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ON THE BOUNDARY BEHAVIOUR OF FUNCTIONS IN THE DJRBASHYAN CLASSES U_α AND A_α

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Abstract. Nevanlinna factorization theorem was essentially extended in a series of papers by M.M. Djrbashyan for classes A_α and U_α introduced by him, see [2], [3]. In this paper we pay particular attention to non vanishing functions $f \in A_\alpha(-1 < \alpha < 0)$ and show that for any θ except at most a set of zero $(1+\alpha)$ - capacity we have $|\ln|f(z)|| = o((1-|z|)^{1+\alpha})$ as $z \to e^{i\theta}$.

1 Introduction

We start with some preliminary information on the M.M. Djrbashyan classes A_α and U_α (see [2], Chapter IX). The Riemann-Liouville fractional integral of order α $(0 < \alpha < \infty)$ of a function $f(z)$ given in the unit circle $U = \{z; |z| < 1\}$ is defined as follows:

$$D^{-\alpha} f(re^{i\theta}) = \frac{1}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} f(te^{i\theta}) dt.$$

For $\alpha = 0$ D^0 is defined as the identity operator, i.e.

$$D^0 f(z) = f(z), \ z \in U.$$

The fractional derivative of order α $(0 < \alpha < \infty)$ is defined as

$$D^\alpha f(re^{i\theta}) = \frac{\partial^p}{\partial r^p} \{D^{-(p-\alpha)} f(re^{i\theta})\}, \ re^{i\theta} \in U,$$

where $p \geq 1$ is the so-called upper integer part of the number α, i.e. the integer defined by the inequalities $p - 1 < \alpha \leq p$.

For any values of the parameter $\alpha \in (-1, \infty)$ we consider the following kernels of Cauchy, Schwarz and Poisson:

$$C_\alpha(z) = \frac{\Gamma(1+\alpha)}{(1-z)^{1+\alpha}}, \ z \in U.$$

$$S_\alpha(z) = 2C_\alpha(z) - C_\alpha(0) = \Gamma(1+\alpha)\left\{\frac{2}{(1-z)^{1+\alpha}} - 1\right\},\ z \in U$$

$$P_\alpha(\varphi, r) = \Re S_\alpha(re^{i\varphi}), re^{i\varphi} \in U$$

The class $A_\alpha(-1 < \alpha < \infty)$ is defined as a class of all functions $f(z)$ analytic in the circle U, for which the integrals

$$m_\alpha(r,f) = \frac{r^{-\alpha}}{2\pi}\int\limits_0^{2\pi} D_{(+)}^{-\alpha} \ln|f(re^{i\theta})|d\theta$$

are bounded as $r \to 1-0$, where

$$D_{(+)}^{-\alpha}\varphi(r) = \max\left\{D^{-\alpha}\varphi(r), 0\right\}.$$

With the increase of the parameter α, the classes $A_\alpha(-1 < \alpha < \infty)$ monotonically expand, thus

$$A_\alpha \subset A_0, (-1 < \alpha < 0),$$

where $A_0 \equiv N$ is the well known Nevanlinna class of analytic functions.

The classes $U_\alpha(-1 < \alpha < \infty)$ of functions $u(z)$ harmonic in the circle U are defined by the condition

$$\sup_{0<r<1}\left\{\int\limits_{-\pi}^{\pi}\left|u_\alpha(re^{i\varphi})\right| d\varphi\right\} = M_\alpha < +\infty,$$

where $u_\alpha(re^{i\varphi}) = r^{-\alpha}D^{-\alpha}u(re^{i\varphi})$.

It is known that the class U_α $(0 < \alpha < \infty)$ coincides with the set of all harmonic functions $u(z)$, represented in the form

$$u(z) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} P_\alpha(\varphi - \theta, r)d\psi(\theta), z = re^{i\varphi} \in U. \qquad (A)$$

where $\psi(\theta)$ is a real-valued function with a finite variation on $[-\pi, \pi]$.

