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THE SMALL PARAMETER METHOD FOR REGULAR LINEAR DIFFERENTIAL EQUATIONS ON UNBOUNDED DOMAINS

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Key words: regular operator, hypoelliptic operator, boundary layer, regular degeneration, singular perturbation, uniform solvability.

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Abstract. Algorithms for the asymptotic expansion of the solution to the Dirichlet problem for a regular equation with a small parameter ε ($\varepsilon > 0$) at higher derivatives on an unbounded domain (the whole space, the half space and a strip), based on the solution to the degenerate (as $\varepsilon \to 0$) Dirichlet problem for a regular hypoelliptic equation of the lower order, are described. Estimates for remainder terms of those expansions are obtained.

Introduction

The degeneration of the Dirichlet problem $\mathfrak{D}_\varepsilon$ for a regular (in the sense of Mikhailov - Nikol'skii [10], [11], [13]-[15]) equation with a small parameter ε ($\varepsilon > 0$) at higher derivatives to the Dirichlet problem $\mathfrak{D}_0$ for a regular hypoelliptic equation (introduced by Hörmander [5]) in the Sobolev anisotropic spaces $\mathbb{W}_2^{\mathscr{M}}(G)$ (generated by a regular polyhedron $\mathscr{M}$ and by unbounded domain G) is considered. The methods for constructing the asymptotic expansion of the solution to Problem $\mathfrak{D}_\varepsilon$ based on Lindshted-Poincare's method, Prandell's boundary layer method (for references and for more details about those methods see [1], [6], [7], [8], [12], [19], [22]), Lusternik-Vishik's method [23] and Newton's polyhedron method [21] are described.

Note that the degenerate Problem $\mathfrak{D}_0$ can be solved by Bubnov-Galerkin's method (see Ghazaryan and Karapetyan [3]) by choosing anisotropic B-splines as base functions (see [18]).

1 Basic notation and terminology

Throughout the paper, we use the following standard notation: $\mathbb{N}$ is the set of all natural numbers, $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, $\mathbb{R}$ is the set of all real numbers. For $n \in \mathbb{N}$, $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, $\beta = (\beta_1, ..., \beta_n) \in \mathbb{N}_0^n$, $\mathscr{M} \subset \mathbb{N}_0^n$ and $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$ we denote

$$|x| = \left(x_1^2 + ... + x_n^2\right)^{\frac{1}{2}}, \qquad x^{(j)} = (x_1, ..., x_{j-1}, x_{j+1}, ..., x_n) \ (1 \le j \le n),$$

$$\alpha! = \alpha_1!...\alpha_n!, \qquad |\alpha| = \alpha_1 + ... + \alpha_n, \qquad \beta \le \alpha \Leftrightarrow \beta_j \le \alpha_j \ (1 \le j \le n),$$

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!\,(\alpha-\beta)!}\ (\beta \le \alpha)\,, \qquad \alpha\beta = \alpha_1\beta_1 + ... + \alpha_n\beta_n,$$

$$\mathscr{M}^2 \equiv \mathscr{M} \times \mathscr{M} \equiv \{(\alpha,\beta) : \alpha, \beta \in \mathscr{M}\}\,,$$

$$\mathscr{M} + \mathscr{M} \equiv \{\alpha + \beta : \alpha, \beta \in \mathscr{M}\}\,,$$

$$\xi^\alpha = \xi_1^\alpha ... \xi_n^\alpha, \qquad D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n},$$

where $D_j = \frac{\partial}{\partial x_j}$ $(1 \le j \le n)$.

We denote by $\mathbb{C}(G)$ the space of all functions f uniformly continuous on the domain $G \subset \mathbb{R}^n$ with the norm

$$\|f\|_{\mathbb{C}(G)} \equiv \sup_{x\in G}|f(x)|.$$

$$\mathbb{W}_2^{(p)}(\Omega) = \left\{ f \in \mathbb{L}_P(\Omega) : \sum_{|\alpha|\le p} \|D^\alpha f\|_{\mathbb{L}_p(\Omega)} < \infty \right\}.$$

For a finite set of multi-indices $\mathscr{M} \subset \mathbb{N}_0^n$ and a domain $G \subseteq \mathbb{R}^n$ we also denote

$$\mathbb{W}_2^{\mathscr{M}}(G) \equiv \left\{ f \in \mathbb{L}_2(G) : \|f\|_{\mathbb{W}_2^{\mathscr{M}}(G)} \equiv \sum_{\alpha \in \langle \mathscr{M} \cup \{0\}\rangle} \|D^\alpha f\|_{\mathbb{L}_2(G)} < \infty \right\},$$

where $\langle \mathscr{M} \bigcup \{0\}\rangle$ is the convex hull of the collection $\mathscr{M} \bigcup \{0\}$, and by $\mathring{\mathbb{H}}_{\mathscr{M}}(G)$ we denote the closure of the set $\mathbb{C}_0^\infty(G)$ with respect to the norm $\|.\|_{\mathbb{W}_2^{\mathscr{M}}(G)}$. Let ∂G be the boundary of G, let Ξ be Nikol'skii's skeleton (see [14]) of the collection $\langle \mathscr{M} \bigcup \{0\}\rangle$ and

$$\mathring{\mathbb{W}}_2^{\mathscr{M}}(G) \equiv \left\{ f \in \mathbb{W}_2^{\mathscr{M}}(G) : D^\alpha f|_{\partial G} = 0, \quad \forall \alpha \in \Xi \right\}.$$

In a Hilbert space $\mathbb{H}$ the inner product will be denoted by $(.,.)_{\mathbb{H}}$.

We consider only real function spaces.

2 Setting of the problem

Let $\Omega \subset \mathbb{R}^n$ be a domain, $\mathscr{N} \subset \mathbb{N}_0^n$ and $\mathscr{N}_0 \subset \mathscr{N}$ be finite collections of multi-indices, $\bar{\varepsilon} \in (0,1)$. Let ψ be a non-negative function defined on $\mathscr{N} \times \mathscr{N}$, and let

$$L_\varepsilon \equiv L_\varepsilon(x,D) \equiv \sum_{\alpha,\beta\in\mathscr{N}} \varepsilon^{\psi(\alpha,\beta)} D^\alpha \left(\eta_{\alpha,\beta}(x,\varepsilon) D^\beta\right) \qquad (\eta_{\alpha,\beta}(x,\varepsilon) \not\equiv 0,\ \alpha,\beta \in \mathscr{N}) \tag{2.1}$$

and

$$L_0 \equiv L_0(x,D) \equiv \sum_{\alpha,\beta\in\mathscr{N}_0} D^\alpha \left(\eta_{\alpha,\beta}(x,0) D^\beta\right) \qquad (\eta_{\alpha,\beta}(x,0) \not\equiv 0,\ \alpha,\beta \in \mathscr{N}_0) \tag{2.2}$$

be linear differential operators with real-valued coefficients defined on $\overline{\Omega} \times [0,\bar{\varepsilon}]$.

Consider the following boundary problems:

Problem $\mathfrak{D}_0$. Find a solution $u \in \mathring{\mathbb{W}}_2^{\mathscr{N}_0}(\Omega)$ to the equation

$$L_0 u = h, \qquad h \in \mathbb{W}_2^{\infty}(\Omega) \equiv \bigcap_{p=1}^{\infty} \mathbb{W}_2^{(p)}(\Omega). \tag{2.3}$$

Problem $\mathfrak{D}_\varepsilon$. Find a solution $u_\varepsilon \in \mathring{\mathbb{W}}_2^{\mathscr{N}}(\Omega)$ to the equation

$$L_\varepsilon u_\varepsilon = h, \qquad h \in \mathbb{W}_2^{\infty}(\Omega). \tag{2.4}$$

In the sequel the following notation will be used:

$$\varphi(\nu) \equiv \min_{\substack{\alpha,\beta\in\mathscr{N} \\ \alpha+\beta=\nu}} \psi(\alpha,\beta) \qquad \nu \in \mathscr{N}+\mathscr{N},$$

and

$$\varphi_{\mathscr{M}}^{opt}\left(\alpha^0\right) \equiv \min\left\{q \in \mathbb{R} : \forall\varepsilon \in (0,\overline{\varepsilon}], \forall\xi \in \mathbb{R}^n, \xi \geq \mathbf{0}, \varepsilon^q \xi^{\alpha^0} \leq \sum_{\alpha\in\mathscr{M}} \varepsilon^{\varphi(\alpha)} \xi^{\alpha}\right\} \tag{2.5}$$

for $\mathscr{M} \subseteq \mathscr{N}+\mathscr{N}$, $\alpha^0 \in \langle\mathscr{M}\rangle$.

