \text{ for all } j \text{ and } \eta \in (2,\infty)$$

and

$$|f(x)| \le t \text{ for } \mu\text{-almost every } x \in \mathcal{X}\setminus(\cup_j 6B_j);$$

(ii) *for each j, let R_j be a $(108,(216)^\nu)$-doubling ball of the family $\{(108)^k B_j\}_{k\in\mathbb{N}}$, and*

$$\omega_j := \chi_{6B_j}\Big/\left(\sum_k \chi_{6B_k}\right).$$

Then there exists a family $\{\varphi_j\}_j$ *of functions such that, for each* j, $\operatorname{supp}(\varphi_j) \subset R_j$, φ_j *has a constant sign on* R_j,

$$\int_{\mathcal{X}} \varphi_j(x)\, d\mu(x) = \int_{6B_j} f(x)\omega_j(x)\, d\mu(x)$$

and

$$\sum_j |\varphi_j(x)| \le \gamma t \text{ for } \mu\text{-almost every } x \in \mathcal{X},$$

where γ *is a positive constant depending only on* $(\mathcal{X}, \mu)$ *and there exists a positive constant* C, *independent of* f, t *and* j, *such that, if* $p = 1$, *then*

$$\|\varphi_j\|_{L^\infty(\mu)} \mu(R_j) \le C \int_{\mathcal{X}} |f(x)\omega_j(x)|\, d\mu(x)$$

and, if $p \in (1, \infty)$, *then*

$$\left[\int_{R_j} |\varphi_j(x)|^p\, d\mu(x)\right]^{1/p} [\mu(R_j)]^{1/p'} \le \frac{C}{t^{p-1}} \int_{\mathcal{X}} |f(x)\omega_j(x)|^p\, d\mu(x);$$

(iii) *when* $p \in (1, \infty)$, *if, for any* j, *choosing* R_j *to be the smallest* $(108, (216)^\nu)$-*doubling ball of the family* $\{(108)^k B_j\}_{k\in\mathbb{N}}$, *then*

$$h := \sum_j (f\omega_j - \varphi_j) \in H^1(\mu)$$

and there exists a positive constant C, *independent of* f *and* t, *such that*

$$\|h\|_{H^1(\mu)} \le \frac{C}{t^{p-1}} \|f\|^p_{L^p(\mu)}.$$

Using Theorem 3.1, Bui and Duong [1] established the following interpolation result for linear operators.

Theorem 3.2. *Let* T *be a linear operator which is bounded from* $H^1(\mu)$ *into* $L^1(\mu)$ *and from* $L^\infty(\mu)$ *into* $\mathrm{RBMO}(\mu)$. *Then* T *can be extended to a bounded linear operator on* $L^p(\mu)$ *for all* $p \in (1, \infty)$.

In [23], Lin and Yang obtained the following interpolation theorem which is useful in the study for the boundedness of sublinear operators.

Theorem 3.3. *Let* T *be a sublinear operator that is bounded from* $L^\infty(\mu)$ *into* $\mathrm{RBMO}(\mu)$ *and from* $H^1(\mu)$ *into* $L^{1,\infty}(\mu)$. *Then* T *can be extended to a bounded sublinear operator on* $L^p(\mu)$ *for all* $p \in (1, \infty)$.

Now we state the following results for the boundedness of the Calderón-Zygmund operator associated with kernel K satisfying (3.1) and (3.2) obtained by Hytönen, Da. Yang and Do. Yang [20] and, independently, Bui and Duong [1].

Theorem 3.4. *Let T be a Calderón-Zygmund operator associated with kernel K satisfying* (3.1) *and* (3.2), *which is bounded on $L^2(\mu)$. Then the following conclusions hold true:*
(i) *T is of weak type $(1,1)$;*
(ii) *T is bounded from $L^\infty(\mu)$ into $\mathrm{RBMO}(\mu)$;*
(iii) *T is bounded from $H^1(\mu)$ into $L^1(\mu)$;*
(iv) *T is bounded on $L^p(\mu)$ for all $p \in (1,\infty)$.*

Moreover, it was shown by Hytönen, Liu, Da. Yang and Do. Yang in [18] that the $L^2(\mu)$-boundedness of a Calderón-Zygmund operator T is equivalent to that of T on $L^p(\mu)$ for all $p \in (1,\infty)$ and its endpoint estimate as follows.

Theorem 3.5. *Let T be a Calderón-Zygmund operator associated with kernel K satisfying* (3.1) *and* (3.2). *Then the following statements are equivalent:*
(i) *T is bounded on $L^2(\mu)$;*
(ii) *T is bounded on $L^p(\mu)$ for all $p \in (1,\infty)$;*
(iii) *T is of weak type $(1,1)$.*

Now we investigate the equivalent characterization of the boundedness of T on $L^2(\mu)$ and from $H^1(\mu)$ to $L^1(\mu)$. Let $\mu(\mathcal{X}) = \infty$ and the kernel K of T satisfy (3.1) and the *Hörmander condition* that there exists a positive constant C such that, for all $x \neq \widetilde{x}$,

$$\int_{\{y\in\mathcal{X}:\ d(x,y)\geq 2d(x,\widetilde{x})\}} [|K(x,y)-K(\widetilde{x},y)|+|K(y,x)-K(y,\widetilde{x})|]\,d\mu(y) \leq C. \tag{3.5}$$

Liu, Da. Yang and Do. Yang [27] obtained the following conclusion.