Any non-negative, normalized, i.e. $\mu[0, 2\pi] = 1$ totally additive set function μ defined on $\{B\}$ is said to be a measure. Consider the system of all Borel measurable sets $\{B\}$ that lie in $[0, 2\pi]$. See, for example [3].

We say that the measure μ is concentrated on $E \subset [0.2\pi]$, and write $\mu \prec E$, if

$$\int\limits_E d\mu = \int\limits_0^{2\pi} d\mu = \mu(E) = 1.$$

The potentials are defined as usual

$$U_0^{[\mu]}(z) = \int\limits_E \ln\frac{1}{|z-\xi|}d\mu(\xi),$$

$$U_\alpha^{[\mu]}(z) = \int\limits_E \frac{1}{|z-\xi|^\alpha} d\mu(\xi), \quad (0 < \alpha < 2).$$

Let

$$V_\alpha = \inf_\mu \sup_{z \in U} U_\alpha^{(\mu)}(z), \quad (0 \le \alpha < 2).$$

The capacity of the order α of a set E is defined as

$$C_\alpha(E) = \frac{1}{V_\alpha}.$$

If $V_\alpha = \infty$, then the corresponding set has zero α-capacity.

2 Main results

First, we prove the following lemma.

Lemma 2.1. *Let the point $z \in U$ tends to the boundary point $e^{i\vartheta}$ on the ray that forms a certain angle $0 \le \beta < \frac{\pi}{2}$ with passing through the $e^{i\vartheta}$ radius of the circle U, i.e. $|\vartheta - \arg(e^{i\vartheta} - z)| = \beta$, and $v(z) = (1-|z|)^{1+\alpha} \Re \left[\frac{2e^{i\vartheta(\alpha+1)}}{(e^{i\vartheta}-z)^{1+\alpha}} - 1 \right]$. Then*

$$\lim_{n\to\infty} v(z) = 2(\cos\beta)^{1+\alpha} \cos[(1+\alpha)\beta], \ (-1 < \alpha < \infty) \tag{2.1}$$

Proof. Without loss of generality we can assume that $\vartheta = 0$. Then the equation of the ray, passing through the point 1 and forming the angle β with the radius connecting the points 0 and 1, takes the form

$$z = 1 + i(1-t)\tan\beta, \quad 0 \le t \le 1.$$

Denoting this ray by L_β, we note that if $z \to 1$ through this ray, then $t \to 1 - 0$. It is easily seen that

$$\lim_{z\to 1, z\in L_\beta} v(z) = \lim_{z\to 1, z\in L_\beta} (1-|z|)^{1+\alpha} \Re \left[\frac{2}{(1-z)^{1+\alpha}} - 1 \right]$$

$$= \lim_{t\to 1^-} \left(1 - \sqrt{t^2 + (1-t)^2\tan^2\beta} \right)^{1+\alpha} \Re \left[\frac{2}{(1-t-i(1-t)\tan\beta)^{1+\alpha}} - 1 \right]$$

$$= \lim_{t\to 1^-} \left(\frac{1 - t^2 - (1-t)^2\tan^2\beta}{1 + \sqrt{t^2 + (1-t)^2\tan^2\beta}} \right)^{1+\alpha}$$

$$\times \Re \left[\frac{2(1+i\tan\beta)^{1+\alpha} - (1-t)^{1+\alpha}(1+\tan^2\beta)}{(1-t)^{1+\alpha}(1+\tan^2\beta)^{1+\alpha}} - 1 \right]$$

$$= 2(\cos\beta)^{1+\alpha} \Re[\cos\beta + i\sin\beta]^{1+\alpha} = 2(\cos\beta)^{1+\alpha}\cos[(1+\alpha)\beta].$$

□

Theorem 2.1. *Let $f(z) \in A_\alpha$, $(-1 < \alpha < 0)$ has no zeros. Then*

$$\lim_{z \to e^{i\theta}} (1 - |z|)^{1+\alpha} \cdot |\ln|f(z)|| = 0 \tag{2.2}$$

for all $\theta \in [0, 2\pi]$, except, perhaps, some unique set with $(1 + \alpha)$ - capacity equal to zero.