We impose the following restrictions on the operators L_0 and L_ε

(A_1) a) the functions $\eta_{\alpha,\beta}(x,\varepsilon)$ $(\alpha,\beta \in \mathscr{N})$ are infinitely differentiable on $\overline{\Omega}\times[0,\overline{\varepsilon}]$;

b) for each $\alpha,\beta \in \mathscr{N}_0$ the functions $\eta_{\alpha,\beta}(x,\varepsilon)$ tend to $\eta_{\alpha,\beta}(x,0)$ as $\varepsilon \to 0$ uniformly with respect to x;

c) for each $\alpha,\beta \in \mathscr{N}_0$ $\psi(\alpha,\beta) = 0$;

(A_2) there exists a constant $\chi_1 > 0$ such that

$$(L_0 w, w) \geq \chi_1 \sum_{\alpha\in\mathscr{N}_0\cup\{0\}} \|D^\alpha w\|^2 \qquad \forall w \in \mathbb{C}_0^\infty(\Omega); \tag{2.6}$$

(A_3) $\{\gamma \in \mathbb{N}_0^n : \gamma \leq \alpha\} \subseteq \langle\mathscr{N}\cup\{0\}\rangle$ for all $\alpha \in \mathscr{N}$;

(A_4) a) the functions $\eta_{\alpha,\beta}(x,\varepsilon)$ are uniformly continuous with respect to x on $\Omega\times(0,\overline{\varepsilon}]$, for $(\alpha,\beta) \in \mathscr{R} \equiv \left\{(\alpha,\beta) \in \mathscr{N}^2\backslash\mathscr{N}_0^2 : |\alpha+\beta| \equiv 0\,(mod2)\right\}$;

b) there exists a constant $\kappa_1 > 0$ such that

$$|\eta_{\alpha,\beta}(x,\varepsilon)| \leq \kappa_1 \qquad \forall x \in \Omega, \forall\varepsilon \in (0,\overline{\varepsilon}], (\alpha,\beta) \in \mathscr{R};$$

c) there exists a constant $\chi_2 > 0$ such that

$$\sum_{(\alpha,\beta)\in\overline{\mathscr{R}}} \varepsilon^{\psi(\alpha,\beta)} \eta_{\alpha,\beta}(x,0)\,(\mathtt{i}\xi)^{\alpha+\beta} \geq \chi_2 \sum_{\alpha\in\mathscr{B}} \varepsilon^{\varphi_{\mathscr{N}+\mathscr{N}}^{opt}(2\alpha)} \xi^{2\alpha} \qquad \forall\xi \in \mathbb{R}^n, \forall\varepsilon \in (0,\overline{\varepsilon}],$$

where

$$\overline{\mathscr{R}} \equiv \left\{(\alpha,\beta) \in \mathscr{R} : \varphi(\alpha+\beta) = \varphi_{\mathscr{N}+\mathscr{N}}^{opt}(\alpha+\beta)\right\},$$

$$\mathscr{V} \equiv \{\alpha \in \mathscr{N}\backslash\mathscr{N}_0 : \alpha \notin \langle(\mathscr{N}\backslash\mathscr{N}_0)\setminus\{\alpha\}\rangle\},$$

$$\mathscr{B} \equiv \mathscr{V} \cup \left\{\alpha \in (\mathscr{N}\backslash\mathscr{N}_0)\setminus\mathscr{V} : \varphi_{(\mathscr{N}\backslash\mathscr{N}_0)\setminus\{\alpha\}}^{opt}(\alpha) > \varphi_{\mathscr{N}\backslash\mathscr{N}_0}^{opt}(\alpha)\right\};$$

d) there exists a constant $\kappa_3 > 0$ such that for every $(\alpha, \beta) \in \mathscr{I} \equiv \{(\alpha, \beta) \in \mathscr{N}^2 \backslash \mathscr{N}_0^2 : |\alpha + \beta| \equiv 1 \,(mod 2)\}$ and $\gamma, \delta \in \mathbb{N}_0^n$, if $\gamma \leq \alpha$, $\delta \leq \beta$ and $\gamma + \delta \neq \alpha + \beta$,

$$\left| D^{\gamma+\delta} \eta_{\alpha,\beta} (x, \varepsilon) \right| \leq \kappa_3 \qquad x \in \Omega, \varepsilon \in (0, \overline{\varepsilon}];$$

(A$_5$) for every $\alpha, \beta \in \mathscr{N} + \mathscr{N}, \alpha \leq \beta, \alpha \neq \beta$

$$\varphi^{opt}_{\mathscr{N}+\mathscr{N}} (\alpha) < \varphi^{opt}_{\mathscr{N}+\mathscr{N}} (\beta) \, ;$$

(A$_6$) for every $(\alpha, \beta) \in \mathscr{N} \times \mathscr{N} \quad \psi (\alpha, \beta) \in \mathbb{N}_0$.

3 Solvability and uniform solvability

Definition 3.1. Problem $\mathfrak{D}_0$ is said to be solvable if for every $h \in \mathbb{L}_2(\Omega)$ the equation $L_0 u = h$ has a unique solution $u_0 \in \mathring{\mathbb{W}}_2^{\mathscr{N}_0} (\Omega)$ such that

$$\|u_0\|_{\mathbb{W}_2^{\mathscr{N}_0}(\Omega)} \leq C \, \|h\|_{\mathbb{L}_2(\Omega)}$$

for some constant $C > 0$ independent of h.

Remark 3.1. (see [13], [14] and [11]). Let Ω be the whole space, the half space or a strip. Then Problem $\mathfrak{D}_0$ is solvable if Condition (A$_2$) holds. If $h \in \mathbb{W}_2^{\infty}(\Omega)$ then the solution w_0 to Problem $\mathfrak{D}_0$ is smooth, i.e. $u_0 \in \mathbb{W}_2^{\infty}(\Omega)$ (see [16]) and hence $D^\alpha u_0 \in \mathbb{C}\left(\overline{\Omega}\right)$ for any $\alpha \in \mathbb{N}_0^n$ by the known embedding theorem (see [2], §9).

Definition 3.2. (see [23]). Problem $\mathfrak{D}_\varepsilon$ is said to be uniformly solvable if there exists a number $\varepsilon_0 > 0$ for which

a) Problem $\mathfrak{D}_\varepsilon$ is solvable for $\varepsilon \in (0, \varepsilon_0]$, i.e., for every $h \in \mathbb{L}_2(\Omega)$ the equation $L_\varepsilon u = h$ has a unique solution $u_\varepsilon \in \mathring{\mathbb{W}}_2^{\mathscr{N}} (\Omega)$;

b) there exists a number $C_0 > 0$, and for each $\varepsilon \in (0, \varepsilon_0]$ a normed function space $B_\varepsilon \left(\mathring{\mathbb{W}}_2^{\mathscr{N}} (\Omega) \subset B_\varepsilon \right)$ with the norm $\|.\|_{B_\varepsilon}$ such that for all $h \in L_2 (\Omega)$

$$\|u_\epsilon\|_{B_\varepsilon} \leq C_0 \, \|h\|_{\mathbb{L}_2(\Omega)} \, , \qquad \varepsilon \in (0, \varepsilon_0].$$

Remark 3.2. (see [13], [14] and [11]). Let Ω be the whole space, the half space or a strip. Then Problem $\mathfrak{D}_\varepsilon$ is solvable for any fixed $\varepsilon \in (0, \varepsilon_0]$ if Conditions (A$_1$) $-$ (A$_4$) hold.

Theorem 3.1. *(see [14] and [4]). Let Ω be the whole space, the half space or a strip. If Condition (A$_3$) holds then $\mathring{\mathbb{W}}_2^{\mathscr{N}} (\Omega) = \mathring{\mathbb{H}}_{\mathscr{N}} (\Omega)$.*

Theorem 3.2. *(see [20]). Let $\mathscr{N} \subset \mathbb{N}_0^n$, $\langle \mathscr{N} \rangle$ be a completely regular polyhedron, $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the shift conditions (for example, see [2] or [4]), and the operator L_ε satisfy Conditions $(A_1) - (A_6)$. Then Problem $\mathfrak{D}_\varepsilon$ is uniformaly solvable. Moreover, there exist constants $\overline{\overline{\varepsilon}} \in (0, \overline{\varepsilon}]$ and $C_1 > 0$ such that for all $u \in \mathring{\mathbb{W}}_2^{\mathscr{N}} (\Omega)$*

$$\|u\|^2_\varepsilon \equiv \sum_{\alpha \in \langle \mathscr{N} \rangle \backslash \langle \mathscr{N}_0 \rangle} \varepsilon^{\varphi^{opt}_{\mathscr{N}+\mathscr{N}}(2\alpha)} \|D^\alpha u\|^2 + \sum_{\alpha \in \langle \mathscr{N}_0 \cup \{0\} \rangle} \|D^\alpha u\|^2 \leq C_1 \left(L_\varepsilon u, u\right) \qquad \forall \varepsilon \in (0, \overline{\overline{\varepsilon}}].$$