Theorem 3.6. *Let $p \in (1,\infty]$ and T be a Calderón-Zygmund operator associated with kernel K satisfying* (3.1) *and* (3.5). *If $\mu(\mathcal{X}) = \infty$, then the following statements are equivalent:*
(i) *T is bounded on $L^2(\mu)$;*
(ii) *T is bounded from $H^1(\mu)$ into $L^1(\mu)$;*
(iii) *T is bounded from $H^1(\mu)$ into $L^{1,\infty}(\mu)$.*

Remark 3.2. It is still unclear whether the conclusions of Theorem 3.6 hold true or not, if we replace the coefficient $K_{B,S}$ by its discrete counterpart $\widetilde{K}^{(\alpha)}_{B,S}$. Precisely, since, for any balls $B \subset S$ with $\mu(2S\backslash B) = 0$, it is unclear whether there exists a positive constant C, independent of B and S, such that $\widetilde{K}^{(\alpha)}_{B,S} \leq C$ or not, the method used in the proof of [27, Lemma 3.1] does not apply to $\widetilde{K}^{(\alpha)}_{B,S}$.

Using Theorems 3.6 and 2.3, Hu, Meng and Yang [15] further obtained the following characterizations for the boundedness of Calderón-Zygmund operators.

Theorem 3.7. *Let $\rho \in (1,\infty)$ and T be a Calderón-Zygmund operator associated with kernel K satisfying* (3.1) *and* (3.2). *Then the following seven statements are equivalent:*
(i) *T is bounded from $H^1(\mu)$ into $L^1(\mu)$;*
(ii) *T is bounded from $H^1(\mu)$ into $L^{1,\infty}(\mu)$;*

(iii) *for some* $\nu \in (0,\infty)$, *there exists a positive constant* C *such that, for all* ε, $t \in (0,\infty)$, *balls* B *and bounded functions* f *with* $\operatorname{supp}(f) \subset B$,

$$\mu(\{x \in B:\ |T_\varepsilon(f)(x)| > t\}) \le Ct^{-\nu}\mu(\rho B)\|f\|^{\nu}_{L^\infty(\mu)};$$

(iv) *for some* $\sigma \in (0,1)$, *there exists a positive constant* C *such that, for all* $\varepsilon \in (0,\infty)$, *balls* B *and bounded functions* f *with* $\operatorname{supp}(f) \subset B$,

$$\frac{1}{\mu(\rho B)}\int_B |T_\varepsilon(f)(x)|^\sigma\, d\mu(x) \le C\|f\|^{\sigma}_{L^\infty(\mu)};$$

(v) T *is bounded from* $L^\infty(\mu)$ *into* $\mathrm{RBMO}(\mu)$*;*
(vi) T *is bounded on* $L^p(\mu)$ *for some* $p \in (1,\infty)$*;*
(vii) T *is bounded on* $L^p(\mu)$ *for all* $p \in (1,\infty)$.

Remark 3.3. For the same reason as in Remark 3.2, the results in Theorem 3.7 are also unknown when $K_{B,S}$ is replaced by $\widetilde{K}^{(\alpha)}_{B,S}$.

Now we turn our attention to the boundedness of T on $H^1(\mu)$ and $\widetilde{H}^1(\mu)$ established in [10]. To be precise, let T be bounded on $L^2(\mu)$ and $T^*1 = 0$, where, by $T^*1 = 0$, we mean that, for any $g \in L^\infty_b(\mu)$ and $\int_{\mathcal{X}} g(y)\, d\mu(y) = 0$, it holds true that

$$\int_{\mathcal{X}} Tg(x)\, d\mu(x) = 0.$$

Theorem 3.8. *Let* $p \in (1,\infty)$. *Suppose that* T *is a Calderón-Zygmund operator associated with kernel* K *satisfying* (3.1) *and* (3.2), *which is bounded on* $L^2(\mu)$, *and* $T^*1 = 0$. *Then there exists a positive constant* C *such that, for all* $f \in \widetilde{H}^1(\mu)$, $Tf \in \widetilde{H}^1(\mu)$ *and*

$$\|Tf\|_{\widetilde{H}^1(\mu)} \le C\|f\|_{\widetilde{H}^1(\mu)}.$$

As a corollary of Theorem 3.8, we obtain the boundedness of Calderón-Zygmund operators on $\widetilde{\mathrm{RBMO}}(\mu)$ in [10] immediately.

Corollary 3.1. *Let* T *be as in Theorem 3.8 and* T^* *the adjoint operator of* T. *Then there exists a positive constant* C *such that, for all* $f \in \widetilde{\mathrm{RBMO}}(\mu)$, $T^*f \in \widetilde{\mathrm{RBMO}}(\mu)$ *and*

$$\|T^*f\|_{\widetilde{\mathrm{RBMO}}(\mu)} \le C\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}.$$

We remark that the method used in the proof of Theorem 3.8 does not work for the boundedness of T on $H^1(\mu)$. Moreover, it is still unknown in general whether T is bounded on $H^1(\mu)$ or not. However, if the dominating function λ satisfies the weak reverse doubling condition (see Definition 2.4), then we have the following conclusion; see [10].

Corollary 3.2. *Let* $(\mathcal{X}, d, \mu)$ *be a non-homogeneous space with the dominating function satisfying the weak reverse doubling condition.*
(i) *If* T *is as in Theorem 3.8, then* T *is bounded on* $H^1(\mu)$.
(ii) *If* T *is as in Theorem 3.8 and* T^* *the adjoint operator of* T, *then* T^* *is bounded on* $\mathrm{RBMO}(\mu)$.

Remark 3.4. (i) When $(\mathcal{X}, d, \mu)$ is as in Remark 2.4(i), by the same reason as therein, we see that all the conclusions of Corollary 3.2 are true in this case.

