ISSN 2077–9879

Eurasian Mathematical Journal

2013, Volume 4, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)

Published by
the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ will also publish survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Mathematical Reviews, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Honorarium. An honorarium will be paid to the authors (at 50 $ per page, but no more than 2000 $ for a paper) and to the reviewers (at 50 $ per review). All papers will be edited with regard to usage of English. If severe editing is required, the honorarium may be reduced to 40 $ per page. Papers may be submitted in Russian. They will be translated into English, and in this case the honorarium will be 25 $ per page.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors. Only original papers are accepted for consideration. In particular, translations into English of papers already published in another language are not accepted. No other forms of scientific misconduct are allowed, such as plagiarism, falsification, incorrect interpretation of other works, incorrect citations, etc.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University,
Main Building
Room 355
Tel.: +7-7172-709500 extension 31313
2 Mirzoyan Street
010008 Astana
Kazakhstan

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 4, Number 2 (2013), 82 – 103

ON THE NATURAL q-ANALOGUES OF THE CLASSICAL ORTHOGONAL POLYNOMIALS

L. Khériji, P. Maroni

Communicated by E.S. Smailov

Key words: q-derivative operator, q-classical forms, integral representation, discrete measure.

AMS Mathematics Subject Classification: 33C45, 42C05.

Abstract. We study the natural q-analogues of the Hermite, Laguerre, Jacobi and Bessel D-classical forms. Their moments and their discrete measure and integral representations are given.

1 Introduction and preliminary results

Let $\mathcal{P}$ be the vector space of all polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of a form $u \in \mathcal{P}'$ (linear functional) on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$ the moments of u. Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg P_n = n,\ n \geq 0$ (MPS). A (MPS) $\{P_n\}_{n\geq 0}$ is called orthogonal (MOPS) if there exists a unique form u satisfying $(u)_0 = 1$ and a sequence of numbers $\{r_n\}_{n\geq 0}$ ($r_n \neq 0,\ n \geq 0$) such that

$$\langle u, P_m P_n\rangle = r_n \delta_{n,m}\ , \quad n, m \geq 0.$$

The form u is then called regular. The (MOPS) $\{P_n\}_{n\geq 0}$ satisfies the second order recurrence relation

$$\begin{cases} P_0(x) = 1\ ,\ P_1(x) = x - \beta_0\ , \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x)\ ,\ n \geq 0, \end{cases} \tag{1.1}$$

where $\quad \beta_n = \dfrac{\langle u, xP_n^2\rangle}{r_n} \quad ; \quad \gamma_{n+1} = \dfrac{r_{n+1}}{r_n} \neq 0, \quad n \geq 0.$

The regular form u is positive definite if and only if $\forall n \geq 0,\ \beta_n \in \mathbb{R},\ \gamma_{n+1} > 0$. Also, its corresponding (MOPS) $\{P_n\}_{n\geq 0}$ is symmetric if and only if $\beta_n = 0,\ n \geq 0$ or, equivalently $(u)_{2n+1} = 0,\ n \geq 0$.

Let us introduce some useful operations in $\mathcal{P}'$. For any form u, any polynomial g and any $(a, b, c) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^2$, we let $Du,\ H_q u,\ gu,\ h_a u,\ \tau_b u$ and δ_c, be the forms defined by duality in the following way

$$\langle Du, f\rangle := -\langle u, f'\rangle\ ,\ \langle H_q u, f\rangle := -\langle u, H_q f\rangle\ ,\ \langle gu, f\rangle := \langle u, gf\rangle\ ,\ f \in \mathcal{P},$$

$$\langle h_a u, f\rangle := \langle u, h_a f\rangle\ ,\ \langle \tau_b u, f\rangle := \langle u, \tau_{-b} f\rangle\ ,\ \langle \delta_c, f\rangle := f(c)\ ,\ f \in \mathcal{P},$$

where

$$(H_q f)(x) = \frac{f(qx)-f(x)}{(q-1)x},\ q \in \widetilde{\mathbb{C}} := \Big\{z \in \mathbb{C},\ z \neq 0, z^n \neq 1, n \geq 1\Big\},$$

$(h_a f)(x) = f(ax)$, $(\tau_{-b} f)(x) = f(x+b)$ [9,11]. Furthermore, the orthogonality is kept by shifting: let $\{\widetilde{P}_n := a^{-n}(h_a \circ \tau_{-b} P_n)\}_{n\geq 0}$, $a \neq 0$, $b \in \mathbb{C}$ be orthogonal with respect to $\widetilde{u} = h_{a^{-1}} \circ \tau_{-b} u$, then the recurrence elements $\widetilde{\beta}_n$, $\widetilde{\gamma}_{n+1}$, $n \geq 0$ of the sequence $\{\widetilde{P}_n\}_{n\geq 0}$ are

$$\widetilde{\beta}_n = \frac{\beta_n - b}{a} \quad , \quad \widetilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2},\ n \geq 0.$$

Let $\{P_n\}_{n\geq 0}$ be a (MPS) and O be a lowering operator on $\mathcal{P}$. The sequence $\{P_n\}_{n\geq 0}$ is said to be O-classical when $\{P_n\}_{n\geq 0}$ is orthogonal and $\{OP_{n+1}\}_{n\geq 0}$ is also orthogonal (the Hahn property). The concept of O-classical orthogonal polynomials was extensively studied by the second author and his coworkers for $O = D$ the derivative operator [11,12], $O = D_\omega$ the divided difference operator [1], and $O = H_q$ the q-derivative one [9] via the following distributional equation satisfied by the regular form u associated with such a sequence:

$$O(\Phi(x)u) + \Psi(x)u = 0 \tag{1.2}$$

where Φ is a monic polynomial, $\deg \Phi \leq 2$ and Ψ a polynomial $\deg \Psi = 1$.

Now, let us recall the D-classical forms; there are four canonical situations [11,12].

1) The Hermite form $\mathcal{H}$ and its (MOPS) having the following properties

$$\begin{cases} \beta_n = 0 \quad , \quad \gamma_{n+1} = \frac{n+1}{2}, \quad n \geq 0, \\ D(\mathcal{H}) + 2x\mathcal{H} = 0, \\ (\mathcal{H})_{2n+1} = 0 \quad , \quad (\mathcal{H})_{2n} = \frac{(2n)!}{2^{2n} n!}, \quad n \geq 0, \\ \langle \mathcal{H}, f\rangle = \frac{1}{\sqrt{\pi}} \displaystyle\int_{-\infty}^{+\infty} \exp(-x^2) f(x)dx, \quad f \in \mathcal{P}. \end{cases} \tag{1.3}$$

2) The Laguerre form $\mathcal{L}(\alpha)$ ($\alpha \neq -n-1$, $n \geq 0$) and its (MOPS) satisfying

$$\begin{cases} \beta_n = 2n + \alpha + 1 \quad , \quad \gamma_{n+1} = (n+1)(n+\alpha+1), \quad n \geq 0, \\ D(x\mathcal{L}(\alpha)) + (x - 1 - \alpha)\mathcal{L}(\alpha) = 0, \\ (\mathcal{L}(\alpha))_n = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}, \quad n \geq 0, \\ \langle \mathcal{L}(\alpha), f\rangle = \frac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^{+\infty} x^\alpha \exp(-x) f(x)dx, \quad f \in \mathcal{P}, \quad \Re\alpha > -1. \end{cases} \tag{1.4}$$

3) The shifted Jacobi form $h_{-\frac{1}{2}} \circ \tau_{-1} \mathcal{J}(\alpha, \beta) := \widetilde{\mathcal{J}}(\alpha, \beta)$ ($\alpha \neq -n-1$, $\beta \neq -n-$

$1,\ \alpha+\beta\neq -n-2,\ n\geq 0)$ and its (MOPS) satisfying

$$\begin{cases} \widetilde{\beta}_0=\frac{\beta+1}{\alpha+\beta+2} \quad , \quad \widetilde{\beta}_{n+1}=\frac{1}{2}-\frac{\alpha^2-\beta^2}{2(2n+\alpha+\beta+2)(2n+\alpha+\beta+4)},\ n\geq 0, \\ \widetilde{\gamma}_{n+1}=\frac{(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}, \quad n\geq 0, \\ D\big(x(x-1)\widetilde{\mathcal{J}}(\alpha,\beta)\big)+\big(-(\alpha+\beta+2)x+\beta+1\big)\widetilde{\mathcal{J}}(\alpha,\beta)=0, \\ (\widetilde{\mathcal{J}}(\alpha,\beta))_n=\frac{\Gamma(n+\beta+1)\Gamma(\alpha+\beta+2)}{\Gamma(n+\alpha+\beta+2)\Gamma(\beta+1)}, \quad n\geq 0, \\ \langle\widetilde{\mathcal{J}}(\alpha,\beta),f\rangle=\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_0^{+1}x^\beta(1-x)^\alpha f(x)dx,\ f\in\mathcal{P},\ \Re\alpha>-1,\ \Re\beta>-1. \end{cases} \tag{1.5}$$

4) The regular Bessel form $\mathcal{B}(\alpha)$ ($\alpha\neq -\dfrac{n}{2}$, $n\geq 0$) and its (MOPS) having the following properties

$$\begin{cases} \beta_0=-\frac{1}{\alpha} \quad ,\beta_{n+1}=\frac{1-\alpha}{(n+\alpha)(n+\alpha+1)}, \quad n\geq 0, \\ \gamma_{n+1}=-\frac{(n+1)(n+2\alpha-1)}{(2n+2\alpha-1)(n+\alpha)^2(2n+2\alpha+1)}, \quad n\geq 0, \\ D(x^2\mathcal{B}(\alpha))-2(\alpha x+1)\mathcal{B}(\alpha)=0, \\ (\mathcal{B}(\alpha))_n=(-1)^n2^n\frac{\Gamma(2\alpha)}{\Gamma(n+2\alpha)}, \quad n\geq 0, \\ \langle\mathcal{B}(\alpha),f\rangle=S_\alpha^{-1}\int_0^{+\infty}\frac{1}{x^2}\int_x^{+\infty}\Big(\frac{x}{t}\Big)^{2\alpha}\exp\big(\frac{2}{t}-\frac{2}{x}\big)s(t)dt\,f(x)\,dx, \\ \text{for } f\in\mathcal{P}, \quad \alpha\geq 6(\frac{2}{\pi})^4 \end{cases} \tag{1.6}$$

with

$$S_\alpha=\int_0^{+\infty}\frac{1}{x^2}\int_x^{+\infty}\Big(\frac{x}{t}\Big)^{2\alpha}\exp\big(\frac{2}{t}-\frac{2}{x}\big)s(t)\,dt\,dx$$

and s given by (1.13), see below.