4 Poincare method on $\mathbb{R}^n$

Theorem 4.1. *Let $\Omega = \mathbb{R}^n$, $m \in \mathbb{N}_0$ and*

I. *a) Conditions (A_1) and (A_6) hold;*

b) the coefficients $\eta_{\alpha,\beta}(x,\varepsilon)$ $(\alpha,\beta \in \mathscr{N})$ of the operator L_ε are bounded together with their derivatives in x_n up to order $m+1$ on $\mathbb{R}^n \times [0,\overline{\varepsilon}]$;

II. *a) Problem $\mathfrak{D}_0$ is solvable;*

b) the solution w_0 of Problem $\mathfrak{D}_0$ is smooth, i.e. $w_0 \in \mathbb{W}_2^\infty(\mathbb{R}^n)$;

III. *Problem $\mathfrak{D}_\varepsilon$ is uniformly solvable;*

Then the solution u_ε to Problem $\mathfrak{D}_\varepsilon$ admits the following asymptotic expansion:

$$u_\varepsilon = w_0 + \sum_{i=1}^{m} \varepsilon^i w_i + z_m, \tag{4.1}$$

where w_0 is the solution to Problem $\mathfrak{D}_0$, w_i $(i = 1,...,m)$ are the solutions to the $\mathfrak{D}_0$ type problems, and the remainder term z_m satisfies the following estimate:

$$\|z_m\|_\varepsilon = O\left(\varepsilon^{m+1}\right) \tag{4.2}$$

($\|.\|_{B_\varepsilon}$ is the norm in Condition III, see Definition 3.2).

Proof. Let $N \in \mathbb{N}_0$. By Condition (A_1,a) the coefficients $\eta_{\alpha,\beta}$ can be represented as a finite power series with respect to ε with the remainder term of the order $(N+1)$:

$$\eta_{\alpha,\beta}(x,\varepsilon) = \eta^{(0)}_{\alpha,\beta}(x) + \sum_{i=1}^{N} \varepsilon^i \eta^{(i)}_{\alpha,\beta}(x) + \varepsilon^{N+1} \bar{\eta}^{(N+1)}_{\alpha,\beta}(x,\varepsilon) \quad (\alpha,\beta \in \mathscr{N}), \tag{4.3}$$

where

$$\eta^{(i)}_{\alpha,\beta}(x) = \frac{1}{i!} \frac{\partial^i \eta_{\alpha,\beta}(x,\epsilon)}{\partial \epsilon^i}\bigg|_{\epsilon=0},$$

$$\bar{\eta}^{(N+1)}_{\alpha,\beta}(x,\epsilon) = \frac{1}{(N+1)!} \frac{\partial^{N+1} \eta_{\alpha,\beta}(x,\epsilon)}{\epsilon^{N+1}}\bigg|_{\epsilon=\bar{\epsilon}},$$

$$(\eta^{(0)}_{\alpha,\beta}(x) \equiv \eta_{\alpha,\beta}(x,0)).$$

Then by Conditions (A_1,b), (A_1,c) and (A_6) .

$$L_\varepsilon = \sum_{s=0}^{N} \varepsilon^s L^{(s)} + \varepsilon^{N+1} L^{(N+1)}, \tag{4.4}$$

where

$$L^{(0)} \equiv L_0, \qquad L^{(s)} \equiv L^{(s)}(D,x) \equiv \sum_{\substack{\alpha,\beta\in\mathscr{N} \\ 0\le j\le N \\ \psi(\alpha,\beta)+j=s}} D^\alpha \eta^{(j)}_{\alpha,\beta}(x) D^\beta \qquad (s = 1,...,N), \tag{4.5}$$

$$L^{(N+1)} \equiv L^{(N+1)}(D,x,\varepsilon) \equiv \sum_{\substack{\alpha,\beta\in\mathscr{N} \\ 0\le s\le N \\ \psi(\alpha,\beta)+s\ge N+1}} \varepsilon^{\psi(\alpha,\beta)+s-N-1} D^\alpha \eta^{(s)}_{\alpha,\beta}(x) D^\beta$$

$$+\sum_{\alpha,\beta\in\mathcal{N}}\varepsilon^{\psi(\alpha,\beta)}D^{\alpha}\bar{\eta}_{\alpha,\beta}^{(N+1)}(x,\varepsilon)\,D^{\beta}. \tag{4.6}$$

Let $N=m$ and let w_0 be the solution of Problem $\mathfrak{D}_0$, and let $w_i\in\mathring{\mathbb{W}}_2^{\mathcal{N}_0}(\mathbb{R}^n)$ $(i=1,...,m)$ be the solution of the equation

$$L_0w_i=-\sum_{s=1}^{i}L^{(s)}w_{i-s} \tag{4.7}$$

It is obvious that by Condition III

$$w_i\in\mathbb{W}_2^{\infty}(\mathbb{R}^n)\qquad i=1,...,m. \tag{4.8}$$

Denote

$$u^{(m)}\equiv w_0+\sum_{i=1}^{m}\varepsilon^{i}w_i.$$

Thus

$$L_\varepsilon u^{(m)}=L_0w_0+\sum_{i=1}^{m}\varepsilon^{i}\left(L_0w_i+\sum_{s=1}^{i}L^{(s)}w_{i-s}\right)+\varepsilon^{N+1}\sum_{i=0}^{m}\sum_{r=0}^{i}\varepsilon^{i-r}L^{(N+1-r)}w_i. \tag{4.9}$$

It is not difficult to see (using expressions (4.5) and (4.6), by Conditions I, II and (4.8)) that there exists a number $M>0$ such that

$$\left\|L^{(N+1-r)}w_i\right\|\leq M\qquad(r=0,...,i;i=0,...,m)\,,$$

hence from (4.9) by (4.7) it follows that there exists a number $K>0$ such that

$$\left\|L_\varepsilon u^{(m)}-h\right\|\leq K\varepsilon^{m+1}.$$

Let u_ε be the solution to Problem $\mathfrak{D}_\varepsilon$, and let $z_m=u_\varepsilon-u^{(m)}$ (it is easy to see that $z_m\in\mathring{\mathbb{W}}_2^{\mathcal{N}}(\mathbb{R}^n)$). Then by Condition III

$$\|z_m\|_\varepsilon\leq(L_\varepsilon z_m,z_m)=(L_\varepsilon u_\varepsilon,z_m)-\left(L_\varepsilon u^{(m)},z_m\right)=-(L_\varepsilon u^{(m)}-h,z_m),$$

so by Cauchy type inequality for any $\omega>0$

$$\|z_m\|_\varepsilon\leq\frac{1}{2}\left(\omega\left\|L_\varepsilon u^{(m)}-h\right\|+\frac{1}{\omega}\|z_m\|\right),$$

therefore

$$\|z_m\|_\varepsilon=O\left(\varepsilon^{m+1}\right).$$

□

Corollary 4.1. *Under Conditions $(A_1)-(A_6)$ the solution u_ε admits asymptotic expansion (4.1) where w_0 is the solution of Problem $\mathfrak{D}_0$, and $w_i\in\mathring{\mathbb{W}}_2^{\mathcal{N}_0}(\mathbb{R}^n)$ $(i=1,...,m)$ is the solution to equation (4.7) and the remainder z_m satisfies estimate (4.2).*

5 Regular degeneration

Denote

$$k_n \equiv \max_{\alpha\in\mathscr{N}_0} \alpha_n, \qquad l_n \equiv \max_{\alpha\in\mathscr{N}} \alpha_n - k_n,$$

$$e^n \equiv (0,...,0,1), \qquad q_n \equiv \psi\left((k_n+l_n)e^n,(k_n+l_n)e^n\right).$$

We impose the following additional restriction on the coefficients of the operator L_ε.