(ii) When $(\mathcal{X}, d, \mu)$ is as in Remark 2.4(ii), by the same reason as therein, we know that all the conclusions of Corollary 3.2 are also true in this case.

3.2 Boundedness of maximal Calderón-Zygmund operators

This subsection is devoted to the $L^p(\mu)$-boundedness, with $p \in (1, \infty)$, of the maximal Calderón-Zygmund operator $T^\#$ and its endpoint estimates. We start with the following result established in [18].

Theorem 3.9. *Let T be a Calderón-Zygmund operator associated with kernel K satisfying* (3.1) *and* (3.2)*, which is bounded on $L^2(\mu)$, and $T^\#$ the maximal Calderón-Zygmund operator associated with T. Then the following statements hold true:*

(i) *$T^\#$ is bounded on $L^p(\mu)$ for all $p \in (1, \infty)$;*

(ii) *for a measure $\nu \in \mathscr{M}(\mathcal{X})$, let*

$$\|\nu\| := \int_{\mathcal{X}} |d\nu(x)|.$$

Then, there exists a positive constant $\widetilde{c}$ such that, for all $\nu \in \mathscr{M}(\mathcal{X})$,

$$\|T^\#\nu\|_{L^{1,\infty}(\mu)} \le \widetilde{c}\|\nu\|.$$

Moreover, $T^\#$ is of weak type $(1, 1)$.

For the maximal Calderón-Zygmund operator associated with kernel K satisfying (3.1) and (3.5), Liu, Meng and Yang [26] obtained the following conclusions.

Theorem 3.10. *Let $T^\#$ be the maximal Calderón-Zygmund operator as in* (3.4) *associated with kernel K satisfying* (3.1) *and* (3.5)*. Then the following statements are equivalent:*

(i) *$T^\#$ is bounded on $L^{p_0}(\mu)$ for some $p_0 \in (1, \infty)$;*

(ii) *$T^\#$ is of weak type* $(1, 1)$*;*

(iii) *$T^\#$ is bounded on $L^p(\mu)$ for all $p \in (1, \infty)$.*

Using Theorem 3.10, Liu, Meng and Yang [26] further showed the following result, which is an improvement of Theorem 3.9.

Theorem 3.11. *Let T be an $L^2(\mu)$-bounded Calderón-Zygmund operator associated with kernel K satisfying* (3.1) *and* (3.5)*, and $T^\#$ the maximal operator associated with T. Then the following statements hold true:*

(i) *$T^\#$ is bounded on $L^p(\mu)$ for all $p \in (1, \infty)$;*

(ii) *$T^\#$ is of weak type* $(1, 1)$.

On the other hand, in the case when $p = \infty$, Lin and Yang [22] showed that $T^\#$ is bounded from $L^\infty(\mu)$ into $\mathrm{RBLO}(\mu)$, which is stated as follows.

Theorem 3.12. *Let T be a Calderón-Zygmund operator as in* (3.3) *associated with kernel K satisfying* (3.1) *and* (3.2)*, which is bounded on $L^2(\mu)$. Then the maximal operator $T^\#$ is bounded from $L^\infty(\mu)$ to* $\mathrm{RBLO}(\mu)$.

3.3 Weighted estimates for multilinear Calderón-Zygmund operators

In this subsection, we review a weighted norm inequality for the multilinear Calderón-Zygmund operator obtained by Hu, Meng and Yang [16]. We first recall some notation and notions.

Let m be a positive integer, $\Delta := \{(x, \ldots, x) : \ x \in \mathcal{X}\}$ and $K(x, y_1, \ldots, y_m)$ a μ-locally integrable function mapping from $(\mathcal{X} \times \cdots \times \mathcal{X}) \backslash \Delta$ to $\mathbb{C}$, which satisfies the *size condition* that there exists a positive constant C such that, for all x, y_1, $\ldots$, $y_m \in \mathcal{X}$ with $x \neq y_j$ for some j,

$$|K(x, y_1, \ldots, y_m)| \leq C \frac{1}{[\sum_{i=1}^m \lambda(x, d(x, y_i))]^m}, \tag{3.6}$$

and the *regularity condition* that there exist some positive constants τ and C such that, for all x, $\widetilde{x}$, y_1, $\ldots$, $y_m \in \mathcal{X}$ with $\max\{d(x, y_1), \ldots, d(x, y_m)\} \geq 2d(x, \widetilde{x})$,

$$\begin{aligned} &|K(x, y_1, \ldots, y_m) - K(\widetilde{x}, y_1, \ldots, y_m)| \\ &\quad \leq C \frac{[d(x, \widetilde{x})]^\tau}{[\sum_{i=1}^m d(x, y_i)]^\tau [\sum_{i=1}^m \lambda(x, d(x, y_i))]^m}. \end{aligned} \tag{3.7}$$

A multilinear operator T associated with kernel K is called a *multilinear Calderón-Zygmund operator*, if it is bounded from $L^1(\mu) \times \cdots \times L^1(\mu)$ into $L^{1/m, \infty}(\mu)$ and satisfies that, for all $f_1, \ldots, f_m \in L^\infty_b(\mu)$ and μ-almost every $x \in \mathcal{X} \backslash (\cap_{j=1}^m \operatorname{supp}(f_j))$,

$$\begin{aligned} &T(f_1, \ldots, f_m)(x) \\ &\quad := \int_{\mathcal{X}} \cdots \int_{\mathcal{X}} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) \, d\mu(y_1) \cdots d\mu(y_m). \end{aligned} \tag{3.8}$$

When $m = 1$, the operator T defined by (3.8) is a version of the Calderón-Zygmund operator. When $m \geq 2$ and $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, dx)$, the operator defined by (3.8) is just the classical multilinear Calderón-Zygmund operator. The study of multilinear Calderón-Zygmund operators is not motivated only by a mere quest to generalize the classical Calderón-Zygmund theory, but rather by their natural appearance in analysis. In what follows, we always assume $m = 2$ for brevity.