Our objective is to highlight the natural q-analogues of the Hermite, Laguerre, Jacobi and Bessel D-classical forms defined in [5, 9], to calculate their moments, and to derive discrete measure and integral representations. In particular, when $q\to 1$ we recover formulas (1.3)-(1.6). In fact, the problem of defining q-analogue of orthogonal sequences has been of interest for several authors, see [2, 3, 10, 13, 14, 15].

A form u is called H_q-classical when it is regular and there exist two polynomials Φ and Ψ, Φ monic, $\deg\Phi\leq 2$, $\deg\Psi=1$ such that [9]

$$H_q(\Phi u)+\Psi u=0 \tag{1.7}$$

with $\Psi'(0)-\frac{1}{2}\Phi''(0)[n]_q\neq 0,\ n\geq 0$ where

$$[n]_q:=\frac{q^n-1}{q-1},\ n\geq 0. \tag{1.8}$$

By using (1.7) it easy to see that the moments of u satisfy the second order q-difference equation

$$\Psi'(0)(u)_1+\Psi(0)=0,$$

$$\Big\{\frac{1}{2}\Phi''(0)[n+1]_q - \Psi'(0)\Big\}(u)_{n+2}+$$

$$+\Big\{\Phi'(0)[n+1]_q - \Psi(0)\Big\}(u)_{n+1} + \Phi(0)[n+1]_q(u)_n = 0\,,\ n \geq 0. \tag{1.9}$$

Concerning integral representations via weight functions for a H_q-classical form u satisfying (1.7), we look for a function U such that

$$\langle u, f\rangle = \int_{-\infty}^{+\infty} U(x)f(x)dx,\ \ f \in \mathcal{P}, \tag{1.10}$$

where we suppose that U is as regular as may be required. According to (1.7), we get [9]

$$\int_{-\infty}^{+\infty}\Big\{q^{-1}\big(H_{q^{-1}}(\Phi U)\big)(x) + \Psi(x)U(x)\Big\}f(x)dx = 0,\ \ f \in \mathcal{P},$$

with the additional condition [9]

$$\lim_{\epsilon\to+0}\int_{\epsilon}^{1}\frac{U(x)-U(-x)}{x}dx \text{ exists or } \ U \text{ is continuous at the origin.} \tag{1.11}$$

Therefore

$$q^{-1}\big(H_{q^{-1}}(\Phi U)\big)(x) + \Psi(x)U(x) = \lambda g(x), \tag{1.12}$$

where $\lambda \in \mathbb{C}$ and g is a locally integrable function with rapid decay representing the null form. For instance the function

$$s(x) = \begin{cases} 0, & x \leq 0, \\ \exp(-x^{\frac{1}{4}})\sin x^{\frac{1}{4}}, & x > 0, \end{cases} \tag{1.13}$$

which was introduced by Stieltjes, represents the null form [16]. When $\lambda = 0$, equation (1.12) becomes

$$\Phi(q^{-1}x)U(q^{-1}x) = \big\{\Phi(x) + (q-1)x\Psi(x)\big\}U(x),$$

so that, if $q > 1$, we have

$$U(q^{-1}x) = \frac{\Phi(x) + (q-1)x\Psi(x)}{\Phi(q^{-1}x)}U(x),\ \ x \in \mathbb{R}, \tag{1.14}$$

and if $0 < q < 1$, replacing x by qx, we have

$$U(qx) = \frac{\Phi(x)}{\Phi(qx) + (q-1)qx\Psi(qx)}U(x),\ \ x \in \mathbb{R}. \tag{1.15}$$

2 The natural q-analogues of the Hermite, Laguerre and Jacobi D-classical forms

First, let us recall the following standard definitions and facts [4, 9], and the following technical lemma needed in the sequel and easy to establish:

$$(a;q)_0 := 1 \; ; \; (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}) \; , \; n \geq 1 \; , \tag{2.1}$$

$$(a;q)_\infty := \prod_{k=0}^{+\infty}(1-aq^{k}) \; , \; |q|<1 \; , \tag{2.2}$$

$$(a;q)_n = \begin{cases} \dfrac{(a;q)_\infty}{(aq^n;q)_\infty} & , \; 0<q<1, \\[2mm] \dfrac{(aq^{-1}q^n;q^{-1})_\infty}{(aq^{-1};q^{-1})_\infty} & , \; q>1, \end{cases} \tag{2.3}$$

$$(a;q)_n = (-1)^n a^n (a^{-1};q^{-1})_n q^{\frac{1}{2}n(n-1)} \; , \; n \geq 0, \tag{2.4}$$

$$q^{\frac{1}{2}n(n-1)} = \begin{cases} \dfrac{(-1;q)_\infty(-q;q)_\infty}{(-q^n;q)_\infty(-q^{-n+1};q)_\infty} & , \; 0<q<1, \\[2mm] \dfrac{(-q^{-n};q^{-1})_\infty(-q^{n-1};q^{-1})_\infty}{(-1;q^{-1})_\infty(-q^{-1};q^{-1})_\infty} & , \; q>1, \end{cases} \tag{2.5}$$

the q-binomial theorem

$$\sum_{k=0}^{+\infty}\frac{(a;q)_k}{(q;q)_k}z^k = \frac{(az;q)_\infty}{(z;q)_\infty} \; , \; |z|<1, \; |q|<1, \tag{2.6}$$

the q-analogue of the exponential function

$$\sum_{k=0}^{+\infty}\frac{q^{\frac{1}{2}k(k-1)}}{(q;q)_k}z^k = (-z;q)_\infty \; , \; |q|<1, \tag{2.7}$$

$$\int_0^{+\infty} t^{x-1}\frac{(-at;q)_\infty}{(-t;q)_\infty}dt = \tag{2.8}$$

$$= \begin{cases} \dfrac{\pi}{\sin(\pi x)}\dfrac{(a;q)_\infty}{(aq^{-x};q)_\infty}\dfrac{(q^{1-x};q)_\infty}{(q;q)_\infty}, & x \in \mathbb{R}_+ \setminus \mathbb{N}, |a|<q^x, 0<q<1, \\[2mm] \dfrac{(-q)^m}{1-q^m}\dfrac{(q^{-1};q^{-1})_m}{(aq^{-1};q^{-1})_m}\ln(q^{-1}), & x=m \in \mathbb{N}^\star, |a|<q^m, 0<q<1. \end{cases}$$

Lemma 2.1. *Let*

$$\xi_\gamma(q) = 1+(1+\gamma)(1-q), \quad q>0, \quad \gamma>-1, \tag{2.9}$$

and

$$q_{(\gamma,\omega)} = 1+\frac{\omega}{1+\gamma} \; , \; \omega \in \mathbb{R}. \tag{2.10}$$

We have

$$\begin{aligned}\xi_\gamma(q)=1 &\Longleftrightarrow q=1,\\ \xi_\gamma(q)=0 &\Longleftrightarrow q=q_{(\gamma,1)},\\ \xi_\gamma(q)=-1 &\Longleftrightarrow q=q_{(\gamma,2)},\\ \xi_\gamma(q)<-1 &\Longleftrightarrow q\in]q_{(\gamma,2)},+\infty[,\\ -1<\xi_\gamma(q)<0 &\Longleftrightarrow q\in]q_{(\gamma,1)},q_{(\gamma,2)}[,\\ 0<\xi_\gamma(q)<1 &\Longleftrightarrow q\in]1,q_{(\gamma,1)}[,\\ \xi_\gamma(q)>1 &\Longleftrightarrow q\in]0,1[.\end{aligned} \tag{2.11}$$

2.1 The natural q-analogue of the Hermite form

The natural q-analogue of the Hermite form is the H_q-classical form $\mathbf{H}(q)$ defined by

$$\begin{cases}\beta_n=0 \quad , \quad \gamma_{n+1}=\frac{1}{2}q^n[n+1]_q, \quad n\geq 0,\\ H_q(\mathbf{H}(q))+2x\mathbf{H}(q)=0.\end{cases}$$

(See [9].) When $q\to 1$, we recover the Hermite D-classical form $\mathcal{H}$ see (1.3).

In [9], we calculated the moments and derived integral representations for $\mathbf{H}(q)$:

$$(\mathbf{H}(q))_{2n}=\frac{1}{2^n}\frac{(q;q^2)_n}{(1-q)^n} \quad , \quad (\mathbf{H}(q))_{2n+1}=0, \quad n\geq 0,$$

for $f\in\mathcal{P}$, $q>1$

$$\langle\mathbf{H}(q),f\rangle=\frac{\sqrt{2}}{\pi}(q-1)^{1/2}\frac{(q^{-2};q^{-2})_\infty}{(q^{-1};q^{-2})_\infty}\int_{-\infty}^{+\infty}\frac{f(x)}{\left(-2(q-1)x^2;q^{-2}\right)_\infty}dx, \tag{2.12}$$

$$\langle\mathbf{H}(q),f\rangle=K\int_{-\frac{1}{q\sqrt{2(1-q)}}}^{+\frac{1}{q\sqrt{2(1-q)}}}\left(2q^2(1-q)x^2;q^2\right)_\infty f(x)dx,\ f\in\mathcal{P},\ 0<q<1, \tag{2.13}$$

where $K=\frac{1}{2}\Big(\int_0^{+\frac{1}{q\sqrt{2(1-q)}}}\left(2q^2(1-q)x^2;q^2\right)_\infty dx\Big)^{-1}$.