(A_7) For every $\alpha,\beta\in\mathscr{N}+\mathscr{N}$

$$\psi(\alpha,\beta)\geq \frac{(\alpha_n+\beta_n-2k_n)\,q_n}{2l_n} \quad \text{with } \alpha+\beta=(\alpha_n+\beta_n)\,e^n,$$

$$\psi(\alpha,\beta)> \frac{(\alpha_n+\beta_n-2k_n)\,q_n}{2l_n} \quad \text{with } \alpha+\beta\neq(\alpha_n+\beta_n)\,e^n.$$

Let $\Omega=\mathbb{R}^n_+\equiv\{x\in\mathbb{R}^n: x_n>0\}$, $\varkappa\in\mathbb{N}$, $N\in\mathbb{N}_0$ and $t=x_n\varepsilon^{-\varkappa}$. Then, under Condition (A_1) the coefficients $\eta_{\alpha,\beta}$ can be represented as in formula (4.3), and in addition the functions $\eta^{(i)}_{\alpha,\beta}$ can be represented as a finite power series with respect to x_n:

$$\eta^{(i)}_{\alpha,\beta}(x)=\eta^{(i,0)}_{\alpha,\beta}\left(x^{(n)}\right)+\sum_{j=1}^{N}x_n^j\eta^{(i,j)}_{\alpha,\beta}\left(x^{(n)}\right)+x_n^{N+1}\bar\eta^{(i,N+1)}_{\alpha,\beta}(x) \qquad (\alpha,\beta\in\mathscr{N};\, i=0,1,...,N),$$

where $\eta^{(i,0)}_{\alpha,\beta}\left(x^{(n)}\right)\equiv\eta^{(i)}_{\alpha,\beta}\left(x^{(n)},0\right)$.

Since

$$\frac{\partial^s}{\partial x_n^s}=\varepsilon^{-s\varkappa}\frac{\partial^s}{\partial t^s}\ (s\geq 1),$$

for $\alpha,\beta\in\mathscr{N};\, i=0,1,...,N;\, j=0,1,...,N$ we get

$$D_x^\alpha\left(x_n^j\eta^{(i,j)}_{\alpha,\beta}\left(x^{(n)}\right)\right)D_x^\beta=\varepsilon^{-\varkappa(\alpha_n+\beta_n)+\varkappa j}D_y^\alpha\left(t^j\eta^{(i,j)}_{\alpha,\beta}(x)\right)D_y^\beta, \tag{5.1}$$

where $y\equiv\left(x^{(n)},t\right)$.

Using (5.1), the operator L_ε can be represented as follows:

$$L_\varepsilon=\sum_{\alpha,\beta\in\mathscr{N}}\left(\sum_{i=0}^{N}\left(\sum_{j=0}^{N}\varepsilon^{i-\varkappa(\alpha_n+\beta_n)+\varkappa j+\psi(\alpha,\beta)}D_y^\alpha\left(t^j\eta^{(i,j)}_{\alpha,\beta}\left(x^{(n)}\right)\right)D_y^\beta+\right.\right.$$
$$\left.\left.+\ D^\alpha\left(x_n^{N+1}\bar\eta^{(i,N+1)}_{\alpha,\beta}(x)\right)D^\beta\right)+\varepsilon^{N+1}D^\alpha\bar\eta^{(N+1)}_{\alpha,\beta}(x,\varepsilon)D^\beta\right). \tag{5.2}$$

Denote

$$\gamma\equiv\max_{\alpha,\beta\in\mathscr{N}}\left(\psi(\alpha,\beta)-\varkappa(\alpha_n+\beta_n)\right).$$

From (5.2), combining terms with equal powers of ε, we get:

$$L_\varepsilon=\varepsilon^\gamma\left\{M_0+\sum_{s=1}^{N}\varepsilon^sR_s+\varepsilon^{N+1}R_{N+1}\right\}, \tag{5.3}$$

where

$$M_0 \equiv \sum_{\substack{\alpha,\beta\in\mathcal{N}\\ \psi(\alpha,\beta)-\varkappa(\alpha_n+\beta_n)=\gamma}} D_y^\alpha \eta_{\alpha,\beta}^{(0,0)}\left(x^{(n)}\right) D_y^\beta = \sum_{\substack{\alpha,\beta\in\mathcal{N}\\ \psi(\alpha,\beta)-\varkappa(\alpha_n+\beta_n)=\gamma}} D_y^\alpha \eta_{\alpha,\beta}\left(x^{(n)},0,0\right) D_y^\beta, \tag{5.4}$$

$$R_s \equiv \sum_{\substack{\alpha,\beta\in\mathcal{N}\\ 0\le i\le N, 0\le j\le N\\ \psi(\alpha,\beta)-\varkappa(\alpha_n+\beta_n)+i+\varkappa j=\gamma+s}} D_y^\alpha \left(t^j \eta_{\alpha,\beta}^{(i,j)}\left(x^{(n)}\right)\right) D_y^\beta \qquad (s=1,...,N), \tag{5.5}$$

and

$$R_{N+1} \equiv \sum_{\substack{\alpha,\beta\in\mathcal{N}\\ 0\le i\le N, 0\le j\le N\\ \psi(\alpha,\beta)-\varkappa(\alpha_n+\beta_n)+i+\varkappa j>\gamma+N}} \varepsilon^{i-\varkappa(\alpha_n+\beta_n)+\varkappa j+\psi(\alpha,\beta)-N-1} D_y^\alpha \left(t^j \eta_{\alpha,\beta}^{(i,j)}\left(x^{(n)}\right)\right) D_y^\beta +$$

$$+\sum_{\substack{\alpha,\beta\in\mathcal{N}\\ 0\le i\le N}} \varepsilon^{-\gamma} D_x^\alpha \left(x_n^{N+1}\bar{\eta}_{\alpha,\beta}^{(i,N+1)}(x)\right) D_x^\beta + \sum_{\alpha,\beta\in\mathcal{N}} \varepsilon^{-\gamma} D_x^\alpha \bar{\eta}_{\alpha,\beta}^{(N+1)}(x,\varepsilon) D_x^\beta.$$

Proposition 5.1. *For M_0 to be an ordinary differential operator of order $2(k_n+l_n)$ with a minor member of order $2k_n$, it is necessary and sufficient that*

1^0) $\gamma = -\frac{k_n q_n}{l_n}$;

2^0) $\varkappa = \frac{q_n}{2l_n}$ is a natural number;

3^0) $\psi(\alpha,\beta) \ge \frac{(\alpha_n+\beta_n-2k_n)q_n}{2l_n}$ for $\alpha+\beta = (\alpha_n+\beta_n)e^n$,

$\psi(\alpha,\beta) > \frac{(\alpha_n+\beta_n-2k_n)q_n}{2l_n}$ for $\alpha+\beta \ne (\alpha_n+\beta_n)e^n$.

Proof. It is easy to see that the derivative $\frac{\partial^{2k_n}}{\partial t^{2k_n}}$ presents in M_0 if and only if $\gamma = \psi(k_n e^n, k_n e^n) - 2\varkappa k_n = -2\varkappa k_n$, and the derivative $\frac{\partial^{2(k_n+l_n)}}{\partial t^{2(k_n+l_n)}}$ presents in M_0, if and only if $\gamma = \psi((k_n+l_n)e^n, (k_n+l_n)e^n) - 2\varkappa(k_n+l_n) = q - 2\varkappa(k_n+l)$, which are equivalent to conditions 1^0) and 2^0). M_0 was an ordinary differential operator if and only if $\psi(\alpha,\beta) - \varkappa(\alpha_n+\beta_n) > \gamma$ for $\alpha+\beta \ne (\alpha_n+\beta_n)e^n$ and $\psi(\alpha,\beta)-\varkappa(\alpha_n+\beta_n) \ge \gamma$ for $\alpha+\beta = (\alpha_n+\beta_n)e^n$, which are equivalent to condition 3^0). □

Remark 5.1. Note that under Condition (A_6) we can assume that $\varkappa = \frac{q_n}{2l_n}$ is a natural number and is equal to 1 (otherwise we can obtain this by the change of the variable).

Remark 5.2. Under Conditions (A_1), (A_6) and (A_7) (in respect to Remark 5.1) if $\varkappa = \frac{q_n}{2l_n}$ then the operator M_0 is an ordinary differential operator.

Let M_0 (introduced in (5.4)) is an ordinary differential operator and satisfies the conditions of Proposition 5.1. We introduce the following equation (which is the characteristic equation of the operator M_0):

$$\lambda^{2\varkappa k_n} Q(\lambda) \equiv \lambda^{2\varkappa k_n} \sum_{\substack{\alpha_n e^n,\beta_n e^n\in\mathcal{N}\\ \psi(\alpha_n e^n,\beta_n e^n)-\varkappa(\alpha_n+\beta_n)=\gamma}} \eta_{\alpha_n e^n,\beta_n e^n}\left(x^{(n)},0,0\right) \lambda^{\alpha_n+\beta_n-2\varkappa k_n} = 0. \tag{5.6}$$

Definition 5.1. The degeneration of the Problem $\mathfrak{D}_\varepsilon$ into Problem $\mathfrak{D}_0$ is called regular if Conditions (A_1), (A_6) and (A_7) hold and the characteristic polynomial $Q(\lambda)$ has exactly l_n pairwise different roots with negative real parts.

Later, we use the following result.