Let $\rho \in [1, \infty)$, $\vec{P} := (p_1, p_2)$ with p_1, $p_2 \in [1, \infty)$ and $1/p := 1/p_1 + 1/p_2$. A map $\vec{\omega} := (\omega_1, \omega_2)$ is said to belong to $A^\rho_{\vec{P}}(\mu)$ if ω_1 and ω_2 are nonnegative μ-measurable functions and there exists a positive constant C such that, for all balls $B \subset \mathcal{X}$,

$$\frac{1}{\mu(\rho B)} \int_B v_{\vec{\omega}}(x) \, d\mu(x) \prod_{j=1}^2 \left\{ \frac{1}{\mu(\rho B)} \int_B [w_j(x)]^{1-p_j'} \, d\mu(x) \right\}^{p/p_j'} \leq C,$$

where, for all $x \in \mathcal{X}$,

$$v_{\vec{\omega}}(x) := \prod_{j=1}^2 [\omega_j(x)]^{p/p_j}$$

and, when $p_j = 1$,

$$\left\{ \frac{1}{\mu(\rho B)} \int_B [w_j(x)]^{1-p_j'} \, d\mu(x) \right\}^{1/p_j'}$$

is understood as $(\inf_B \omega_j)^{-1}$ for $j \in \{1, 2\}$.

The following result was obtained by Hu, Meng and Yang [16].

Theorem 3.13. *Let K be a μ-locally integrable function mapping from $(\mathcal{X}\times\mathcal{X}\times\mathcal{X})\setminus\Delta$ to $\mathbb{C}$, which satisfies (3.6) and (3.7) with $m = 2$, and T as in (3.8). Then, for all $\vec{P} := (p_1, p_2)$ with p_1, $p_2 \in [1,\infty)$, $1/p := 1/p_1 + 1/p_2$ and $\vec{\omega} := (\omega_1, \omega_2) \in A^{\rho}_{\vec{P}}(\mu)$ with $\rho \in [1,\infty)$, T can be extended to be a bounded bilinear operator from $L^{p_1}(\omega_1) \times L^{p_2}(\omega_2)$ into $L^p(v_{\vec{\omega}})$ and, moreover, there exists a positive constant C such that, for all $f_1 \in L^{p_1}(\omega_1)$ and $f_2 \in L^{p_2}(\omega_2)$,*

$$\|T(f_1, f_2)\|_{L^{p,\infty}(v_{\vec{\omega}})} \le C\|f_1\|_{L^{p_1}(\omega_1)}\|f_2\|_{L^{p_1}(\omega_2)}.$$

Remark 3.5. (i) In the case of non-homogeneous spaces, it is still unknown whether T is bounded from $L^1(\mu) \times L^1(\mu)$ into $L^{1/2,\infty}(\mu)$ or not, if we only assume that T is bounded from $L^{q_1}(\mu) \times L^{q_2}(\mu)$ into $L^{q,\infty}(\mu)$ for some q_1, $q_2 \in (1,\infty)$ and $q \in (0,\infty)$ with $1/q := 1/q_1 + 1/q_2$ and that there exist positive constants C and τ such that, for all x, y, z, $\widetilde{y} \in \mathcal{X}$ with $\max\{d(x, y),\, d(x, z)\} \ge 2d(y, \widetilde{y})$,

$$\begin{aligned}&|K(x,y,z) - K(x,\widetilde{y},z)| + |K(x,z,y) - K(x,z,\widetilde{y})|\\ &\quad\le C\frac{[d(y,\widetilde{y})]^{\tau}}{[d(x,y)+d(x,z)]^{\tau}[\lambda(x,d(x,y))+\lambda(x,d(x,z))]^2}.\end{aligned}$$

Even in the case when $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, \mu)$ with μ as in (1.2), this is also unknown.

(ii) In [16], the authors concluded that, if we only assume that T is bounded from $L^{q_1}(\mu) \times L^{q_2}(\mu)$ into $L^{q,\infty}(\mu)$ for some q_1, $q_2 \in (1,\infty)$ and q with $1/q := 1/q_1 + 1/q_2$, then the ranges of indices p_1 and p_2 in Theorem 3.13 are $p_1 \in [q_1,\infty)$ and $p_2 \in [q_2,\infty)$, which are narrower than Theorem 3.13, and $\vec{\omega}$ belongs to some smaller weight class than Theorem 3.13.

3.4 Boundedness of multilinear commutators

In this subsection, we mainly discuss the boundedness of the multilinear commutator, generated by the Calderón-Zygmund operator with any RBMO(μ) function, and its weak-type endpoint estimate.

Let $b \in$ RBMO(μ) and T be a Calderón-Zygmund operator as in (3.3) associated with kernel K satisfying (3.1) and (3.2). For any $f \in L^\infty_b(\mu)$ and $x \in \mathcal{X}\setminus \operatorname{supp}(f)$, the commutator $[b, T]$ is defined by setting

$$[b,T](f)(x) := b(x)T(f)(x) - T(bf)(x). \tag{3.9}$$

Under the *additional assumption*: there exists $m \in (0,\infty)$ such that

$$\lambda(x, ar) = a^m\lambda(x, r) \text{ for all } x \in \mathcal{X} \text{ and } a,\ r \in (0,\infty), \tag{3.10}$$

where λ is the dominating function of the measure μ, Bui and Duong [1] obtained the following $L^p(\mu)$-boundedness, with $p \in (1,\infty)$, of $[b, T]$.