In [8], the authors give the following discrete measure representations of $\mathbf{H}(q)$:

$$\mathbf{H}(q)=\frac{1}{2(q^{-1};q^{-2})_\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k q^{-k^2}}{(q^{-2};q^{-2})_k}\Big\{\delta_{\frac{-iq^k}{\sqrt{2(q-1)}}}+\delta_{\frac{iq^k}{\sqrt{2(q-1)}}}\Big\},\ q>1,$$

and

$$\mathbf{H}(q)=\frac{(q;q^2)_\infty}{2}\sum_{k=0}^{+\infty}\frac{q^k}{(q^2;q^2)_k}\Big\{\delta_{\frac{-q^k}{\sqrt{2(1-q)}}}+\delta_{\frac{q^k}{\sqrt{2(1-q)}}}\Big\},\ 0<q<1.$$

Remarks. 1. By using the above representations of $\mathbf{H}(q)$ and (2.12)-(2.13), one can write

$$2\langle\mathbf{H}(q),f\rangle=\frac{\sqrt{2}}{\pi}(q-1)^{1/2}\frac{(q^{-2};q^{-2})_\infty}{(q^{-1};q^{-2})_\infty}\int_{-\infty}^{+\infty}\frac{f(x)}{\left(-2(q-1)x^2;q^{-2}\right)_\infty}dx+$$

$$+\frac{1}{2(q^{-1};q^{-2})_\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k q^{-k^2}}{(q^{-2};q^{-2})_k}\Big\langle \delta_{\frac{-iq^k}{\sqrt{2(q-1)}}}+\delta_{\frac{iq^k}{\sqrt{2(q-1)}}},f\Big\rangle,\ f\in\mathcal{P},\ q>1$$

and

$$2\langle \mathbf{H}(q),f\rangle = K\int_{-\frac{1}{q\sqrt{2(1-q)}}}^{+\frac{1}{q\sqrt{2(1-q)}}}\big(2q^2(1-q)x^2;q^2\big)_\infty f(x)dx+$$

$$+\frac{(q;q^2)_\infty}{2}\sum_{k=0}^{+\infty}\frac{q^k}{(q^2;q^2)_k}\Big\langle \delta_{\frac{-q^k}{\sqrt{2(1-q)}}}+\delta_{\frac{q^k}{\sqrt{2(1-q)}}},f\Big\rangle,\ f\in\mathcal{P},\ 0<q<1.$$

2. Denoting by U_q the weight function given either by (2.12) or by (2.13), we have (see (1.3))

$$\lim_{q\to1}\int_{-\infty}^{+\infty}U_q(x)f(x)dx=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\exp(-x^2)f(x)dx=\langle\mathcal{H},f\rangle,\ \ f\in\mathcal{P}.$$

2.2 The natural q-analogue of the Laguerre form

The natural q-analogue of the Laguerre form is the H_q-classical form $\mathbf{L}(\alpha,q)$ ($\alpha\neq -[n]_q-1,\ n\geq 0$) defined by

$$\begin{cases}\beta_n=q^n\big\{(1+q^{-1})[n]_q+1+\alpha\big\},\ \ n\geq0,\\ \gamma_{n+1}=q^{2n}[n+1]_q\big\{[n]_q+1+\alpha\big\},\ \ n\geq 0,\\ H_q(x\mathbf{L}(\alpha,q))+(x-1-\alpha)\mathbf{L}(\alpha,q)=0.\end{cases}\tag{2.14}$$

(See [5,9].) Formula (2.14) implies that the form $\mathbf{L}(\alpha,q)$ is positive definite if and only if $q>0,\ \alpha>-1$.

When $q\to1$, we recover the Laguerre D-classical form $\mathcal{L}(\alpha)$ ($\alpha\neq-n-1,\ n\geq0$), see (1.4).

By the regularity condition $\alpha\neq-[n]_q-1,\ n\geq0$ of the form $\mathbf{L}(\alpha,q)$ we get

$$\xi_\alpha(q)\neq q^n,\ \ n\geq0.\tag{2.15}$$

Proposition 2.1. *The form $\mathbf{L}(\alpha,q)$ has the following properties.*

(1) The moments of $\mathbf{L}(\alpha,q)$ are: for $n\geq0$

$$(\mathbf{L}(\alpha,q))_n=\begin{cases}\dfrac{q_{(\alpha,1)}^{\frac{(n-1)n}{2}}}{(q_{(\alpha,1)}-1)^n}, & q=q_{(\alpha,1)}\\ \left(\dfrac{\xi_\alpha(q)}{1-q}\right)^n\big((\xi_\alpha(q))^{-1};q\big)_n, & q\in]0,+\infty[\setminus\{1,q_{(\alpha,1)}\}.\end{cases}$$

(2) For all $0<q<1$ and $\alpha>-1$, the form $\mathbf{L}(\alpha,q)$ has the following discrete measure representation

$$\mathbf{L}(\alpha,q)=\big((\xi_\alpha(q))^{-1};q\big)_\infty\sum_{k=0}^{+\infty}\frac{\big(\xi_\alpha(q)\big)^{-k}}{(q;q)_k}\ \delta_{\frac{q^k\xi_\alpha(q)}{1-q}}.$$

(3) For all $q \in]1,+\infty[\setminus\{q_{(\alpha,1)}\}$ and $\alpha > -1$, the form $\mathbf{L}(\alpha,q)$ has the following discrete measure representation

$$\mathbf{L}(\alpha,q) = \frac{1}{\big(q^{-1}(\xi_\alpha(q))^{-1};q^{-1}\big)_\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k q^{-\frac{1}{2}k(k+1)}\big(\xi_\alpha(q)\big)^{-k}}{(q^{-1};q^{-1})_k}\, \delta_{\frac{q^k \xi_\alpha(q)}{1-q}}.$$

(4) For $q = q_{(\alpha,1)}$ and $\alpha > -1$, the form $\mathbf{L}(\alpha, q_{(\alpha,1)})$ has the following discrete measure representation

$$(-1;q^{-1}_{(\alpha,1)})_\infty(-q^{-1}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_\infty \mathbf{L}(\alpha,q_{(\alpha,1)}) =$$
$$= \sum_{k=0}^{+\infty} q_{(\alpha,1)}^{-\frac{1}{2}k(k+1)} \sum_{l=0}^{k} \frac{q_{(\alpha,1)}^{-l^2+l(k-1)}}{(q^{-1}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_l(q^{-1}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_{k-l}} \delta_{\frac{q^{k-2l}_{(\alpha,1)}}{q_{(\alpha,1)}-1}}.$$

(5) For all $f \in \mathcal{P}$, $0 < q < 1$ and $\alpha > -1$, the form $\mathbf{L}(\alpha,q)$ has the following integral representation

$$\langle \mathbf{L}(\alpha,q), f\rangle = K\int_0^{\frac{q^{-1}\xi_\alpha(q)}{1-q}} x^{-\frac{\ln \xi_\alpha(q)}{\ln q}-1} \big((1-q)(\xi_\alpha(q))^{-1}qx;q\big)_\infty f(x)\, dx, \tag{2.16}$$

where $K^{-1} = \displaystyle\int_0^{\frac{q^{-1}\xi_\alpha(q)}{1-q}} x^{-\frac{\ln \xi_\alpha(q)}{\ln q}-1} \big((1-q)(\xi_\alpha(q))^{-1}qx;q\big)_\infty\, dx.$

(6) For all $1 < q < q_{(\alpha,1)}$ and $\alpha > -1$, the form $\mathbf{L}(\alpha,q)$ has the following integral representation

$$\langle \mathbf{L}(\alpha,q), f\rangle = K\int_0^{+\infty} \frac{x^{-\frac{\ln \xi_\alpha(q)}{\ln q}-1}}{\big(-(q-1)(\xi_\alpha(q))^{-1}x;q^{-1}\big)_\infty} f(x)\, dx,\ \ f\in\mathcal{P}, \tag{2.17}$$

where $K^{-1} = \displaystyle\int_0^{+\infty} \frac{t^{-\frac{\ln \xi_\alpha(q)}{\ln q}-1}}{\big(-(q-1)(\xi_\alpha(q))^{-1}t;q^{-1}\big)_\infty}\, dt$ *is given by (2.8).*

(7) For $q = q_{(\alpha,1)}$, $\alpha > -1$ and for all $f \in \mathcal{P}$, the form $\mathbf{L}(\alpha,q_{(\alpha,1)})$ has the following integral representation

$$\langle \mathbf{L}(\alpha,q_{(\alpha,1)}), f\rangle = \frac{q_{(\alpha,1)}^{-\frac{1}{8}} \exp\Big(-\frac{\ln^2(q_{(\alpha,1)}-1)}{2\ln q_{(\alpha,1)}}\Big)}{\sqrt{2\pi(q_{(\alpha,1)}-1)\ln q_{(\alpha,1)}}}$$
$$\times \int_0^{+\infty} x^{-\frac{3}{2}-\frac{\ln(q_{(\alpha,1)}-1)}{\ln q_{(\alpha,1)}}} \exp\Big(-\frac{\ln^2 x}{2\ln q_{(\alpha,1)}}\Big) f(x)\, dx.$$

Proof. By (2.14) we get

$$\Phi(x) = x \quad ; \quad \Psi(x) = x-1-\alpha. \tag{2.18}$$

(1.9) becomes

$$(\mathbf{L}(\alpha,q))_{n+1} = \big([n]_q+1+\alpha\big)(\mathbf{L}(\alpha,q))_n\ ,\ \ n\geq 0 \quad ; \quad (\mathbf{L}(\alpha,q))_0 = 1$$

and, as a consequence,

$$(\mathbf{L}(\alpha,q))_n=\prod_{k=1}^{n}\big([k-1]_q+1+\alpha\big)\;,\;\; n\geq 1 \quad ; \quad (\mathbf{L}(\alpha,q))_0=1.$$

Replacing $[k-1]_q$ by $\dfrac{q^{k-1}-1}{q-1}$, using the definitions in (2.9) and (2.1), the above relation yields the desired result taking into account the properties in (2.11) and (2.15).

To establish (2)-(4), by virtue of (2.3) and (2.5) we may write the moments of $\mathbf{L}(\alpha,q)$ from (1) in the following way

$$(\mathbf{L}(\alpha,q))_n=\begin{cases}\big((\xi_\alpha(q))^{-1};q\big)_\infty\left(\frac{\xi_\alpha(q)}{1-q}\right)^n\frac{1}{\big((\xi_\alpha(q))^{-1}q^n;q\big)_\infty} & ,\; 0<q<1,\\ \frac{1}{\big((\xi_\alpha(q))^{-1}q^{-1};q^{-1}\big)_\infty}\left(\frac{\xi_\alpha(q)}{1-q}\right)^n\big((\xi_\alpha(q))^{-1}q^{n-1};q^{-1}\big)_\infty, & q>1,\; q\neq q_{(\alpha,1)},\\ \frac{1}{(-1;q^{-1}_{(\alpha,1)})_\infty(-q^{-1}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_\infty}\frac{(-q^{-n}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_\infty(-q^{n-1}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_\infty}{(q_{(\alpha,1)}-1)^n}\;, & q=q_{(\alpha,1)}.\end{cases} \tag{2.19}$$

Moreover, by the q-binomial theorem (2.6), we get

$$\frac{1}{\big((\xi_\alpha(q))^{-1}q^n;q\big)_\infty}=\sum_{k=0}^{+\infty}\frac{(\xi_\alpha(q))^{-k}}{(q;q)_k}\,q^{nk},\;\; 0<q<1,\;\; \alpha>-1 \tag{2.20}$$

since $\forall n\geq 0,\; \big|(\xi_\alpha(q))^{-1}q^n\big|<1$ (see the last inequality in (2.11)).