Proof. If for some point $e^{i\theta}$ equality (2) does not take place, then there exists a sequence $\{z_n\} \subset U$, such that $z_n \to e^{i\theta}$ as $n \to \infty$ and

$$\lim_{z \to \infty} (1 - |z_n|)^{1+\alpha} \cdot |\ln|f(z_n)|| = d \neq 0 \tag{2.3}$$

It is known that (see [1]) if for harmonic function $\ln|f(z)| \in U_\alpha$ equality (2.3) takes place, then $\ln|f(z)|$ has the following representation

$$\ln|f(z)| = u(z) + s\Gamma(1+\alpha)\Re\left[\frac{2e^{i(1+\alpha)\theta}}{(e^{i\theta} - z)^{1+\alpha}} - 1\right], \tag{2.4}$$

where $u(z)$ is a harmonic function from the class U_α, such that

$$\lim_{z \to e^{i\theta}} (1 - |z|)^{1+\alpha} u(z) = 0 \tag{2.5}$$

and $2\pi s$ is the jump of the function $\psi(y)$ at the point θ in the presentation

$$\ln|f(z)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_\alpha(\varphi - \gamma, r) d\psi(\gamma), \;\; z = re^{i\varphi} \in U$$

For Nevanlinna classes $A_0 \equiv N$ this result is obtained in [4].

In view of (2.5), multiplying (2.4) by $(1 - |z|)^{1+\alpha}$ we have

$$(1 - |z|)^{1+\alpha} \ln|f(z)| = s \cdot \Gamma(1+\alpha)(1 - |z|)^{1+\alpha} \cdot Re\left[\frac{2e^{i(1+\alpha)\theta}}{(e^{i\theta} - z)^{1+\beta}} - 1\right]$$

Hence, by lemma 1, we obtain

$$\lim (1 - |z|)^{1+\alpha} \ln|f(z)| = 2s\Gamma(1+\alpha)$$

where $z \to e^{i\theta}$ through the radius $(0, e^{i\theta})$.

It is shown in [6] that for existence of a function in a class $A_\alpha(-1 < \alpha < 0)$, for which radial boundary values on Borel measurable set E on a circle does not exist or are equal to zero, it is necessary and sufficient that the $(1+\alpha)$- capacity of E is equal to zero. □

Analogously we prove the following statement.

Theorem 2.2. *Let the function $V(z)$ be harmonic in the U. If for some $-1 < \alpha < 0$*

$$\overline{lim} (1 - |z|)^{1+\alpha} \cdot |V(z)| = d > 0 \tag{2.6}$$

as $z \to e^{i\theta}$ along the tangent to the circle $|z| = 1$, then $V(z)$ does not belong to U_α.

Proof. Without loss of generality we assume $\theta = 0$. Thus $z \to 1$ along the ray L_β forming the angle $0 < \beta < \frac{\pi}{2}$ with the radius passing through the point 1. Therefore, in view of Lemma 1, relation (l) takes place. Now assume the contrary, i.e. $V(z) \in N_\alpha$. Then it follows (see [4]) that $V(z)$ can be represented in the form

$$V(z) = V_1(z) + s\Gamma(1+\alpha)\Re\left[\frac{2}{(1-z)^{1+\alpha}} - 1\right], \tag{2.7}$$

where $2\pi s$ is the jump of the function $\psi(\theta)$ at the point $\theta = 0$ and (A) is such a function that

$$\lim_{z\to 1}(1-|z_n|)^{1+\alpha}|V_1(z)| = 0. \tag{2.8}$$

Multiplying (2.7) by $(1-|z|)^{1+\alpha}$ we get

$$(1-|z|)^{1+\alpha}V(z) = (1-|z|)^{1+\alpha}V_1(z) + s\Gamma(1+\alpha)(1-|z|)^{1+\alpha}\Re\left[\frac{2}{(1-z)^{1+\alpha}} - 1\right].$$