Lemma 5.1. *(see Lemma 4 in [23]). Let $m, M \in \mathbb{N}_0$ and let*

$$P(t) = \sum_{j=2m}^{2M} a_j t^j \qquad (a_{2m} \neq 0, a_{2M} \neq 0)$$

be a polynomial with real coefficients. If there exists $C > 0$ such that for all $\xi \in \mathbb{R}$

$$Re\left(P\left(\mathtt{i}\xi\right)\right) \equiv \sum_{j=m}^{M} (-1)^j a_{2j} \xi^{2j} \geq C\left(\xi^{2m} + \xi^{2M}\right)$$

then P has exactly $(M - m)$ roots with negative real parts.

For the complete symbol of the operator L_ε, we introduce the notation

$$L_\varepsilon\left(x, \mathtt{i}\xi\right) \equiv \sum_{\alpha,\beta \in \mathscr{N}} \varepsilon^{\psi(\alpha,\beta)} \eta_{\alpha,\beta}\left(x, \varepsilon\right) \left(\mathtt{i}\xi\right)^{\alpha+\beta}.$$

Theorem 5.1. *Let Conditions (A_1), (A_4.c), (A_6) and (A_7) hold. Then Q (defined in (5.6)) has exactly l_n roots with negative real parts.*

Proof. It follows by Condition (A_4.c) that there is a constant $\chi_2 > 0$ such that for all $\xi_n \in \mathbb{R}$ and $\varepsilon \in (0, \overline{\varepsilon}]$

$$\sum_{(\alpha_n e^n, \beta_n e^n) \in \overline{\mathscr{R}}} \varepsilon^{\psi(\alpha_n e^n, \beta_n e^n)} \eta_{\alpha_n e^n, \beta_n e^n}\left(x^{(n)}, 0, 0\right) \left(\mathtt{i}\xi_n\right)^{\alpha_n+\beta_n} \geq \chi_2 \sum_{\alpha_n e^n \in \mathscr{B}} \varepsilon^{\varphi^{opt}_{\mathscr{N}+\mathscr{N}}(2\alpha_n e^n)} \xi_n^{2\alpha}. \tag{5.7}$$

Clearly,

$$\sum_{\substack{\alpha_n e^n, \beta_n e^n \in \mathscr{N} \\ \psi(\alpha_n e^n, \beta_n e^n) - \varkappa(\alpha_n+\beta_n) = \gamma}} \varepsilon^{\psi(\alpha_n e^n, \beta_n e^n)} \eta_{\alpha_n e^n, \beta_n e^n}\left(x^{(n)}, 0, 0\right) \left(\mathtt{i}\xi_n\right)^{\alpha_n+\beta_n} =$$

$$= \sum_{\substack{\alpha_n e^n, \beta_n e^n \in \mathscr{N} \\ \psi(\alpha_n e^n, \beta_n e^n) - \varkappa(\alpha_n+\beta_n) = \gamma}} \varepsilon^{\gamma} \eta_{\alpha_n e^n, \beta_n e^n}\left(x^{(n)}, 0, 0\right) \left(\mathtt{i}\xi_n \varepsilon^{\varkappa}\right)^{\alpha_n+\beta_n} = \left(\mathtt{i}\xi_n\right)^{2\varkappa k_n} Q\left(\mathtt{i}\xi_n \varepsilon^{\varkappa}\right).$$

It is not hard to check that

$$Re\left(\mathtt{i}\xi_n\right)^{2\varkappa k_n} Q\left(\mathtt{i}\xi_n \varepsilon^{\varkappa}\right) = \sum_{(\alpha_n e^n, \beta_n e^n) \in \overline{\mathscr{R}}} \varepsilon^{\psi(\alpha_n e^n, \beta_n e^n)} \eta_{\alpha_n e^n, \beta_n e^n}\left(x^{(n)}, 0, 0\right) \left(\mathtt{i}\xi_n\right)^{\alpha_n+\beta_n}.$$

Then from condition (5.7), by Lemma 5.1, it immediately follows that the polynomial Q has exactly l_n roots with negative real parts. $\square$

Remark 5.3. (see [9]). Let $u \in \mathbb{W}_2^{\infty}(\mathbb{R}_+^n)$. For $u \in \mathring{\mathbb{W}}_2^{\mathscr{N}_0}\left(\mathbb{R}_+^n\right)$ to be true, it is necessary and sufficient that

$$\left.\frac{\partial^s u}{\partial x_n^s}\right|_{x_n=0} = 0 \qquad (s = 0, 1, ..., k_n - 1)\,, \tag{5.8}$$

and for $u \in \mathring{\mathbb{W}}_2^{\mathscr{N}}\left(\mathbb{R}_+^n\right)$, to be true it is necessary and sufficient conditions (5.8) are satisfied and

$$\left.\frac{\partial^{k_n+s} u}{\partial x_n^{k_n+s}}\right|_{x_n=0} = 0 \qquad (s = 0, 1, ..., l_n - 1)\,. \tag{5.9}$$

6 Boundary layer method on $\mathbb{R}_+^n$ and on a strip

Definition 6.1. (see [23], p. 7). Let $v_\varepsilon(x) = v_\varepsilon(x_1, \ldots, x_n)$ be an s ($s \in \mathbb{N}$) times differentiable function in a domain $Q \subset \mathbb{R}^n$. Then v_ε is called a boundary layer type function of order k ($k < s$), if

1. for every closed subset $\overline{K}$ of the domain Q ($\overline{K} \subset Q$), which does not intersect the boundary ∂Q of the domain Q ($\overline{K} \cap \partial Q = \varnothing$) and for every $\delta > 0$ there exists positive number ε_0 such that

$$|D^\alpha v_\varepsilon(x)| \le \delta \qquad \forall \varepsilon \in (0, \varepsilon_0], \forall x \in \overline{K}, |\alpha| \le s;$$

2. there exist positive numbers M and ε_0 such that

$$|D^\alpha v_\varepsilon(x)| \le M \qquad \forall \varepsilon \in (0, \varepsilon_0], \forall x \in Q, |\alpha| = k;$$

3. for every $\delta > 0$ there exists positive number ε_0 such that

$$|D^\alpha v_\varepsilon(x)| \le \delta \qquad \forall \varepsilon \in (0, \varepsilon_0], \forall x \in \overline{Q}, |\alpha| < k;$$

Example 1. The typical examples of boundary layer type functions of order k on the positive semiaxis are

$$\varepsilon^k e^{-\frac{\lambda t}{\varepsilon}} \text{ and } \varepsilon^k P\left(\frac{t}{\varepsilon}\right) e^{-\frac{\lambda t}{\varepsilon}},$$

where $\lambda > 0$ and P is a polynomial.

Suppose $\tau \in (0, \infty)$, and $\phi\,(y)$ is an infinitely differentiable function of one variable, that equals to 1 when $y \le \frac{\tau}{2}$ and vanishes when $y \ge \tau$.

Theorem 6.1. *Let* $\Omega = \mathbb{R}_+^n$, $m \in \mathbb{N}_0$ *and*

I. *a) Conditions* (A_1) *and* (A_6) *hold;*

b) The coefficients $\eta_{\alpha,\beta}\,(x, \varepsilon)$ $(\alpha, \beta \in \mathscr{N})$ *of the operator* L_ε *are bounded with its derivatives of* x_n *up to order* $m + k_n + 1$ *on* $\overline{\mathbb{R}_+^n} \times [0, \overline{\varepsilon}]$*;*

II. *a) Problem* $\mathfrak{D}_0$ *is solvable;*

b) The solution w_0 *of Problem* $\mathfrak{D}_0$ *is smooth, i.e.* $w_0 \in \mathbb{W}_2^{\infty}\left(\mathbb{R}_+^n\right)$*;*

III. *Problem* $\mathfrak{D}_\varepsilon$ *is uniformly solvable;*

IV. *The degeneration of Problem* $\mathfrak{D}_\varepsilon$ *into Problem* $\mathfrak{D}_0$ *is regular.*

Then the solution u_ε *of Problem* $\mathfrak{D}_\varepsilon$ *admits the following asymptotic expansion:*

$$u_\varepsilon = w_0 + \sum_{i=1}^{m} \varepsilon^i w_i + \sum_{i=0}^{m+k_n} \varepsilon^i \left(v_i + \varepsilon \phi\,(x_n)\,\alpha_i\right) + z_m,$$

where w_0 is the solution of Problem $\mathfrak{D}_0$, w_i $(i=1,...,m)$ is the solution of the $\mathfrak{D}_0$ type problem, $v_i=\varepsilon^{k_n}\overline{v}_i$ $(i=0,...,m+k_n)$ is a boundary layer type function of order k_n, α_i $(i=0,...,m+k_n)$ is a polynomial of degree k_n-1 with respect to x_n, and for the remainder z_m the following estimate holds:

$$\|z_m\|_\varepsilon = O\left(\varepsilon^{m+1}\right)$$

($\|.\|_\varepsilon$ is the norm in Condition III, see Definition 3.2).