Theorem 3.14. *Assume that λ satisfies* (3.10)*. Let $b \in$RBMO (μ) and T be as in* (3.3) *associated with kernel K satisfying* (3.1) *and* (3.2)*, which is bounded on $L^2(\mu)$. Then the commutator $[b,T]$ in* (3.9) *is bounded on $L^p(\mu)$ for all $p \in (1,\infty)$.*

Theorem 3.14 was extended by Fu, Yang and Yuan [8] to the boundedness of the multilinear commutator on the Orlicz space. We first recall some notions and notation from [8].

Let Φ be a *convex Orlicz function* on $[0,\infty)$, namely, a convex increasing function satisfying $\Phi(0)=0$, $\Phi(t)>0$ for all $t \in (0,\infty)$ and $\Phi(t)\to\infty$ as $t\to\infty$. Let

$$a_\Phi := \inf_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)} \quad \text{and} \quad b_\Phi := \sup_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}.$$

The *Orlicz space* $L^\Phi(\mu)$ is defined to be the space of all measurable functions f on $(\mathcal{X},d,\mu)$ such that $\int_{\mathcal{X}} \Phi(|f(x)|)\,d\mu(x) < \infty$; moreover, for any $f \in L^\Phi(\mu)$, its *Luxemburg norm* in $L^\Phi(\mu)$ is defined by

$$\|f\|_{L^\Phi(\mu)} := \inf\left\{t\in(0,\infty):\ \int_{\mathcal{X}} \Phi(|f(x)|/t)\,d\mu(x) \le 1\right\}.$$

For any sequence $\vec{b} := (b_1,\ldots,b_k)$ of functions, the *multilinear commutator* $T_{\vec{b}}$ of the Calderón-Zygmund operator T and $\vec{b}$ is defined by setting, for all suitable functions f and $x\in\mathcal{X}$,

$$T_{\vec{b}}f(x) := [b_k,[b_{k-1},\cdots,[b_1,T]\cdots]]f(x). \tag{3.11}$$

Now we state the following result about the boundedness of multilinear commutators on Orlicz spaces from [8].

Theorem 3.15. *Let $k\in\mathbb{N}$, $b_i \in$ RBMO(μ) for all $i\in\{1,\ldots,k\}$, Φ a convex Orlicz function satisfying that $1<a_\Phi\le b_\Phi<\infty$. Assume that T is as in* (3.3) *associated with kernel K satisfying* (3.1) *and* (3.2)*, which is bounded on $L^2(\mu)$. Then the multilinear commutator $T_{\vec{b}}$ in* (3.11) *is bounded on Orlicz spaces $L^\Phi(\mu)$, namely, there exists a positive constant C such that, for all $f\in L^\Phi(\mu)$,*

$$\|T_{\vec{b}}f\|_{L^\Phi(\mu)} \le C\|b_1\|_{\mathrm{RBMO}(\mu)}\cdots\|b_k\|_{\mathrm{RBMO}(\mu)}\|f\|_{L^\Phi(\mu)}.$$

Let $\Phi_1(t) := t^p$ for all $t\in(0,\infty)$ with $p\in(1,\infty)$. Then Φ_1 is a convex Orlicz function, with $a_{\Phi_1}=b_{\Phi_1}=p\in(1,\infty)$, and $L^{\Phi_1}(\mu)=L^p(\mu)$. In this case, for $k=1$, Theorem 3.15 also essentially improves Theorem 3.14 by removing the assumption (3.10).

The endpoint counterpart of Theorem 3.15 was also considered in [8]. To begin with, we recall the following Orlicz type function space $\mathrm{Osc}_{\exp L^r}(\mu)$.

Definition 3.2. For $r\in[1,\infty)$, a function $f\in L^1_{\mathrm{loc}}(\mu)$ is said to belong to the *space* $\mathrm{Osc}_{\exp L^r}(\mu)$ if there exists a positive constant C such that,

(i) for all balls B,

$$\|f-m_{\widetilde{B}}(f)\|_{\exp L^r,\,B,\,\mu/\mu(2B)}$$

$$:= \inf\left\{\lambda \in (0,\infty) \,:\, \frac{1}{\mu(2B)}\int_B \exp\left(\frac{|f(x)-m_{\widetilde{B}}(f)|}{\lambda}\right)^r d\mu(x) \le 2\right\} \le C;$$

(ii) for all doubling balls $B \subset S$,

$$|m_B(f) - m_S(f)| \le CK_{B,S}.$$

The $\mathrm{Osc}_{\exp L^r}(\mu)$ *norm* of f, $\|f\|_{\mathrm{Osc}_{\exp L^r}(\mu)}$, is then defined to be the infimum of all positive constants C satisfying (i) and (ii).

Now we state the endpoint estimate for the multilinear commutator from [8] as follows.