Also, by the q-analogue of the exponential function (2.7), we may write successively for $q>1,\; q\neq q_{(\alpha,1)},\; \alpha>-1$

$$\big((\xi_\alpha(q))^{-1}q^{n-1};q^{-1}\big)_\infty=\sum_{k=0}^{+\infty}\frac{q^{-\frac{1}{2}k(k-1)}}{(q^{-1};q^{-1})_k}\big(-q^{-1}(\xi_\alpha(q))^{-1}q^n\big)^k, \tag{2.21}$$

for $q=q_{(\alpha,1)},\; \alpha>-1$

$$\begin{cases}(-q^{-n}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_\infty=\displaystyle\sum_{k=0}^{+\infty}\frac{q_{(\alpha,1)}^{-\frac{1}{2}k(k-1)}}{(q^{-1}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_k}\,q_{(\alpha,1)}^{-nk},\\ (-q^{n-1}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_\infty=\displaystyle\sum_{k=0}^{+\infty}\frac{q_{(\alpha,1)}^{-\frac{1}{2}k(k-1)}}{(q^{-1}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_k}\,q_{(\alpha,1)}^{(n-1)k}.\end{cases} \tag{2.22}$$

Using the Cauchy product of the two power series in (2.22) we get for $q=q_{(\alpha,1)},\; \alpha>-1$

$$\begin{aligned}&(-q^{-n}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_\infty\,(-q^{n-1}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_\infty\\ &=\sum_{k=0}^{+\infty}q_{(\alpha,1)}^{-\frac{1}{2}k(k+1)}\sum_{l=0}^{k}\frac{q_{(\alpha,1)}^{-l^2+l(k-1)}}{(q^{-1}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_l(q^{-1}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_{k-l}}\,q_{(\alpha,1)}^{n(k-2l)}.\end{aligned} \tag{2.23}$$

Now, replacing (2.20)-(2.21) and (2.23) in (2.19) we obtain the desired results. Thus, parts (2)-(4) are proved.

To establish the integral representations in (5)-(7), according to (1.10), we look for a function U representing $\mathbf{L}(\alpha, q)$. From the hypothesis of (5), we have $0 < q < 1,\ \alpha > -1$. By virtue of (1.10), (2.18), (2.9) and the q-distributional equation in (2.14), the q-difference equation (1.15) becomes

$$U(qx) = \frac{(q\xi_\alpha(q))^{-1}}{1-(1-q)q(\xi_\alpha(q))^{-1}x}U(x), \tag{2.24}$$

with $\xi_\alpha(q) > 1$ according to (2.11). Consequently, we look for U of the form

$$U(x) = \begin{cases} \big((1-q)q(\xi_\alpha(q))^{-1}x; q\big)_\infty V(x)\,, & 0 < x < \dfrac{q^{-1}\xi_\alpha(q)}{1-q}, \\ 0 & , \quad x \leq 0 \text{ or } x \geq \dfrac{q^{-1}\xi_\alpha(q)}{1-q}. \end{cases}$$

Substituting this U in (2.24) leads to the equality $V(qx) = (q\xi_\alpha(q))^{-1}V(x)$, therefore

$$V(x) = Kx^{-\frac{\ln \xi_\alpha(q)}{\ln q}-1}.$$

Thus, (2.16) follows.

In part (6), we have $1 < q < q_{(\alpha,1)},\ \alpha > -1$. By virtue of (1.10), (2.18), (2.9) and the q-distributional equation in (2.14), the q-difference equation (1.14) becomes

$$U(q^{-1}x) = q\xi_\alpha(q)\big(1+(q-1)(\xi_\alpha(q))^{-1}x\big)U(x) \tag{2.25}$$

with $0 < \xi_\alpha(q) < 1$ according to (2.11). Consequently, we look for U of the form

$$U(x) = \begin{cases} \dfrac{V(x)}{\big(-(q-1)(\xi_\alpha(q))^{-1}x; q^{-1}\big)_\infty} & , \quad x > 0, \\ 0 & , \quad x \leq 0. \end{cases}$$

Substituting this U in (2.25) leads to the equality $V(q^{-1}x) = q\xi_\alpha(q)V(x)$, therefore

$$V(x) = Kx^{-\frac{\ln \xi_\alpha(q)}{\ln q}-1}.$$

This implies (2.17).

In part (7), we have $1 < q = q_{(\alpha,1)},\ \alpha > -1$. By virtue of (1.10), (2.18), (2.9) and the q-distributional equation in (2.14), the q-difference equation (1.14) becomes

$$U(q^{-1}_{(\alpha,1)}x) = q_{(\alpha,1)}(q_{(\alpha,1)}-1)xU(x) \quad , \quad x > 0 \tag{2.26}$$

since $\xi_\alpha(q_{(\alpha,1)}) = 0$ according to (2.11). Consequently, we look for U of the form

$$U(x) = \exp\Big(-\frac{\ln^2 x}{2\ln q_{(\alpha,1)}}\Big)V(x) \quad , \quad x > 0.$$

Substituting this U in (2.26) this leads to the equality $V(q^{-1}_{(\alpha,1)}x) = q^{\frac{3}{2}}_{(\alpha,1)}(q_{(\alpha,1)}-1)V(x)$, therefore

$$V(x) = Kx^{-\frac{3}{2}-\frac{\ln(q_{(\alpha,1)}-1)}{ln q_{(\alpha,1)}}}.$$

Let us now compute the constant K^{-1}:

$$K^{-1}=\int_0^{+\infty} x^{-\frac{3}{2}-\frac{\ln(q_{(\alpha,1)}-1)}{\ln q_{(\alpha,1)}}}\exp\Big(-\frac{\ln^2 x}{2\ln q_{(\alpha,1)}}\Big)\,dx.$$

By the change of variable: $t=\frac{\ln x}{\sqrt{2\ln q_{(\alpha,1)}}}$ we get

$$K^{-1}=\sqrt{2\ln q_{(\alpha,1)}}\int_{-\infty}^{+\infty}\exp\Big(-\frac{\ln\big((q_{(\alpha,1)}-1)^2 q_{(\alpha,1)}\big)}{\sqrt{2\ln q_{(\alpha,1)}}}\;t-t^2\Big)\,dt.$$

Consequently, using the change of variable $u=t+\frac{1}{2}\;\frac{\ln\big((q_{(\alpha,1)}-1)^2 q_{(\alpha,1)}\big)}{\sqrt{2\ln q_{(\alpha,1)}}}$ and taking into account that $\int_{-\infty}^{+\infty}\exp(-u^2)\,du=\sqrt{\pi}$, we get

$$K^{-1}=\sqrt{2\pi(q_{(\alpha,1)}-1)\ln q_{(\alpha,1)}}\;\;q_{(\alpha,1)}^{\frac{1}{8}}\;\exp\Big(\frac{\ln^2(q_{(\alpha,1)}-1)}{2\ln q_{(\alpha,1)}}\Big),$$

whence the desired integral representation in (7). □

Remarks. 1. Taking into account the representations in Proposition 2.2., we may write successively

$$2\langle\mathbf{L}(\alpha,q),f\rangle=K\int_0^{\frac{q^{-1}\xi_\alpha(q)}{1-q}} x^{-\frac{\ln\xi_\alpha(q)}{\ln q}-1}\,\big((1-q)(\xi_\alpha(q))^{-1}qx;q\big)_\infty f(x)\,dx+$$
$$+\big((\xi_\alpha(q))^{-1};q\big)_\infty\sum_{k=0}^{+\infty}\frac{\big(\xi_\alpha(q)\big)^{-k}}{(q;q)_k}\;\langle\delta_{\frac{q^k\xi_\alpha(q)}{1-q}},f\rangle,\;\;f\in\mathcal{P},\;\;0<q<1,\;\;\alpha>-1,$$

$$2\langle\mathbf{L}(\alpha,q),f\rangle=K\int_0^{+\infty}\frac{x^{-\frac{\ln\xi_\alpha(q)}{\ln q}-1}}{\big(-(q-1)(\xi_\alpha(q))^{-1}x;q^{-1}\big)_\infty}f(x)\,dx+\frac{1}{\big(q^{-1}(\xi_\alpha(q))^{-1};q^{-1}\big)_\infty}\times$$
$$\sum_{k=0}^{+\infty}\frac{(-1)^k q^{-\frac{1}{2}k(k+1)}\big(\xi_\alpha(q)\big)^{-k}}{(q^{-1};q^{-1})_k}\;\langle\delta_{\frac{q^k\xi_\alpha(q)}{1-q}},f\rangle,\;f\in\mathcal{P},\;\;1<q<q_{(\alpha,1)},\;\;\alpha>-1,$$

and

$$2\langle\mathbf{L}(\alpha,q_{(\alpha,1)}),f\rangle=\frac{q_{(\alpha,1)}^{-\frac{1}{8}}\exp\Big(-\frac{\ln^2(q_{(\alpha,1)}-1)}{2\ln q_{(\alpha,1)}}\Big)}{\sqrt{2\pi(q_{(\alpha,1)}-1)\ln q_{(\alpha,1)}}}\times$$
$$\int_0^{+\infty}x^{-\frac{3}{2}-\frac{\ln(q_{(\alpha,1)}-1)}{\ln q_{(\alpha,1)}}}\exp\Big(-\frac{\ln^2 x}{2\ln q_{(\alpha,1)}}\Big)f(x)\,dx+\frac{1}{(-1;q^{-1}_{(\alpha,1)})_\infty(-q^{-1}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_\infty}\times$$
$$\sum_{k=0}^{+\infty}q_{(\alpha,1)}^{-\frac{1}{2}k(k+1)}\sum_{l=0}^{k}\frac{q_{(\alpha,1)}^{-l^2+l(k-1)}}{(q^{-1}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_l(q^{-1}_{(\alpha,1)};q^{-1}_{(\alpha,1)})_{k-l}}\langle\delta_{\frac{q_{(\alpha,1)}^{k-2l}}{q_{(\alpha,1)}-1}},f\rangle,\;\;f\in\mathcal{P},\;\;\alpha>-1.$$

2. Denoting by U_q the weight function given either by (2.16) or by (2.17), we have (see (1.4))

$$\lim_{q\to 1}\int_{-\infty}^{+\infty}U_q(x)f(x)dx=\frac{1}{\Gamma(\alpha+1)}\int_0^{+\infty}x^\alpha\exp(-x)f(x)dx=\langle\mathcal{L}(\alpha),f\rangle,\;\;f\in\mathcal{P}.$$