The proof of Theorem 6.1 will be given below.

Denoting the roots of the polynomial Q with negative real parts by $-\lambda_1,...,-\lambda_{l_n}$, by Definition 5.1 we get

$$\lambda_q \neq \lambda_j \qquad (1 \le q \neq j \le l_n). \tag{6.1}$$

Proposition 6.1. *Let w_0 be a solution of Problem $\mathfrak{D}_0$. Under the conditions of Theorem 6.1 there exist functions $c_{0,1}\equiv c_{0,1}\left(x^{(n)},\varepsilon\right),...,c_{0,l_n}\equiv c_{0,l_n}\left(x^{(n)},\varepsilon\right)$ uniformly bounded in $\mathbb{R}^n_+$ (with respect to ε) with their derivatives in any order such that the functions ($t=x_n\varepsilon^{-1}, x_n=\varepsilon t$)*

$$v_0 \equiv \varepsilon^{k_n}\overline{v}_0 \equiv \varepsilon^{k_n}\sum_{s=1}^{l_n} c_{0,s}e^{-\lambda_s t} = \varepsilon^{k_n}\sum_{s=1}^{l_n} c_{0,s}e^{-\lambda_s x_n\varepsilon^{-1}}, \tag{6.2}$$

$$\varepsilon\alpha_0 \equiv -\varepsilon^{k_n}\sum_{q=1}^{l_n} c_{0,q}\sum_{s=0}^{k_n-1}\frac{\left(-\lambda_q t\right)^s}{s!} = -\varepsilon\sum_{q=1}^{l_n} c_{0,q}\sum_{s=0}^{k_r-1}\varepsilon^{k_n-1-s}\frac{\left(-\lambda_q x_n\right)^s}{s!}. \tag{6.3}$$

satisfy the following conditions

1) v_0 is a boundary layer type function of the order k_n;

2) the function $w_0+v_0+\varepsilon\alpha_0$ satisfies the boundary conditions of Problem $\mathfrak{D}_\varepsilon$.

Proof. Statement 1. We keep the requirement that w_0+v_0 has to satisfy conditions (5.9), i.e.

$$\left.\frac{\partial^{k_n+s}\left(w_0+\varepsilon^{k_n}\overline{v}_0\right)}{\partial x_n^{k_n+s}}\right|_{x_n=0} = 0 \qquad (s=0,1,...,l_n-1). \tag{6.4}$$

By Condition II.b in Theorem 6.1 and Remark 3.1, from (6.4) we get

$$\left.\frac{\partial^{k_n+s}\varepsilon^{k_n}\overline{v}_0}{\partial x_n^{k_n+s}}\right|_{x_n=0} = -\left.\frac{\partial^{k_n+s}w_0}{\partial x_n^{k_n+s}}\right|_{x_n=0} \qquad (s=0,1,...,l_n-1), \tag{6.5}$$

or

$$\left.\frac{\partial^{k_n+s}\overline{v}_0}{\partial t^{k_n+s}}\right|_{t=0} = -\varepsilon^s\left.\frac{\partial^{k_n+s}w_0}{\partial x_n^{k_n+s}}\right|_{x_n=0} \qquad (s=0,1,...,l_n-1). \tag{6.6}$$

Substituting representation (6.2) of the function $\overline{v}_0$ into (6.6), we get a system of l_n linear equations with l_n unknown quantities $c_{0,q}=c_{0,q}\left(x^{(n)},\varepsilon\right)$:

$$\sum_{q=1}^{l_n}\left(-\lambda_s\right)^{k_n+s}c_{0,q} = -\varepsilon^s\left.\frac{\partial^{k_n+s}w_0}{\partial x_n^{k_n+s}}\right|_{x_n=0} \qquad (s=0,1,...,l_n-1). \tag{6.7}$$

The determinant of this system is of Vandermonde type and it does not vanish by the conditions (6.1).

Consequently, system (6.7) has a unique solution.

Statement 2. The function $-\varepsilon\alpha_0$ is the sum of the first k_n terms of the Taylor series of v_0 in a neighborhood of $x_n = 0$. Therefore, the function $v_0 + \varepsilon\alpha_0$ satisfies boundary conditions (5.8). On the other hand, $\varepsilon\alpha_0$ is a $k_n - 1$ order polynomial of x_n (or of t). Hence, it automatically satisfies conditions (5.9). Besides, w_0 satisfies boundary conditions (5.8) and w_0+v_0 satisfies boundary conditions (5.9), and therefore the function $w_0 + v_0 + \varepsilon\alpha_0$ satisfies boundary conditions (5.8) and (5.9).

Note that functions w_0, α_0 and their derivatives in any order are uniformly bounded in $\mathbb{R}^n_+$ (with respect to ε), and $M_0 v_0 = 0$ in $\mathbb{R}^n_+$. □

Remark 6.1. If $d_1, \ldots, d_{l_n}$ is a solution of the system

$$\begin{cases} \sum\limits_{q=1}^{l_n} (-\lambda_s)^{k_n} d_q = -\left.\frac{\partial^{k_n} w_0}{\partial x_n^{k_n}}\right|_{x_n=0}, \\ \sum\limits_{q=1}^{l_n} (-\lambda_s)^{k_n+s} d_q = 0 \quad (s = 1, \ldots l_n - 1). \end{cases}$$

then it is not difficult to see that the solution $c_{0,1}, \ldots, c_{0,l_n}$ to system (6.7) can be represented in the form

$$c_{0,q}\left(x^{(n)}, \varepsilon\right) = d_q\left(x^{(n)}\right) + \sum_{s=1}^{l_r-1} g_{q,s}\left(x^{(n)}\right)\varepsilon^s,$$

where $g_{q,s}$ $(q = 1, \ldots, l_n)$ are some functions independent of ε.

For $t \in R$ and $1 \le j \le n$ we set $\left(x^{(j)}, t\right) \equiv (x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)$.

Proposition 6.2. *Under the conditions of Theorem 6.1 there exist functions* $c_{i,1} \equiv c_{i,1}\left(x^{(n)}, t, \varepsilon\right), \ldots, c_{i,l_n} \equiv c_{i,l_n}\left(x^{(n)}, t, \varepsilon\right)$ $(0 < i \le m + k_n)$ *uniformly bounded on* $\mathbb{R}^n_+$ *(with respect to* ε*) with their derivatives of any order such that the functions*

$$v_i \equiv \varepsilon^{k_n}\overline{v}_i \equiv \varepsilon^{k_n}\sum_{q=1}^{l_n} c_{i,q} e^{-\lambda_q t}, \tag{6.8}$$

and

$$\varepsilon\alpha_i \equiv -\varepsilon^{k_n}\sum_{q=1}^{l_n} c_{i,q}\left(x^{(n)}, t, \varepsilon\right)\sum_{s=0}^{k_n-1}\frac{(-\lambda_q t)^s}{s!} =$$

$$= -\varepsilon\sum_{q=1}^{l_n} c_{i,q}\left(x^{(n)}, t, \varepsilon\right)\sum_{s=0}^{k_n-1}\varepsilon^{k_n-1-s}\frac{(-\lambda_q x_n)^s}{s!}. \tag{6.9}$$

satisfy the following conditions

1) α_i *and their derivatives of any order uniformly bounded on* $\mathbb{R}^n_+$ *(with respect to* ε*)*

2) the solution $w_i \in \mathring{\mathbb{W}}_2^{\mathcal{N}_0}\left(\mathbb{R}^n_+\right)$ $(i=1,...,m)$ *of the equation*

$$L_0 w_i = h_i \equiv -\sum_{s=1}^{i} L^{(s)} w_{i-s} - \sum_{s=0}^{i-1} L^{(s)}\left(\phi\left(x_n\right)\alpha_{i-s-1}\right) \qquad (i=1,...,m), \tag{6.10}$$

and their derivatives of any order uniformly bounded on $\mathbb{R}^n_+$ *(with respect to* ε*)*

3) $c_{i,q}\left(x^{(n)},t,\varepsilon\right)$ *are polynomials of* t*;*

4) v_i *is a boundary layer type function of order* k_n *such that*

$$M_0 \overline{v}_i = -\sum_{s=1}^{i} R_s \overline{v}_{i-s} \qquad (i>0), \tag{6.11}$$

5) the function $w_i + v_i + \varepsilon\alpha_i$ *satisfies the boundary conditions of Problem* $\mathfrak{D}_\varepsilon$*.*

Before proving Proposition 6.2, we give the following obvious lemma without a proof.