Hence

$$(1-|z|)^{1+\alpha}|V(z)| \le (1-|z|)^{1+\alpha}|V_1(z)| + s\Gamma(1+\alpha)(1-|z|)^{1+\alpha}\left|\Re\left[\frac{2}{(1-z)^{1+\alpha}} - 1\right]\right|. \tag{2.9}$$

Further, in view of (2.6), there is a sequence $\{z_n\}_1^\infty \subset U$ such that $z_n \to 1$ as $n \to 1$ and all terms $z_n,\ n = 1, 2, ...$ lie on some tangent to the circle $|z| = 1$, and

$$\lim_{z\to\infty}(1-|z_n|)^{1+\alpha}|V(z_n)| = d > 0. \tag{2.10}$$

By (2.8) it follows that

$$\lim_{z\to\infty}(1-|z_n|)^{1+\alpha}|V(z_n)| = 0. \tag{2.11}$$

Now, we calculate the limit

$$\lim_{z\to\infty}(1-|z_n|)^{1+\alpha}\left|\Re\left[\frac{2}{(1-|z_n|)^{1+\alpha}} - 1\right]\right|. \tag{2.12}$$

It is easily seen, that every point $z_n,\ n = l, 2, ...$ lies on some ray L_{β_n}, which was defined in the proof of Lemma 2.1. Since $z_n \to 1$ as $n \to \infty$ along the tangent to the circle $|z| = 1$ then $\beta_n \to \frac{\pi}{2}$. Thereby using (2.l) we have

$$(1-|z_n|)^{1+\alpha}\left|\Re\left[\frac{2}{(1-|z_n|)^{1+\alpha}} - 1\right]\right| \sim 2(\cos\beta_n)|\cos[(1+\alpha)\beta_n]| \quad \text{as } n \to \infty.$$

Therefore, in view of (2.9), for the sequence z_n we obtain

$$(1-|z_n|)^{1+\alpha}|v(z_n)| \le (1-|z_n|)^{1+\alpha}|v_1(z_n)| + 2s\Gamma(1+\alpha)(\cos\beta_n)|\cos[(1+\alpha)\beta_n]|,$$

$n = 1, 2, ...$, which implies that

$$s \ge \frac{(1-|z_n|)^{1+\alpha}|v(z_n)| - (1-|z_n|)^{1+\alpha}|v_1(z_n)|}{2\Gamma(1+\alpha)(\cos\beta_n)^{1+\alpha}|\cos[(1+\alpha)\beta_n]|}, \quad n = 1, 2, ...$$

As we mentioned $\beta_n \to \frac{\pi}{2}$ as $n \to \infty$. Thus, in view of (8) and (10), passing to the limit as $n \to \infty$ we obtain $s = +\infty$, but the jump of a function with a finite variation cannot be infinite. Therefore we arrive at a contradiction which completes the proof of theorem 2.2. $\square$

Remark. For Nevanlinna classes $A_0 \equiv N$ this theorem was proved in [4].

By this theorem and Lemma 1 the following statements hold.

Theorem 2.3. *If $f(z) \in A_\alpha$ $(-1 < \alpha < 0)$ has no zeros and $z = re^{i\varphi} \to e^{i\theta}$ along any tangent path, then*

$$\overline{lim}(1 - |z|)^{1+\alpha} \cdot \ln |f(z)| = 0$$

for all $\theta \in [0, 2\pi]$.

Theorem 2.4. *Let $f(z) \in A_\alpha$ $(-1 < \alpha < 0)$ and have no zeros. If*

$$\overline{lim}(1 - |z|)^{1+\alpha} \cdot \ln |f(z)| = d \neq 0,$$

then the angular boundary values (see [5]) of the function $(1 - |z|)^{1+\alpha} \cdot In|f(z)|$ as $z \to e^{i\theta}$ do not exist. Furthermore, the $(1 + \alpha)$ - capacity of the set where angular boundary values do not exist is equal to zero.

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