Theorem 3.16. *Let $k \in \mathbb{N}$, $r_i \in [1,\infty)$ and $b_i \in \mathrm{Osc}_{\exp L^{r_i}}(\mu)$ for $i \in \{1,\ldots,k\}$. Let T be as in* (3.3) *associated with kernel K satisfying* (3.1) *and* (3.2)*, and $T_{\vec{b}}$ as in* (3.11)*, respectively. If T is bounded on $L^2(\mu)$, then there exists a positive constant C such that, for all $t \in (0,\infty)$ and $f \in L_b^\infty(\mu)$,*

$$\mu(\{x \in \mathcal{X} \,:\, |T_{\vec{b}}f(x)| > t\}) \le C\Phi_{1/r}(\|b_1\|_{\mathrm{Osc}_{\exp L^{r_1}}(\mu)} \cdots \|b_k\|_{\mathrm{Osc}_{\exp L^{r_k}}(\mu)}) \times \int_{\mathcal{X}} \Phi_{1/r}\left(\frac{|f(y)|}{t}\right) d\mu(y),$$

where $1/r := 1/r_1 + \cdots + 1/r_k$ and, for all $s \in (0,\infty)$ and $t \in (0,\infty)$,

$$\Phi_s(t) := t\log^s(2+t).$$

4 Generalized fractional integrals and Marcinkiewicz integrals

In this section, we review the results on the equivalent boundedness of the generalized fractional integral, the boundedness of the multilinear commutator associated with the generalized fractional integral. The equivalent characterization of the boundedness of the Marcinkiewicz integral and some endpoint estimates for the Marcinkiewicz integral are also considered in this section.

4.1 Generalized fractional integrals

This subsection is devoted to the equivalent characterization of the $(L^p(\mu), L^q(\mu))$-boundedness of the generalized fractional integral T_α, and the boundedness of the multilinear commutator associated with T_α on Orlicz spaces and its endpoint estimate established by Fu, Yang and Yuan in [9]. We begin with the notion of the generalized fractional integral.

Definition 4.1. Let $\alpha \in (0,1)$. A function $K_\alpha \in L^1_{\mathrm{loc}}(\mathcal{X}\times\mathcal{X} \setminus \{(x,x) :\ x \in \mathcal{X}\})$ is called a *generalized fractional integral kernel* if there exists a positive constant $C(K_\alpha)$, depending on K_α, such that,

(i) for all x, $y \in \mathcal{X}$ with $x \ne y$,

$$|K_\alpha(x,y)| \le C(K_\alpha)\frac{1}{[\lambda(x,d(x,y))]^{1-\alpha}}; \tag{4.1}$$

(ii) there exists a positive constant $\delta \in (0,1]$ such that, for all x, $\widetilde{x}$, $y \in \mathcal{X}$ with $d(x,y) \geq C(K_\alpha)d(x,\widetilde{x})$,

$$|K_\alpha(x,y)-K_\alpha(\widetilde{x},y)|+|K_\alpha(y,x)-K_\alpha(y,\widetilde{x})| \leq C(K_\alpha)\frac{[d(x,\widetilde{x})]^\delta}{[d(x,y)]^\delta[\lambda(x,d(x,y))]^{1-\alpha}}. \tag{4.2}$$

A linear operator T_α is called a *generalized fractional integral* with kernel K_α satisfying (4.1) and (4.2) if, for all $f \in L_b^\infty(\mu)$ and $x \notin \operatorname{supp}(f)$,

$$T_\alpha f(x) := \int_{\mathcal{X}} K_\alpha(x,y)f(y)\,d\mu(y). \tag{4.3}$$

Remark 4.1. It was shown in [9] that there exists a specific example of the generalized fractional integral, which is a natural variant of the so-called Bergman-type operator; see [9] for the details.

Then we state the following result on the equivalent characterizations for the boundedness of the generalized fractional integral over $(\mathcal{X},d,\mu)$ from [9].

Theorem 4.1. *Let $\alpha \in (0,1)$ and T_α be a generalized fractional integral as in* (4.3) *with kernel K satisfying* (4.1) *and* (4.2)*. Then the following statements are equivalent:*

(i) *T_α is bounded from $L^p(\mu)$ into $L^q(\mu)$ for all $p \in (1, 1/\alpha)$ and $1/q := 1/p - \alpha$;*

(ii) *T_α is bounded from $L^1(\mu)$ into $L^{1/(1-\alpha),\infty}(\mu)$;*

(iii) *there exists a positive constant C such that, for all $f \in L^{1/\alpha}(\mu)$ with $T_\alpha f$ being finite almost everywhere,*

$$\|T_\alpha f\|_{\mathrm{RBMO}(\mu)} \leq C\|f\|_{L^{1/\alpha}(\mu)};$$

(iv) *T_α is bounded from $H^1(\mu)$ into $L^{1/(1-\alpha)}(\mu)$;*

(v) *T_α is bounded from $H^1(\mu)$ into $L^{1/(1-\alpha),\infty}(\mu)$.*

Remark 4.2. By the same reason as in Remark 3.2, we see that the results in Theorem 4.1 are also unknown for $\widetilde{K}_{B,S}^{(\alpha)}$ instead of $K_{B,S}$.

Now we turn our attention to the boundedness of the multilinear commutator of the generalized fractional integral. For any sequence $\vec{b} := (b_1,\ldots,b_k)$ of functions, the *multilinear commutator* $T_{\alpha,\vec{b}}$ of the generalized fractional integral T_α with $\vec{b}$ is defined by setting, for all suitable functions f,

$$T_{\alpha,\vec{b}}f := [b_k,\cdots,[b_1,T_\alpha]\cdots]f, \tag{4.4}$$

where

$$[b_1,T_\alpha]f := b_1T_\alpha f - T_\alpha(b_1 f).$$

The following theorem was established in [9].