Lemma 6.1. *Let* Q *be a domain in* $\mathbb{R}$*, let* $p \in \mathbb{N}_0$*,* $b_i(t) \in \mathbb{C}^p(Q)$ $(i=1,\dots,n)$ *and let* $A \in \mathbb{R}^{n\times n}$ *be a matrix with* $\det A \neq 0$*. Then:*

a) the system of equations $A\left(x_1(t),\dots,x_n(t)\right)^T = \left(b_1(t),\dots,b_n(t)\right)$*, has a unique solution, such that* $x_r(t) \in \mathbb{C}^p(Q)$ $(r=1,\dots,n)$*, and*

b) if

$$\left.\frac{\partial^s}{\partial t^s} b_r(t)\right|_{t=t_0} = 0 \qquad (s=0,...,p;\ r=1,...,n),$$

then

$$\left.\frac{\partial^s}{\partial t^s} x_r(t)\right|_{t=t_0} = 0 \qquad (s=0,...,p;\ r=1,...,n).$$

Proof of Proposition 6.2: Suppose that $w_i\,(i\le m)$ is a solution of equation (6.10), satisfying boundary conditions (5.8).

We keep the requirement that $w_i + \varepsilon^{k_n}\overline{v}_i$ has to satisfy the conditions (5.9), i.e.

$$\left.\frac{\partial^{k_n+s}\overline{v}_i}{\partial t^{k_n+s}}\right|_{t=0} = -\varepsilon^s \left.\frac{\partial^{k_n+s} w_i}{\partial x_n^{k_n+s}}\right|_{x_n=0} \qquad (s=0,1,...,l_n-1). \tag{6.12}$$

where it is assumed that $w_i \equiv 0$ when $i>m$.

The remainder part of the proof is similar to Proposition 6.1. We prove statements 1) and 2) by induction on i. Obviously, the function w_0 satisfies 1) (see Conditions I and IV, Remark 3.1 and Definition 3.1). Consequently, the function $c_{0,q}\left(x^{(n)},t\right)$ also satisfies 1) by Lemma 6.1, and hence the function α_0 satisfies 1) (see representation (6.3)).

By the induction assumption, all coefficients $c_{i-s,q}$ $(0<s\le i)$ are polynomials of t, all functions $\overline{v}_{i-s}$ $(0<s\le i)$ are of form (6.8) and the operator $R_{r,s+k_r}$ $(s>0)$ is independent of D_r (or $\frac{\partial}{\partial t}$, see formula (5.5)). Therefore, the right-hand side of (6.11) is of the form

$$\sum_{s=1}^{l_n} F_s e^{-\lambda_s t},$$

where $F_s = F_s\left(x^{(n)}, t\right)$ is a polynomial of t. Consequently, the solution $\overline{v}_i$ of equation (6.11) has the form

$$\overline{v}_i = \varphi_i + \theta_i,$$

where $\varphi_i = \varphi_i\left(x^{(n)}, t\right)$ is a partial solution of nonhomogeneous equation (6.11), which can be deduced by the uncertain coefficient method (see [17]) and has the form

$$\sum_{s=1}^{l_n} K_s e^{-\lambda_s t}$$

where $K_s = K_s\left(x^{(n)}, t\right)$ is a polynomial t order. Its order is higher by 1 than the order of $F_s\left(x^{(n)}, t\right)$ of t (see condition (6.1)), and $\theta_i = \theta_i\left(x^{(n)}, t\right)$ is a solution of form (6.8) for the corresponding homogeneous equation, satisfying the boundary conditions

$$\left.\frac{\partial^{k_n+s}\theta_i}{\partial t^{k_n+s}}\right|_{t=0} = -\varepsilon^s \left.\frac{\partial^{k_n+s} w_i}{\partial x_n^{k_n+s}}\right|_{x_n=0} - \left.\frac{\partial^{k_n+s}\varphi_i}{\partial t^{k_n+s}}\right|_{t=0} \qquad (s = 0, 1, ..., l_n - 1),$$

Consequently, the function v_i is of the form (6.8).

By the induction assumption for $0 \leq j < i$ the functions w_j, α_j and $c_{j,q}$ satisfy 1). Hence, the function h_i (see (6.10)) and its derivatives of any order are uniformly bounded in Ω with respect to ε. Consequently, by Remark 3.2 the function w_i satisfies 1). By Lemma 6.1, it is not difficult to see that the functions $c_{i,q}$ satisfies 1), and hence also α_i satisfies 1).

Statement 3) follows from 1) and 2), and the statement 4) immediately follows by Proposition 6.2 and the definition of the function ϕ.

Proof of theorem 6.1: Let functions w_i $(i = 1, ..., m)$, v_i and α_i $(i = 1, ..., m + k_n)$ satisfy the conditions of Propositions 6.1 and 6.2. Denote

$$u^{(m)} \equiv w_0 + \sum_{i=1}^{m} \varepsilon^i w_i + \sum_{i=0}^{m+k_n} \varepsilon^i \left(v_i + \varepsilon\phi\left(x_n\right)\alpha_i\right).$$

Thus by using forms (4.4) (assuming that $N = m$) and (5.3) (assuming that $N = m + k_n$) we get

$$L_\varepsilon u^{(m)} = \left\{\left(L_0 + \sum_{s=1}^{m} \varepsilon^s L^{(s)} + \varepsilon^{N+1} L^{(N+1)}\right)\left(w_0 + \sum_{i=1}^{m} \varepsilon^i w_i + \sum_{i=0}^{m+k_n} \varepsilon^{i+1}\phi\left(x_n\right)\alpha_i\right)\right\} +$$

$$+\varepsilon^\gamma \left\{\left(M_0 + \sum_{s=1}^{m+k_n} \varepsilon^s R_s + \varepsilon^{m+k_n+1} R_{m+k_n+1}\right)\left(\sum_{i=0}^{m+k_n} \varepsilon^i v_i\right)\right\},$$

Hence

$$L_\varepsilon u^{(m)} = L_0 w_0 + \left\{\sum_{i=1}^{m} \varepsilon^i \left(L_0 w_i + \sum_{s=1}^{i} L^{(s)} w_{i-s} + \sum_{s=0}^{i} L^{(s)}\left(\phi\left(x_n\right)\alpha_{i-s-1}\right)\right) +\right.$$

$$\left. + \sum_{s=0}^{m+1} \varepsilon^s L^{(s)} \left(\sum_{i=m+1-s}^{m} \varepsilon^i w_i + \sum_{i=m-s}^{m+k_n} \varepsilon^{i+1}\phi\left(x_n\right)\alpha_i\right)\right\} +$$

$$+\varepsilon^{\gamma}\left\{M_0 v_0+\sum_{i=1}^{m+k_n}\varepsilon^{i}\left(M_0 v_i+\sum_{s=1}^{i}R_s v_{i-s}\right)+\sum_{s=1}^{m+k_n+1}\varepsilon^{s}R_s\sum_{i=m+k_n+1-s}^{m+k_n}\varepsilon^{i}v_i\right\}.$$

By virtue of (6.10) and (6.11) we get

$$L_\varepsilon u^{(m)}=h+\sum_{s=0}^{m+1}\left(\sum_{i=m+1-s}^{m}\varepsilon^{i+s}L^{(s)}w_i+\sum_{i=N-s}^{m+k_n}\varepsilon^{i+1+s}L^{(s)}\left(\phi\left(x_n\right)\alpha_i\right)\right)$$

$$+\sum_{s=1}^{m+k_n+1}\sum_{i=m+k_n+1-s}^{m+k_n}\varepsilon^{i+s}R_s v_i. \tag{6.13}$$

It is not hard to see that by Propositions 6.1 and 6.2 it follows that there exists $M>0$ such that

$$\left\|L^{(N+1-r)}w_i\right\|\le M \qquad (r=0,...,i;i=0,...,m),$$

$$\left\|L^{(N+1-r)}\left(\phi\left(x_n\right)\alpha_i\right)\right\|\le M \qquad (r=0,...,i;i=0,...,m+k_n),$$

$$\left\|R_{N+1-r}v_i\right\|\le M \qquad (r=0,...,i;i=0,...,m+k_n).$$

Hence from (6.13) that there exists $K>0$ such that

$$\left\|L_\varepsilon u^{(m)}-h\right\|\le K\varepsilon^{m+1}.$$

Similar to the proof of Theorem 4.1 we can show that

$$\left\|z_m\right\|_\varepsilon=O\left(\varepsilon^{m+1}\right).$$

7 Newton's polyhedron method on $\mathbb{R}^n_+$ and on a strip

In this Section we will impose the following restriction instead of (A_7):