2.3 The natural q-analogue of the shifted Jacobi form $\widetilde{\mathcal{J}}(\alpha,\beta)$

The natural q-analogue of this shifted form is the H_q-classical one $\mathbf{J}(\alpha,\beta,q)$ ($\alpha+\beta\neq\frac{3-2q}{q-1}$, $\alpha+\beta\neq-[n]_q-2$, $\beta\neq-[n]_q-1$, $\alpha+\beta+2-(\beta+1)q^n+[n]_q\neq 0$, $n\geq 0$) satisfying [5,9]

$$\begin{cases} \beta_n = q^{n-1}\frac{(1+q)(\alpha+\beta+2+[n-1]_q)(\beta+1+[n]_q)-(\beta+1)(\alpha+\beta+2+[2n]_q)}{(\alpha+\beta+2+[2n-2]_q)(\alpha+\beta+2+[2n]_q)}, \quad n\geq 0, \\ \gamma_{n+1} = q^{2n}\frac{[n+1]_q(\alpha+\beta+2+[n-1]_q)([n]_q+\beta+1)([n]_q+(\beta+1)(1-q^n)+\alpha+1)}{(\alpha+\beta+2+[2n-1]_q)(\alpha+\beta+2+[2n]_q)^2(\alpha+\beta+2+[2n+1]_q)}, \quad n\geq 0, \\ H_q(x(x-1)\mathbf{J}(\alpha,\beta,q)) + (-(\alpha+\beta+2)x+\beta+1)\mathbf{J}(\alpha,\beta,q) = 0. \end{cases} \tag{2.27}$$

Formula (2.27) implies that the form $\mathbf{J}(\alpha,\beta,q)$ is positive definite if and only if

$$0<q<1,\ \alpha>-1,\ \beta>-1 \quad ; \quad 1<q<q_{(\alpha+\beta+1,1)},\ \alpha>-1,\ \beta>-1.$$

When $q\to 1$, we recover the D-classical form $\widetilde{\mathcal{J}}(\alpha,\beta)$, see (1.5). In fact, for the cases of positive definiteness of $\mathbf{J}(\alpha,\beta,q)$, according to Lemma 2.1, one may deduce the following

$$\begin{cases} 1<q_{(\alpha+\beta+1,1)}<q_{(\alpha+\beta+1,2)}<q_{(\beta,2)}, \\ 1<q_{(\alpha+\beta+1,1)}<q_{(\beta,1)}<q_{(\beta,2)}, \\ \xi_\beta(q)\neq q^n \quad , \quad \xi_{\alpha+\beta+1}(q)\neq q^n \quad , \quad n\geq 0, \\ q\neq q_{(\alpha+\beta+1,2)}, \\ \xi_\beta(q)=0 \Longleftrightarrow \xi_{\alpha+\beta+1}(q)\neq 0. \end{cases} \tag{2.28}$$

Next, we will calculate the moments and derive integral representations and discrete measure representations of the H_q-classical form $\mathbf{J}(\alpha,\beta,q)$ in the cases of positive definiteness.

Proposition 2.2. *The form $\mathbf{J}(\alpha,\beta,q)$ has the following properties.*

(1) The moments of $\mathbf{J}(\alpha,\beta,q)$ are: for $n\geq 0$

$$(\mathbf{J}(\alpha,\beta,q))_n = \begin{cases} \dfrac{q_{(\beta,1)}^{\frac{1}{2}(n-1)n}}{\left(-\xi_{\alpha+\beta+1}(q_{(\beta,1)})\right)^n\left(\left(\xi_{\alpha+\beta+1}(q_{(\beta,1)})\right)^{-1};q_{(\beta,1)}\right)_n}, & q=q_{(\beta,1)}, \\ \dfrac{\left(-\xi_\beta(q_{(\alpha+\beta+1,1)})\right)^n\left(\left(\xi_\beta(q_{(\alpha+\beta+1,1)})\right)^{-1};q_{(\alpha+\beta+1,1)}\right)_n}{q_{(\alpha+\beta+1,1)}^{\frac{1}{2}(n-1)n}}, & q=q_{(\alpha+\beta+1,1)}, \\ \left(\dfrac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}\right)^n\dfrac{\left((\xi_\beta(q))^{-1};q\right)_n}{\left((\xi_{\alpha+\beta+1}(q))^{-1};q\right)_n}, & q\in]0,+\infty[\backslash\{1,q_{(\alpha+\beta+1,1)},q_{(\beta,1)}\}. \end{cases}$$

(2) For all $0<q<1$, $\alpha>-1$, $\beta>-1$, the form $\mathbf{J}(\alpha,\beta,q)$ has the following discrete measure representation

$$\mathbf{J}(\alpha,\beta,q) = \frac{((\xi_\beta(q))^{-1};q)_\infty}{((\xi_{\alpha+\beta+1}(q))^{-1};q)_\infty}\times$$

$$\sum_{k=0}^{+\infty}\frac{(-1)^k q^{\frac{1}{2}k(k-1)}}{(\xi_{\alpha+\beta+1}(q))^k}\sum_{\nu=0}^{k}\Big(-\frac{\xi_{\alpha+\beta+1}(q)}{q^k\xi_\beta(q)}\Big)^\nu\frac{q^{\frac{1}{2}\nu(\nu+1)}}{(q;q)_\nu(q;q)_{k-\nu}}\ \delta_{\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}q^k}.$$

(3) For all $q\in]1,q_{(\alpha+\beta+1,1)}[\cup]q_{(\alpha+\beta+1,1)},q_{(\beta,2)}[,\ \alpha>-1,\ \beta>-1,$ *the form* $\mathbf{J}(\alpha,\beta,q)$ *has the following discrete measure representation*

$$\mathbf{J}(\alpha,\beta,q)=\frac{(\xi_\beta(q);q^{-1})_\infty}{(\xi_{\alpha+\beta+1}(q);q^{-1})_\infty}\times$$

$$\sum_{k=0}^{+\infty}(-\xi_{\alpha+\beta+1}(q))^k q^{-\frac{1}{2}k(k-1)}\sum_{\nu=0}^{k}\Big(\frac{q^k\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}\Big)^\nu\frac{(-1)^\nu q^{-\frac{1}{2}\nu(\nu+1)}}{(q;q)_\nu(q;q)_{k-\nu}}\ \delta_{q^{-k}}.$$

In particular, for $q=q_{(\beta,1)}$ *and for all* $\alpha>-1,\ \beta>-1,$ *the form* $\mathbf{J}(\alpha,\beta,q_{(\beta,1)})$ *has the following discrete measure representation*

$$\mathbf{J}(\alpha,\beta,q_{(\beta,1)})=\frac{1}{\big(\xi_{\alpha+\beta+1}(q_{(\beta,1)});q^{-1}_{(\beta,1)}\big)_\infty}\sum_{k=0}^{+\infty}\frac{(-\xi_{\alpha+\beta+1}(q_{(\beta,1)}))^k q_{(\beta,1)}^{-\frac{1}{2}k(k-1)}}{\big(q^{-1}_{(\beta,1)};q^{-1}_{(\beta,1)}\big)_k}\delta_{q^{-k}_{(\beta,1)}}.$$

(4) For $q=q_{(\alpha+\beta+1,1)}$ *and for all* $\alpha>-1,\ \beta>-1,$ *the form* $\mathbf{J}(\alpha,\beta,q_{(\alpha+\beta+1,1)})$ *has the following discrete measure representation*

$$\mathbf{J}(\alpha,\beta,q_{(\alpha+\beta+1,1)})=\big(\xi_\beta(q_{(\alpha+\beta+1,1)});q^{-1}_{(\alpha+\beta+1,1)}\big)_\infty\times$$

$$\sum_{k=0}^{+\infty}\frac{\big(\xi_\beta(q_{(\alpha+\beta+1,1)})\big)^k}{\big(q^{-1}_{(\alpha+\beta+1,1)};q^{-1}_{(\alpha+\beta+1,1)}\big)_k}\delta_{q^{-k}_{(\alpha+\beta+1,1)}}.$$

(5) For $\alpha\geq 0,\ \beta>-1,\ q_{(\alpha+\beta+1,-\alpha)}<q<1$ *and for all* $f\in\mathcal{P}$*, the form* $\mathbf{J}(\alpha,\beta,q)$ *has the following integral representation*

$$\langle\mathbf{J}(\alpha,\beta,q),f\rangle=K\int_0^{\frac{\xi_\beta(q)}{q\xi_{\alpha+\beta+1}(q)}}x^{-\frac{\ln\xi_\beta(q)}{\ln q}-1}\frac{\big(\frac{q\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x;q\big)_\infty}{(x;q)_\infty}f(x)\ dx,\tag{2.29}$$

where $K^{-1}=\displaystyle\int_0^{\frac{\xi_\beta(q)}{q\xi_{\alpha+\beta+1}(q)}}x^{-\frac{\ln\xi_\beta(q)}{\ln q}-1}\frac{\big(\frac{q\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x;q\big)_\infty}{(x;q)_\infty}\ dx.$

(6) For $-1<\alpha<0,\ \beta>-1,\ 0<q<1$ *and for all* $f\in\mathcal{P}$*, the form* $\mathbf{J}(\alpha,\beta,q)$ *has the following integral representation*

$$\langle\mathbf{J}(\alpha,\beta,q),f\rangle=K\int_0^1 x^{-\frac{\ln\xi_\beta(q)}{\ln q}-1}\frac{\big(\frac{q\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x;q\big)_\infty}{(x;q)_\infty}\left|\sin\Big(2\pi\frac{\ln x}{\ln q}\Big)\right|f(x)\ dx,\tag{2.30}$$

where $K^{-1}=\displaystyle\int_0^1 x^{-\frac{\ln\xi_\beta(q)}{\ln q}-1}\frac{\big(\frac{q\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x;q\big)_\infty}{(x;q)_\infty}\left|\sin\Big(2\pi\frac{\ln x}{\ln q}\Big)\right|dx.$

(7) For $\alpha \geq 0,\ \beta > -1,\ 1 < q < q_{(\alpha+\beta+1,1)}$ and for all $f \in \mathcal{P}$, the form $\mathbf{J}(\alpha,\beta,q)$ has the following integral representation

$$\langle \mathbf{J}(\alpha,\beta,q), f\rangle = K \int_0^q x^{-\frac{\ln \xi_\beta(q)}{\ln q}-1} \frac{(q^{-1}x;q^{-1})_\infty}{\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x;q^{-1}\right)_\infty}\ f(x)\ dx, \tag{2.31}$$

where $K^{-1} = \displaystyle\int_0^q x^{-\frac{\ln \xi_\beta(q)}{\ln q}-1} \frac{(q^{-1}x;q^{-1})_\infty}{\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x;q^{-1}\right)_\infty}\ dx.$