Theorem 4.2. *Let $\alpha \in (0,1)$, $k \in \mathbb{N}$ and $b_j \in \mathrm{RBMO}(\mu)$ for all $j \in \{1,\ldots,k\}$. Let Φ be a convex Orlicz function and Ψ defined, via its inverse, by setting, for all $t \in (0,\infty)$,*

$$\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha},$$

where

$$\Phi^{-1}(t) := \inf\{s \in (0,\infty) :\ \Phi(s) > t\}.$$

Suppose that T_α is as in (4.3), with kernel K_α satisfying (4.1) and (4.2), which is bounded from $L^p(\mu)$ into $L^q(\mu)$ for all $p \in (1, 1/\alpha)$ and $1/q := 1/p - \alpha$. If $1 < a_\Phi \le b_\Phi < \infty$ and $1 < a_\Psi \le b_\Psi < \infty$, then the multilinear commutator $T_{\alpha,\vec{b}}$ as in (4.4) is bounded from $L^\Phi(\mu)$ to $L^\Psi(\mu)$, namely, there exists a positive constant C such that, for all $f \in L^\Phi(\mu)$,

$$\|T_{\alpha,\vec{b}} f\|_{L^\Psi(\mu)} \le C \prod_{j=1}^k \|b_j\|_{\mathrm{RBMO}(\mu)} \|f\|_{L^\Phi(\mu)}.$$

Now we consider the endpoint counterpart of Theorem 4.2. For $i \in \{1,\dots,k\}$, the *family of all finite subsets $\sigma := \{\sigma(1),\dots,\sigma(i)\}$ of $\{1,\dots,k\}$ with i different elements* is denoted by C_i^k. For any $\sigma \in C_i^k$, the *complementary sequence* σ' is defined by $\sigma' := \{1,\dots,k\} \setminus \sigma$. For any $\sigma := \{\sigma(1),\dots,\sigma(i)\} \in C_i^k$ and k-tuple $r := (r_1,\dots,r_k)$, we write that

$$1/r_\sigma := 1/r_{\sigma(1)} + \cdots + 1/r_{\sigma(i)} \quad \text{and} \quad 1/r_{\sigma'} := 1/r - 1/r_\sigma,$$

where $1/r := 1/r_1 + \cdots + 1/r_k$.

For $r \in [1,\infty)$, let $\mathrm{Osc}_{\exp L^r}(\mu)$ be as in Definition 3.2. Then we are ready to state the result in [9].

Theorem 4.3. *Let $\alpha \in (0,1)$, $k \in \mathbb{N}$, $r_j \in [1,\infty)$ and $b_j \in \mathrm{Osc}_{\exp L^{r_j}}(\mu)$ for $j \in \{1,\dots,k\}$. Let T_α and $T_{\alpha,\vec{b}}$ be, respectively, as in (4.3) and (4.4) with kernel K_α satisfying (4.1) and (4.2). Suppose that T_α is bounded from $L^p(\mu)$ into $L^q(\mu)$ for all $p \in (1,1/\alpha)$ and $1/q := 1/p - \alpha$. Then, there exists a positive constant C such that, for all $t \in (0,\infty)$ and $f \in L_b^\infty(\mu)$,*

$$\begin{aligned}
&\mu(\{x \in \mathcal{X} :\ |T_{\alpha,\vec{b}} f(x)| > t\}) \\
&\quad \le C \left[\Phi_{1/r}\left(\prod_{j=1}^k \|b_j\|_{\mathrm{Osc}_{\exp L^{r_j}}(\mu)}\right)\right] \left[\sum_{j=0}^k \sum_{\sigma \in C_j^k} \Phi_{1/r_\sigma}\left(\|\Phi_{1/r_{\sigma'}}(t^{-1}|f|)\|_{L^1(\mu)}\right)\right],
\end{aligned}$$

where, for all $s \in (0,\infty)$, Φ_s is as in Theorem 3.16.

Remark 4.3. For all $\alpha \in (0,1)$, $f \in L_b^\infty(\mu)$ and $x \in \mathcal{X}$, the *fractional integral $I_\alpha f(x)$* is defined by

$$I_\alpha f(x) := \int_{\mathcal{X}} \frac{f(y)}{[\lambda(y, d(x,y))]^{1-\alpha}}\, d\mu(y).$$

It is easy to see that, under the assumption (3.10), the fractional integral I_α is a special case of the generalized fractional integral. Moreover, in [9], the authors showed that all the conclusions of Theorems 4.1, 4.2 and 4.3 hold true, if T_α is replaced by I_α.

4.2 Marcinkiewicz integrals

In this subsection, we discuss equivalent characterizations for the $L^p(\mu)$-boundedness, with $p \in (1, \infty)$, of the Marcinkiewicz integral and its several endpoint estimates obtained by Lin and Yang in [24]. To this end, we first recall the notion of the Marcinkiewicz integral; see [12] for the case $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, \mu)$ with μ as in (1.2).

Definition 4.2. A function $K \in L^1_{\rm loc}(\mathcal{X} \times \mathcal{X} \setminus \{(x,x) : x \in \mathcal{X}\})$ is called a *Marcinkiewicz integral kernel* if there exists a positive constant $C(K)$, depending on K, such that,

(i) for all x, $y \in \mathcal{X}$ with $x \neq y$,

$$|K(x,y)| \le C(K) \frac{d(x,y)}{\lambda(x, d(x,y))}; \tag{4.5}$$

(ii) for all y, $\widetilde{y} \in \mathcal{X}$,

$$\int_{\{x\in\mathcal{X}:\ d(x,y)\ge 2d(y,\widetilde{y})\}} [|K(x,y) - K(x,\widetilde{y})| + |K(y,x) - K(\widetilde{y},x)|] \frac{d\mu(x)}{d(x,y)} \le C(K). \tag{4.6}$$

The *Marcinkiewicz integral* $\mathcal{M}(f)$ *associated to the above kernel* K is defined by setting, for all suitable functions f and $x \notin \mathcal{X}$,

$$\mathcal{M}(f)(x) := \left[\int_0^\infty \left|\int_{\{y\in\mathcal{X}:\ d(x,y)<t\}} K(x,y) f(y)\, d\mu(y)\right|^2 \frac{dt}{t^3}\right]^{1/2}. \tag{4.7}$$

When $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, dx)$, $\mathcal{M}$ is just the classical Marcinkiewicz integral. Thus, $\mathcal{M}$ is a natural generalization of the classical Marcinkiewicz integral in the present setting.