(A_7') there are natural numbers $p\in[1,l_n], s_1,...,s_p$ $(0\equiv s_0<s_1<...<s_p\equiv l_n)$ such that for every $r=1,...,p$

a) for $r<p$ (i.e. for $p=1$ this condition is absent)

$$\frac{\psi_r-\psi_{r-1}}{s_r-s_{r-1}}<\frac{\psi_{r+1}-\psi_r}{s_{r+1}-s_r},$$

where $\psi_j\equiv\psi\left(\left(k_n+s_j\right)e^n,\left(k_n+s_j\right)e^n\right)$ $(j=1,...,p)$,

b) for every $\alpha,\beta\in\mathscr{N}+\mathscr{N}$ if $(\alpha_n+\beta_n)\in[2k_n+2s_{r-1},2k_n+2s_r]$ then

$$\psi\left(\alpha,\beta\right)\ge\frac{\left(\alpha_n+\beta_n-2k_n\right)\left(\psi_r-\psi_{r-1}\right)}{2\left(s_r-s_{r-1}\right)}\quad\text{with } \alpha+\beta=\left(\alpha_n+\beta_n\right)e^n,$$

$$\psi\left(\alpha,\beta\right)>\frac{\left(\alpha_n+\beta_n-2k_n\right)\left(\psi_r-\psi_{r-1}\right)}{2\left(s_r-s_{r-1}\right)}\quad\text{with } \alpha+\beta\ne\left(\alpha_n+\beta_n\right)e^n.$$

Let $\varkappa_r\in\mathbb{N}_0$ $(r=1,...,p)$. For $r=1,...,p$, denote

$$\gamma_r\equiv\max_{\alpha,\beta\in\mathscr{N}}\left(\psi\left(\alpha,\beta\right)-\varkappa_r\left(\alpha_n+\beta_n\right)\right),$$

$$M_r \equiv \sum_{\substack{\alpha,\beta\in\mathcal{N} \\ (\alpha_n+\beta_n)\in[2k_n+2s_{r-1},2k_n+2s_r] \\ \psi(\alpha,\beta)-\varkappa_r(\alpha_n+\beta_n)=\gamma_r}} D_y^{\alpha}\eta_{\alpha,\beta}\left(x^{(n)},0,0\right)D_y^{\beta}, \tag{7.1}$$

Proposition 7.1. *For M_r $(r=1,...,p)$ to be an ordinary differential operator of order $2\left(k_n+l_n\right)$ with a minor member of order $2\left(k_n+s_{r-1}\right)$, it is necessary and sufficient that*

1^0) $\gamma_r=-\frac{k_n(\psi_r-\psi_{r-1})}{s_r-s_{r-1}}$;

2^0) $\varkappa_r=\frac{\psi_r-\psi_{r-1}}{2(s_r-s_{r-1})}$ is a natural number;

3^0) the Condition (A_7'.b) holds.

Proof. Similar to the proof of Proposition 5.1. □

Remark 7.1. Note that under Condition (A_6) we can assume that $\varkappa_r=\frac{\psi_r-\psi_{r-1}}{2(s_r-s_{r-1})}$ $(r=1,...,p)$ are natural numbers (otherwise we can obtain this by the change of the variable).

Remark 7.2. Under Conditions (A_1), (A_6) and (A_7') (with respect to Remark 7.1) if $\varkappa_r=\frac{\psi_r-\psi_{r-1}}{2(s_r-s_{r-1})}$ $(r=1,...,p)$ then the operator M_r is an ordinary differential operator.

Let M_r $(r=1,...,p)$ (introduced in (7.1)) be an ordinary differential operator satisfying the conditions of Proposition 7.1. We introduce the following equation (which is the characteristic equation of the operator M_r):

The theorem remains valid with this definition of regular degeneration.

$$\lambda^{2\varkappa_r k_n}Q_r\left(\lambda\right)\equiv\lambda^{2\varkappa_r k_n}\sum_{\substack{\alpha_n e^n,\beta_n e^n\in\mathcal{N} \\ (\alpha_n+\beta_n)\in[2k_n+2s_{r-1},2k_n+2s_r] \\ \psi(\alpha_n e^n,\beta_n e^n)-\varkappa_r(\alpha_n+\beta_n)=\gamma_r}}\eta_{\alpha_n e^n,\beta_n e^n}\left(x^{(n)},0,0\right)\lambda^{\alpha_n+\beta_n-2\varkappa_r k_n}=0. \tag{7.2}$$

Definition 7.1. The degeneration of the Problem $\mathfrak{D}_\varepsilon$ into Problem $\mathfrak{D}_0$ is called regular if the Conditions (A_1), (A_6) and (A_7') hold and if for every $r=1,...,p$ and $(x_1,...,x_{n-1})\in\mathbb{R}^{n-1}$ the characteristic polynomial $Q_r\left(\lambda\right)$ has exactly s_r pairwise different roots with negative real parts.

Theorem 6.1 remains valid with this definition of regular degeneration.

References

[1] J. Awrejcewicz, V.A Krysko, *Introduction to asymptotic methods,* T&F G., Boca Raton, 2006.

[2] O.V. Besov, V.P. Il'in, S.M. Nikol'skii, *Integral representations of functions and embedding Theorems,* Nauka, Moscow, 1977 (in Russian). English transl. J.Wiley and Sons, New York. Vol. 1,2, 1979.

[3] G.G. Ghazaryan, A.G. Karapetyan, *On the convergence of Galerkin approximations to the solution of the Dirichlet problem for some general equations,* Mat. Sb. (N.S.) 124(166):3(7) (1984), 291–306

[4] G.G. Ghazaryan, *On density of smooth functions in* $\mathring{W}_p^r(\Omega)$*,* Mat. Notes 2 (1967), no.1, 45-52.

[5] L. Hörmander, *The analysis of linear partial differential operators, derivatives,* Sringer-Verlag, Berlin, Heidelberg, New-York, Tokyo, 1983.

[6] E.M. Jager, Jiang Furu, *The theory of singular perturbations,* Elsevier, North-Holland Series in Applied Mathematics and Mechanics 42, Amsterdam, 1996.

[7] R.S. Johnson, *Singular perturbation theory,* Techniques with applications to engineering, Springer, 2005.

[8] G.A. Karapetyan, H.G. Tananyan, *Degeneration of semielliptic equations with constant coefficients in rectangular parallelepipeds,* Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences) 45 (2010), no. 2, 82–93.

[9] G.A. Karapetyan, *Regular equations depending on a parameter,* Izvestiya AN ArmSSR. 25 (1990), no. 2, 192-202. (in Russian).

[10] V.P. Mikhailov, *Behavior at infinity of a certain class of polynomials,* Trudy Mat. Inst. Steklov. 91 (1967), 59–80 (in Russian). English Transl.

[11] V.P. Mikhailov, *The first boundary value problem for quasi-elliptic and quasi-parabolic equations,* Trudy Mat. Inst. Steklov. 91 (1967), 81–99 (in Russian). English transl.

[12] A.H. Nayfeh, *Perturbation methods,* Wiley-VCH, 2004.

[13] S.M. Nikol'skii, *The first boundary problem for a general linear equation,* Doklady AN USSR Ser. Mat. 146 (1962), no. 4, 767 - 769 (in Russian). English transl.

[14] S.M. Nikol'skii, *A proof of the uniqueness of the classical solution of the first boundary-value problem for a general linear partial differential equation in a convex bounded region,* Izvestiya AN USSR Ser. Mat. 27 (1963), no. 5, 1113–1134 (in Russian). English transl.

[15] S.M. Nikol'skii, *A variational problem,* Mat. Sb. (N.S.) 62(104):1 (1963), 53–75 (in Russian). English transl.

[16] E. Pehkonen, *Ein hypoelliptisches Diriclet Problem,* Com. Mat. Phys. 48 (1978), no. 3, 131–143.

[17] L.S. Pontryagin, *Ordinary differential equations,* Nauka, Moscow, 1982. (in Russian).

[18] H.G. Tananyan, *The finite element method for linear differential semielliptic equations,* Vestnik RAU, Physical, Mathematical and Natural Sciences. 2 (2009), 52–60. (in Russian)

[19] H.G. Tananyan, *The small parameter method for semi-elliptic equations with constants coefficients in the half space,* Mathematics in Higher School. 6 (2010), no. 1, 37-46. (in Russian)

[20] H.G. Tananyan, *On the uniform solvability of boundary value problem for one class of singularly perturbed regular equations,* Proc. A. Razmadze Math. Inst., Tbilisi. 152 (2010), 111–128.

[21] V.A. Trenogin, *The development and applications of the asymptotic method of Lyusternik and Vishik,* Russian Math. Surveys. 25 (1970), no. 4, 119–156. (in Russian). English tranls.

[22] F.Verhulst, *Methods and applications of singular perturbations, boundary layers and multiple timescale dynamics,* Springer, 2005.

[23] M.I. Vishik, L.A. Lyusternik, *Regular degeneration and boundary layer for linear differential equations with small parameter,* Uspekhi Mat. Nauk. 12 (77) (1957), no. 7, 3–122. (in Russian). English transl.

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