(8) For $-1 < \alpha < 0,\ \beta > -1,\ 1 < q < q_{(\alpha+\beta+1,1)}$ and for all $f \in \mathcal{P}$, the form $\mathbf{J}(\alpha,\beta,q)$ has the following integral representation

$$\langle \mathbf{J}(\alpha,\beta,q), f\rangle = K$$

$$\times \int_0^{\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}} x^{-\frac{\ln \xi_\beta(q)}{\ln q}-1} \frac{(q^{-1}x;q^{-1})_\infty}{\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x;q^{-1}\right)_\infty} \left|\sin\left(2\pi\frac{\ln\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x\right)}{\ln q^{-1}}\right)\right| f(x)\ dx, \tag{2.32}$$

where $K^{-1} = \displaystyle\int_0^1 x^{-\frac{\ln \xi_\beta(q)}{\ln q}-1} \frac{(q^{-1}x;q^{-1})_\infty}{\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x;q^{-1}\right)_\infty} \left|\sin\left(2\pi\frac{\ln\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x\right)}{\ln q^{-1}}\right)\right| dx.$

Proof. By the q-distributional equation in (2.27) we get

$$\Phi(x) = x(x-1) \quad ; \quad \Psi(x) = -\big((\alpha+\beta+2)x - (\beta+1)\big). \tag{2.33}$$

Therefore, (1.9) becomes

$$(\mathbf{J}(\alpha,\beta,q))_{n+1} = \frac{[n]_q+\beta+1}{[n]_q+\alpha+\beta+1+2}(\mathbf{J}(\alpha,\beta,q))_n\,,\ n \geq 0 \quad ; \quad (\mathbf{J}(\alpha,\beta,q))_0 = 1.$$

Consequently, we get

$$(\mathbf{J}(\alpha,\beta,q))_n = \frac{\displaystyle\prod_{k=1}^n\Big([k-1]_q+\beta+1\Big)}{\displaystyle\prod_{k=1}^n\Big([k-1]_q+\alpha+\beta+1+2\Big)}\,,\ n \geq 1 \quad ; \quad (\mathbf{J}(\alpha,\beta,q))_0 = 1.$$

Replacing $[k-1]_q$ by $\dfrac{q^{k-1}-1}{q-1}$ and, taking into account the definitions in (2.9) and (2.1), the above relation yields the desired result in (1) due to the properties in (2.11) and (2.28).

To prove the equalities in (2)-(4), according to (2.3)-(2.4) we may write the moments in (1) as follows

$$(\mathbf{J}(\alpha,\beta,q))_n =$$

$$\begin{cases} \dfrac{\left((\xi_\beta(q))^{-1};q\right)_\infty}{\left((\xi_{\alpha+\beta+1}(q))^{-1};q\right)_\infty}\left(\dfrac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}\right)^n \dfrac{\left((\xi_{\alpha+\beta+1}(q))^{-1}q^n;q\right)_\infty}{\left((\xi_\beta(q))^{-1}q^n;q\right)_\infty}, & 0<q<1, \\ \dfrac{\left(\xi_\beta(q);q^{-1}\right)_\infty}{\left(\xi_{\alpha+\beta+1}(q);q^{-1}\right)_\infty} \dfrac{\left(\xi_{\alpha+\beta+1}(q)q^{-n};q^{-1}\right)_\infty}{\left(\xi_\beta(q)q^{-n};q^{-1}\right)_\infty}, & q>1. \end{cases} \tag{2.34}$$

For the first case of (2.34), by (2.6)-(2.7) and since $\forall n \geq 0,\ |(\xi_\beta(q))^{-1}q^n| < 1$ (see (2.11)); we get for all $0<q<1,\ \alpha>-1,\ \beta>-1,\ n\geq 0$

$$(\mathbf{J}(\alpha,\beta,q))_n = \frac{\left((\xi_\beta(q))^{-1};q\right)_\infty}{\left((\xi_{\alpha+\beta+1}(q))^{-1};q\right)_\infty}\left(\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}\right)^n \times$$
$$\sum_{k=0}^{+\infty}\frac{(\xi_\beta(q))^{-k}}{(q;q)_k}q^{nk}\ \sum_{k=0}^{+\infty}\frac{(-1)^k q^{\frac{1}{2}k(k-1)}(\xi_{\alpha+\beta+1}(q))^{-k}}{(q;q)_k}q^{nk}.$$

Hence we get the desired discrete measure representation in (2).

For the second case of (2.34), by (2.6)-(2.7) and since $\forall n \geq 0,\ |(\xi_\beta(q))^{-1}q^n| < 1$ (see (2.11)); we get for all $q\in]1,q_{(\beta,2)}[,\ \alpha>-1,\ \beta>-1,\ n\geq 0$

$$(\mathbf{J}(\alpha,\beta,q))_n = \frac{\left(\xi_\beta(q);q^{-1}\right)_\infty}{\left(\xi_{\alpha+\beta+1}(q);q^{-1}\right)_\infty}\times$$
$$\sum_{k=0}^{+\infty}\frac{(\xi_\beta(q))^{k}}{(q^{-1};q^{-1})_k}q^{-nk}\ \sum_{k=0}^{+\infty}\frac{(-1)^k q^{-\frac{1}{2}k(k-1)}(\xi_{\alpha+\beta+1}(q))^{k}}{(q^{-1};q^{-1})_k}q^{-nk}.$$

The Cauchy product implies the desired first representation in (3). Moreover, the second representation in (3) is a direct corollary of the first one with $q=q_{(\beta,1)}$.

On the other hand, taking $q=q_{(\alpha+\beta+1,1)}$ in the second equality of (2.34), it becomes

$$(\mathbf{J}(\alpha,\beta,q_{(\alpha+\beta+1,1)}))_n = \frac{\left(\xi_\beta(q_{(\alpha+\beta+1,1)});q^{-1}_{(\alpha+\beta+1,1)}\right)_\infty}{\left(\xi_\beta(q_{(\alpha+\beta+1,1)})q^{-n}_{(\alpha+\beta+1,1)};q^{-1}_{(\alpha+\beta+1,1)}\right)_\infty} \tag{2.35}$$

because $\xi_{\alpha+\beta+1}(q_{(\alpha+\beta+1,1)})=0$.

According to (2.9)-(2.10) and to the fact that $q_{(\alpha+\beta+1,1)}>1,\ \alpha>-1,\ \beta>-1$ we have

$$\forall n\geq 0,\ \left|\xi_\beta(q_{(\alpha+\beta+1,1)})q^{-n}_{(\alpha+\beta+1,1)}\right| < \left|\xi_\beta(q_{(\alpha+\beta+1,1)})\right| = \frac{\alpha+1}{\alpha+\beta+2}<1.$$

By (2.6), (2.35) becomes

$$(\mathbf{J}(\alpha,\beta,q_{(\alpha+\beta+1,1)}))_n = \left(\xi_\beta(q_{(\alpha+\beta+1,1)});q^{-1}_{(\alpha+\beta+1,1)}\right)_\infty \sum_{k=0}^{+\infty}\frac{\left(\xi_\beta(q_{(\alpha+\beta+1,1)})\right)^k q^{-nk}_{(\alpha+\beta+1,1)}}{\left(q^{-1}_{(\alpha+\beta+1,1)};q^{-1}_{(\alpha+\beta+1,1)}\right)_k}$$

from which we deduce the discrete measure representation in (4).

To establish the integral representations in (5)-(6), taking into account (1.10), we look for a function U representing $\mathbf{J}(\alpha,\beta,q)$.

By virtue of (1.10), (2.33), (2.9) and the q-distributional equation in (2.27), the q-difference equation (1.15) becomes

$$U(qx)=(q\xi_\beta(q))^{-1}\frac{1-x}{1-\frac{q\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x}U(x). \tag{2.36}$$

But, taking $\alpha>-1,\ \beta>-1,\ 0<q<1$ it is quite straightforward to get the following equivalences

$$0<\frac{\xi_\beta(q)}{q\xi_{\alpha+\beta+1}(q)}<1\Longleftrightarrow q>\frac{\beta+2}{\alpha+\beta+2}=q_{(\alpha+\beta+1,-\alpha)},$$

$$0<q_{(\alpha+\beta+1,-\alpha)}<1\Longleftrightarrow \alpha\geq 0, \tag{2.37}$$

and

$$q_{(\alpha+\beta+1,-\alpha)}>1\Longleftrightarrow \alpha<0. \tag{2.38}$$

Consequently, if $\alpha\geq 0,\ \beta>-1,\ q_{(\alpha+\beta+1,-\alpha)}<q<1$, we look for U of the form

$$U(x)=\begin{cases} x^\eta\frac{\left(\frac{q\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x;q\right)_\infty}{(x;q)_\infty}, & 0<x<\frac{\xi_\beta(q)}{q\xi_{\alpha+\beta+1}(q)},\\ 0, & x\leq 0 \text{ or } x\geq\frac{\xi_\beta(q)}{q\xi_{\alpha+\beta+1}(q)}.\end{cases}$$

Substituting this U in (2.36) leads to the equality $\eta=-\frac{\ln\xi_\beta(q)}{\ln q}-1$, and we get the result in (2.29).

Also, if $-1<\alpha<0,\ \beta>-1,\ 0<q<1$, we look for U of the form

$$U(x)=\begin{cases} x^{-\frac{\ln\xi_\beta(q)}{\ln q}-1}\frac{\left(\frac{q\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x;q\right)_\infty}{(x;q)_\infty}V(x), & 0<x<1,\\ 0, & x\leq 0 \text{ or } x\geq 1.\end{cases} \tag{2.39}$$

Substituting this U in (2.36) leads to the equality $V(qx)=V(x)$. Taking into account (2.39), we may choose

$$V(x)=K\left|\sin\left(2\pi\frac{\ln x}{\ln q}\right)\right|.$$

Thus (2.30) follows.