The following conclusion was obtained in [24].

Theorem 4.4. *Let* $\mathcal{M}$ *be a Marcinkiewicz integral as in* (4.7) *associated with kernel* K *satisfying* (4.5) *and* (4.6). *Then the following statements are equivalent:*

(i) $\mathcal{M}$ *is bounded on* $L^{p_0}(\mu)$ *for some* $p_0 \in (1, \infty)$*;*

(ii) $\mathcal{M}$ *is of weak type* $(1, 1)$*;*

(iii) $\mathcal{M}$ *is bounded on* $L^p(\mu)$ *for all* $p \in (1, \infty)$*;*

(iv) $\mathcal{M}$ *is bounded from* $H^1(\mu)$ *into* $L^1(\mu)$*.*

Remark 4.4. By the same reason as in Remark 3.2, we see that the results in Theorem 4.4 are still unknown when $K_{B,S}$ is replaced by $\widetilde{K}^{(\alpha)}_{B,S}$.

Comparing with the corresponding result in [12], Theorem 4.4 makes an essential improvement.

To discuss the corresponding endpoint estimate, we recall the notion of the space of all finite linear combinations of $(p, 1)_\lambda$-atomic blocks.

Definition 4.3. Let $p \in (1, \infty]$. The *space* $H^{1,\,p}_{\mathrm{fin}}(\mu)$ is defined to be the vector space of all finite linear combinations of $(p, 1)_\lambda$-atomic blocks. Moreover, the *norm* of f in $H^{1,\,p}_{\mathrm{fin}}(\mu)$ is defined by

$$\|f\|_{H^{1,\,p}_{\mathrm{fin}}(\mu)} := \inf\left\{\sum_{j=1}^{N} |b_j|_{H^{1,\,p}_{\mathrm{atb}}(\mu)} : \ f = \sum_{j=1}^{N} b_j, \ b_j \text{ is a } (p,1)_\lambda - \text{atomic block}, \ N \in \mathbb{N}\right\}.$$

Now we are ready to state the results on the endpoint estimate of $\mathcal{M}$ in [24].

Theorem 4.5. *Let $\mathcal{M}$ be a Marcinkiewicz integral as in* (4.7) *associated with kernel K satisfying* (4.5) *and* (4.6).

(i) *If $\mathcal{M}$ is bounded from $H^1(\mu)$ into $L^1(\mu)$, then, for $f \in L^\infty(\mu)$, $\mathcal{M}(f)$ is either infinite everywhere or finite μ-almost everywhere; more precisely, if $\mathcal{M}(f)$ is finite at some point $x_0 \in \mathcal{X}$, then $\mathcal{M}(f)$ is finite μ-almost everywhere and*

$$\|\mathcal{M}(f)\|_{\mathrm{RBLO}(\mu)} \le C\|f\|_{L^\infty(\mu)},$$

where C is a positive constant independent of f.

(ii) *If there exists a positive constant C such that, for all $f \in L^\infty_b(\mu)$,*

$$\|\mathcal{M}(f)\|_{\mathrm{RBMO}(\mu)} \le C\|f\|_{L^\infty(\mu)},$$

then $\mathcal{M}$ is bounded from $H^{1,\,\infty}_{\mathrm{fin}}(\mu)$ into $L^1(\mu)$.

Remark 4.5. (i) It was shown in [24] that, if $\mathcal{M}$ is bounded from $H^1(\mu)$ into $L^1(\mu)$, then, for any $f \in L^\infty(\mu)$, $\mathcal{M}(f)$ is either infinite everywhere or

$$\|\mathcal{M}(f)\|_{\mathrm{RBMO}(\mu)} \le C\|f\|_{L^\infty(\mu)},$$

with the positive constant C independent of f, which improves the known corresponding result even on the classical Euclidean space $\mathbb{R}^D$.

(ii) By Theorem 4.4, if $\mathcal{M}$ is bounded from $H^1(\mu)$ into $L^1(\mu)$, then it is also bounded on $L^p(\mu)$ for all $p \in (1, \infty)$ and, for any $f \in L^\infty_b(\mu)$, $\mathcal{M}(f)$ is finite at some point $x_0 \in \mathcal{X}$. This, together with Theorem 4.5(i), further shows that $\mathcal{M}$ is bounded from $L^\infty_b(\mu)$ into $\mathrm{RBLO}(\mu)$ and hence, it is bounded from $L^\infty_b(\mu)$ into $\mathrm{RBMO}(\mu)$.

(iii) In the present setting, it is still unclear whether the uniform boundedness in some Banach space $\mathcal{B}$ of a sublinear operator T on all $(\infty, 1)_\lambda$-atomic blocks can guarantee the boundedness of T from $H^1(\mu)$ into $\mathcal{B}$ or not. Thus, under the assumption of Theorem 4.5, it is still unknown whether the Marcinkiewicz integral $\mathcal{M}$ can be extended boundedly from $H^1(\mu)$ into $L^1(\mu)$ or not.

(iv) The results in Theorem 4.5 are also unknown, when $K_{B,\,S}$ is replaced by $\widetilde{K}^{(\alpha)}_{B,\,S}$, by the same reason as in Remark 3.2.

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