From the hypotheses of (7)-(8), we have $\alpha>-1,\ \beta>-1,\ 1<q<q_{(\alpha+\beta+1,1)}$. By virtue of (1.10), (2.33), (2.9) and the q-distributional equation in (2.27), the q-difference equation (1.14) becomes

$$U(q^{-1}x)=q\xi_\beta(q)\ \frac{1-\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x}{1-q^{-1}x}\ U(x). \tag{2.40}$$

According to (2.11), (2.28) and (2.37)-(2.38), we have

$$0<\xi_{\alpha+\beta+1}(q)<\xi_\beta(q)<1\,,\quad 1<q<q_{(\alpha+\beta+1,1)},$$

$$\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}>q\Longleftrightarrow q>q_{(\alpha+\beta+1,-\alpha)}.$$

Therefore, if $\alpha\geq 0,\ \beta>-1,\ 1<q<q_{(\alpha+\beta+1,1)}$ we get

$$U(x)=\begin{cases} Kx^{-\frac{\ln\xi_\beta(q)}{\ln q}-1}\dfrac{(q^{-1}x;q^{-1})_\infty}{\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x;q^{-1}\right)_\infty}\,, & 0<x<q,\\ 0\,, & x\leq 0 \text{ or } x\geq q,\end{cases}$$

from which we obtain the result in (2.31).
Finally, if $-1<\alpha<0,\ \beta>-1,\ 1<q<\min\big(q_{(\alpha+\beta+1,1)},q_{(\alpha+\beta+1,-\alpha)}\big)=q_{(\alpha+\beta+1,1)}$, we look for U of the form

$$U(x)=\begin{cases} x^{-\frac{\ln\xi_\beta(q)}{\ln q}-1}\dfrac{\left(q^{-1}x;q^{-1}\right)_\infty}{\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x;q^{-1}\right)_\infty}V(x)\,, & 0<x<\dfrac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)},\\ 0\,, & x\leq 0\text{ or } x\geq\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}.\end{cases}\tag{2.41}$$

Substituting this U in (2.40) leads to the equality $V(qx)=V(x)$. Taking into account (2.41), we may choose

$$V(x)=K\left|\sin\left(2\pi\frac{\ln\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}\,x\right)}{\ln q^{-1}}\right)\right|.$$

Thus (2.32) follows. □

Remark. Denoting by U_q the weight function given either by (2.29) or by (2.31), we have for all $f\in\mathcal{P}$ (see (1.5))

$$\lim_{q\to 1}\int_{-\infty}^{+\infty}U_q(x)f(x)dx=\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_0^1 x^\beta(1-x)^\alpha f(x)dx=\langle\widetilde{\mathcal{J}}(\alpha,\beta),f\rangle.$$

3 The natural q-analogue of the Bessel D-classical form

The natural q-analogue of the Bessel form is the H_q-classical form $\mathbf{B}(\alpha,q)$ ($\alpha\neq\frac{1}{2}(q-1)^{-1},\quad \alpha\neq-\frac{1}{2}[n]_q,\ n\geq 0$) satisfying [5,9]

$$\begin{cases}\beta_n=-2q^n\frac{2\alpha+(1+q^{-1})[n-1]_q-q^{-1}[2n]_q}{(2\alpha+[2n-2]_q)(2\alpha+[2n]_q)},\ n\geq 0,\\ \gamma_{n+1}=-4q^{3n}\frac{[n+1]_q(2\alpha+[n-1]_q)}{(2\alpha+[2n-1]_q)(2\alpha+[2n]_q)^2(2\alpha+[2n+1]_q)},\ n\geq 0,\\ H_q(x^2\mathbf{B}(\alpha,q))-2(\alpha x+1)\mathbf{B}(\alpha,q)=0.\end{cases}\tag{3.1}$$

By (3.1), the form $\mathbf{B}(\alpha,q)$ is not positive definite for any value of its parameter α.

When $q\to 1$, we recover the Bessel D-classical form $\mathcal{B}(\alpha)$ in (1.6).

Due to the regularity conditions $\alpha\neq\frac{1}{2}(q-1)^{-1},\ \alpha\neq-\frac{1}{2}[n]_q,\ n\geq 0$ we get

$$\xi_{2\alpha-1}(q)\neq 0\quad;\quad \xi_{2\alpha-1}(q)\neq q^n\,,\quad n\geq 0.\tag{3.2}$$

Proposition 3.1. *The form* $\mathbf{B}(\alpha, q)$ *has the following properties.*

(1) *The moments of* $\mathbf{B}(\alpha, q)$ *are: for* $n \geq 0$

$$(\mathbf{B}(\alpha, q))_n = \left(\frac{2(q-1)}{\xi_{2\alpha-1}(q)}\right)^n \frac{1}{\left((\xi_{2\alpha-1}(q))^{-1}; q\right)_n}.$$

(2) *For all* $0 < q < 1,\ \alpha \neq \frac{1}{2}(q-1)^{-1},\ \alpha \neq -\frac{1}{2}[n]_q,\ n \geq 0$, *the form* $\mathbf{B}(\alpha, q)$ *has the following discrete measure representation*

$$\mathbf{B}(\alpha, q) = \frac{1}{\left((\xi_{2\alpha-1}(q))^{-1}; q\right)_\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k q^{\frac{1}{2}k(k-1)} (\xi_{2\alpha-1}(q))^{-k}}{(q;q)_k} \delta_{\frac{2(q-1)}{\xi_{2\alpha-1}(q)} q^k}.$$

Proof. For (1), it is seen from the q-distributional equation in (3.1) that

$$\Phi(x) = x^2 \quad ; \quad \Psi(x) = -2(\alpha x + 1). \tag{3.3}$$

Therefore, system (1.9) takes the form

$$(\mathbf{B}(\alpha, q))_{n+1} = \frac{-2}{[n]_q + 2\alpha} (\mathbf{B}(\alpha, q))_n,\ n \geq 0 \quad ; \quad (\mathbf{B}(\alpha, q))_0 = 1.$$

Consequently, we get

$$(\mathbf{B}(\alpha, q))_n = \frac{(-2)^n}{\prod_{k=1}^{n} \left([k-1]_q + 2\alpha\right)},\ n \geq 1 \quad ; \quad (\mathbf{B}(\alpha, q))_0 = 1.$$

Replacing $[k-1]_q$ by $\dfrac{q^{k-1}-1}{q-1}$, taking into account the definitions in (2.9) and (2.1), the above relation yields the desired result due to the properties in (2.11) and (3.2).

To establish (2), let $0 < q < 1,\ \alpha \neq \frac{1}{2}(q-1)^{-1},\ \alpha \neq -\frac{1}{2}[n]_q,\ n \geq 0$. According to (2.3) and (3.2) the equality in (1) takes the form

$$(\mathbf{B}(\alpha, q))_n = \left(\frac{2(q-1)}{\xi_{2\alpha-1}(q)}\right)^n \frac{\left((\xi_{2\alpha-1}(q))^{-1} q^n; q\right)_\infty}{\left((\xi_{2\alpha-1}(q))^{-1}; q\right)_\infty},\ n \geq 0.$$

By virtue of (2.7), the above expression gives for all $n \geq 0$

$$(\mathbf{B}(\alpha, q))_n = \left(\frac{2(q-1)}{\xi_{2\alpha-1}(q)}\right)^n \frac{1}{\left((\xi_{2\alpha-1}(q))^{-1}; q\right)_\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k q^{\frac{1}{2}k(k-1)} (\xi_{2\alpha-1}(q))^{-k}}{(q;q)_k} q^{nk}.$$

Thus the discrete measure representation in (2). □

In [12], the second author proposed an integral representation for the non-positive definite Bessel form $\mathcal{B}(\alpha)$ ($\alpha \neq -\dfrac{n}{2},\ n \geq 0$) as a consequence of its D-classical character; see (1.6). In what follows, our goal is to determine an integral representation for the H_q-classical form $\mathbf{B}(\alpha, q)$ when $q > 1$.

For the convenience of the reader, we provide in this paragraph a summary of definitions used in the sequel. Throughout this summary, we will fix $q \in]0,1[$.

The q-Jackson integrals from 0 to b and from 0 to $+\infty$ of a function f are defined in [6] by

$$\int_0^b f(t)\, d_q t := (1-q)b \sum_{n=0}^{+\infty} f(bq^n)q^n \tag{3.4}$$

and

$$\int_0^{+\infty} f(t)\, d_q t := (1-q) \sum_{n=-\infty}^{+\infty} f(q^n)q^n, \tag{3.5}$$

provided the sums converge absolutely.

The q-Jackson integral of a function f in a generic interval $[a,b]$ is given in [6] by

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t. \tag{3.6}$$

We recall that for any function f, we have

$$H_q\left(\int_a^x f(t)\, d_q t\right) = f(x). \tag{3.7}$$

(See [7].) It is clear by (3.4)-(3.5) that the q-integral is an infinite Riemann sum with the nodes forming a geometric progression. We would then expect that

$$\int_a^b f(t)\, d_q t \underset{q \to 1^-}{\longrightarrow} \int_a^b f(t)\, dt$$

for continuous functions.

For the rest of the paper, let

$$\alpha > 0 \quad \text{and} \quad 1 < q < q_{(2\alpha-1,1)}. \tag{3.8}$$

We define the following sequence of numbers

$$x_k(q) = \frac{2(q-1)}{\xi_{2\alpha-1}(q)}\, q^{-k}, \quad k \in \mathbb{N}. \tag{3.9}$$

Taking into account (3.8) and (2.11) we have

$$\forall k \in \mathbb{N}, \quad 0 < x_{k+1}(q) < x_k(q) < x_0(q) = \frac{2(q-1)}{\xi_{2\alpha-1}(q)} \quad ; \quad x_k(q) \underset{k \to +\infty}{\longrightarrow} 0. \tag{3.10}$$

According to (1.10) and (3.8), we look for a function U representing $\mathbf{B}(\alpha, q)$ and satisfying the additional condition

$$\int_{-\infty}^{+\infty} U(x)\, dx \neq 0. \tag{3.11}$$

By virtue of (1.10), (3.3), (2.9), (3.8)-(3.10) and by the q-distributional equation in (3.1), the q-difference equation (1.12) becomes

$$U(q^{-1}x) - q^2\xi_{2\alpha-1}(q)\Big(1 - x_0(q)\ x^{-1}\Big)U(x) = \lambda(1-q)q^2x^{-1}s\big(q(x - x_0(q))\big), \quad (3.12)$$

where $\lambda \neq 0$ and s is the Stieltjes function given by (1.13).

For $\alpha > 0,\ 1 < q < q_{(2\alpha-1,1)}$ and $x > 0,\ x \geq x_0(q)$ denoting

$$[qx, +\infty[_{q^{-1}} := \Big\{xq^n, \quad n \geq 1\Big\}$$

we obtain the definition of the following q^{-1}-integral

$$\int_{qx}^{+\infty} \frac{t^{\frac{\ln \xi_{2\alpha-1}(q)}{\ln q}}}{(x_0(q)t^{-1};q^{-1})_\infty}\, s\big(q(t - x_0(q))\big)\, d_{q^{-1}}t$$

$$:= (1 - q^{-1})\ qx \sum_{k=1}^{+\infty} \frac{q^{k\,\frac{\ln \xi_{2\alpha-1}(q)}{\ln q}}\ x^{\frac{\ln \xi_{2\alpha-1}(q)}{\ln q}}\ s\big(q(q^k\,x - x_0(q))\big)}{(x_k(q)x^{-1};q^{-1})_\infty}\ q^k$$

$$= (q-1)\ x^{1+\frac{\ln \xi_{2\alpha-1}(q)}{\ln q}} \sum_{k=1}^{+\infty} \frac{q^{k\,(1+\frac{\ln \xi_{2\alpha-1}(q)}{\ln q})}\ s\big(q^{k+1}(x - x_k(q))\big)}{(x_k(q)x^{-1};q^{-1})_\infty}. \quad (3.13)$$

It is straightforward to prove by the d'Alembert test and by using (1.13) and (3.9)-(3.10), that the sum in (3.13) converges absolutely for $\alpha > 0,\ 1 < q < q_{(2\alpha-1,1)}$ and $x \geq x_0(q)$.

Consequently, a possible solution of the q-difference equation (3.12) is

$$U(x) = \begin{cases} \frac{-\lambda}{\xi_{2\alpha-1}(q)} \frac{\big(x_1(q)\,x^{-1}\,;\,q^{-1}\big)_\infty}{x^{2+\frac{\ln \xi_{2\alpha-1}(q)}{\ln q}}} \times \\ \displaystyle\int_{qx}^{+\infty} \frac{t^{\frac{\ln \xi_{2\alpha-1}(q)}{\ln q}}}{(x_0(q)t^{-1};q^{-1})_\infty}\, s\big(q(t - x_0(q)\big)\, d_{q^{-1}}t\,, & x \geq x_0(q). \\ 0\,, & x < x_0(q). \end{cases} \quad (3.14)$$

On one hand, we have by (3.13)-(3.14) and (1.13) the following expression of U for $x > x_0(q)$

$$U(x) = \frac{\lambda(1-q)}{\xi_{2\alpha-1}(q)}\,\frac{\big(x_0(q)\,x^{-1};q^{-1}\big)_\infty}{x - x_0(q)} \times$$

$$\sum_{k=1}^{+\infty} \frac{q^{k\,(1+\frac{\ln \xi_{2\alpha-1}(q)}{\ln q})}\ \exp(-q^{\frac{k+1}{4}}\,(x - x_k(q))^{\frac{1}{4}})\ \sin(q^{\frac{k+1}{4}}\,(x - x_k(q))^{\frac{1}{4}})}{(x_k(q)\,x^{-1}\,;\,q^{-1})_\infty}.$$

Taking into account (3.9)-(3.10) and the d'Alembert test one more time, this yields

$$|U(x)| \leq \frac{|\lambda|(q-1)}{\xi_{2\alpha-1}(q)}\,\frac{\exp(-\frac{1}{2}\,(x - x_0(q))^{\frac{1}{4}})}{x - x_0(q)} \times$$

$$\sum_{k=1}^{+\infty} q^{k\,(1+\frac{\ln\xi_{2\alpha-1}(q)}{\ln q})}\exp\big(-\frac{1}{2}\,q^{\frac{k+1}{4}}\,(a-x_0(q))^{\frac{1}{4}}\big)\,,\quad x\geq a,\,\forall a>x_0(q)\,,$$

$$= o\Big(\exp\big(-\frac{1}{2}\,(x-x_0(q))^{\frac{1}{4}}\big)\Big)\,,\qquad x\longrightarrow +\infty.$$

On the other hand, the function in (3.14) is a possible representation of the form $\mathbf{B}(\alpha,q)$ when $\alpha>0$ and $1<q<q_{(2\alpha-1,1)}$. Condition (3.11) now becomes

$$\int_{x_0(q)}^{+\infty} U(x)\,dx = -\frac{\lambda}{\xi_{2\alpha-1}(q)}\,S_\alpha(q)\neq 0$$

where

$$S_\alpha(q)=\int_{x_0(q)}^{+\infty}\frac{\big(x_1(q)\,x^{-1}\,;\,q^{-1}\big)_\infty}{x^{2+\frac{\ln\xi_{2\alpha-1}(q)}{\ln q}}}\Big(\int_{qx}^{+\infty}\frac{t^{\frac{\ln\xi_{2\alpha-1}(q)}{\ln q}}}{(x_0(q)t^{-1};q^{-1})_\infty}\,s\big(q(t-x_0(q)\big)\,d_{q^{-1}}t\Big)\,dx.$$

Furthermore,

$$\int_{qx}^{+\infty}\frac{t^{\frac{\ln\xi_{2\alpha-1}(q)}{\ln q}}}{(x_0(q)t^{-1};q^{-1})_\infty}\,s\big(q(t-x_0(q)\big)\,d_{q^{-1}}t\underset{q\to 1^+}{\longrightarrow}\int_x^{+\infty} t^{-2\alpha}\,\exp\big(\frac{2}{t}\big)\,s(t)\,dt$$

since $t\longmapsto \dfrac{t^{\frac{\ln\xi_{2\alpha-1}(q)}{\ln q}}}{(x_0(q)t^{-1};q^{-1})_\infty}\,s\big(q(t-x_0(q)\big)$ is continuous in $[qx,+\infty[$ which implies that

$$S_\alpha(q)\underset{q\to 1^+}{\longrightarrow} S_\alpha$$

and $S_\alpha\neq 0$ for $\alpha\geq 6\,(\frac{2}{\pi})^4$ [12] (see also (1.6)). Thus we get

$$\forall\alpha\geq 6\left(\frac{2}{\pi}\right)^4,\ \exists q_\alpha>1,\quad \forall\ 1<q<\min(q_\alpha,q_{(2\alpha-1,1)}),\qquad S_\alpha(q)\neq 0.$$

Consequently, for all $\alpha\geq 6(\frac{2}{\pi})^4,\ 1<q<\min(q_\alpha,q_{(2\alpha-1,1)})$ and $f\in\mathcal{P}$, the form $\mathbf{B}(\alpha,q)$ has the following integral representation

$$\langle\mathbf{B}(\alpha,q),f\rangle=(S_\alpha(q))^{-1}\times$$
$$\int_{x_0(q)}^{+\infty}\frac{\big(x_1(q)\,x^{-1};q^{-1}\big)_\infty}{x^{2+\frac{\ln\xi_{2\alpha-1}(q)}{\ln q}}}\Big(\int_{qx}^{+\infty}\frac{t^{\frac{\ln\xi_{2\alpha-1}(q)}{\ln q}}}{(x_0(q)t^{-1};q^{-1})_\infty}\,s\big(q(t-x_0(q)\big)\,d_{q^{-1}}t\Big)\ f(x)\,dx.\quad(3.15)$$

Acknowledgments

The authors would like the thank to anonymous reviewer and the associate editor for their insightful comments, which led to a significantly improved presentation of the manuscript.

References

[1] F. Abdelkarim, P. Maroni, *The D_ω-classical orthogonal polynomials,* Results in Math. 32 (1997), 1-28.

[2] C. Berg, M.E.H. Ismail, *q-Hermite polynomials and classical orthogonal polynomials,* Canad. J. Math. 48 (1996), no. 1, 43-63.

[3] C. Berg, A. Ruffing, *Generalized q-Hermite polynomials,* Commun. Math. Phy. 223 (2004), no. 1, 29-46.

[4] G. Gasper, M. Rahman, *Basic hypergeometric series,* Cambridge University Press, Cambridge, 1990.

[5] A. Ghressi, L. Khériji, *The symmetrical H_q-semiclassical orthogonal polynomials of class one,* SIGMA. 5 (2009), 076, 22 pages.

[6] F.H. Jackson, *On a q-definite integrals,* Quarterly J. Pure Appl. Math. 41 (1910), 193-203.

[7] V.G. Kac, P. Cheung, *Quantum calculus*, Universitext, Springer-Verlag, New York, 2002.

[8] O.F. Kamech, M. Mejri, *Some discrete representations of q-classical linear forms,* Appl. Math. E-notes. 9 (2009), 34-39.

[9] L. Khériji, P. Maroni, *The H_q-classical orthogonal polynomials,* Acta. Appl. Math. 71 (2002), no. 1, 49-115.

[10] R. Koekoek, R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue,* Reports of the Faculty of Technical Mathematics and Informatics. Report 98-17. Delft University of Technology, TU Delft 1998.

[11] P. Maroni, *Variations around classical orthogonal polynomials. Connected problems,* J. Comput. Appl. Math. 48 (1993), 133-155.

[12] P. Maroni, *An integral representation for the Bessel form,* J. Comp. Appl. Math. 57 (1995), 251-260.

[13] P. Maroni, M. Mejri, *Some semiclassical orthogonal polynomials of class one,* Eurasian Math. J. 2 (2011), no.2, 108-128.

[14] M. Mejri, *q-Extension of some symmetrical and semi-classical orthogonal polynomials of class one,* Appl. Anal. Discrete Math. 3 (2009), 78-87.

[15] D.S. Moak, *The q-analogue of the Laguerre polynomials,* J. Math. Anal. Appl. 81 (1981), no. 1, 20-47.

[16] T.J. Stieltjes, *Recherches sur les fractions continues,* Ann. Fac. Sci. Toulouse. 8 (1894), J. 1-122; 9 (1895) A1-A47.

Lotfi Khériji
Institut Supérieur des Sciences Appliquées et de Technologie de Gabès
Rue Omar Ibn El Khattab 6072-Gabès, Tunisia.
E-mail: kheriji@yahoo.fr or Lotfi.Kheriji@issatgb.rnu.tn

Pascal Maroni
UMR 7598 Laboratoire Jacques-Luis Lions
Université Pierre et Marie Curie Paris 6, CNRS
Boite courrier 187
75252 Paris cedex 05, France
E-mail: maroni@ann.jussieu.fr

Received: 25.05.2011

EURASIAN MATHEMATICAL JOURNAL

2013 – Том 4, № 2 – Астана: ЕНУ. – 144 с.

Подписано в печать 5.07.2013 г. Тираж – 150 экз.

Адрес редакции: 010008, Астана, ул. Мирзояна, 2,
Евразийский национальный университет имени Л.Н. Гумилева,
главный корпус, каб. 355
Тел.: +7-7172-709500 добавочный 31313

Дизайн: К. Булан

Отпечатано в типографии ЕНУ имени Л.Н.Гумилева

Свидетельство о постановке на учет печатного издания
Министерства культуры и информации Республики Казахстан
№ 10330 – Ж от 25.09.2009 г.