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ON ANGULAR BOUNDARY LIMITS OF NORMAL SUBHARMONIC FUNCTIONS

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Abstract. The paper considers normal subharmonic functions defined in the unit circle. New necessary and sufficient conditions for the existence of angular limits at arbitrary points of the unit circumference are obtained. The obtained conditions are less strong than in the previous results of different authors. Examples confirming the significance of these conditions are given.

1 Introduction

A number of papers investigate the angular limits at an arbitrary point of the unit circumference for harmonic and subharmonic functions defined in the unit circle, see [2-6, 8-16].

We need the following notation (see [7]). We denote by D the unite circle $|z| < 1$, by Γ denote its circumference $|z| = 1$, by $h(\xi, \varphi)$ the chord of D with the endpoint $\xi = e^{i\theta} \in \Gamma$, and by φ the angle between this chord and the radius at ξ, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$. Let $\Delta(\xi, \varphi_1, \varphi_2)$ be a subdomain of D bounded by the chords $h(\xi, \varphi_1)$ and $h(\xi, \varphi_2)$. The domain $\Delta(\xi, \varphi_1, \varphi_2)$ is called the *Stoltz angle* with the vertex $\xi = e^{i\theta} \in \Gamma$. If the opening of the angle is not important, then for brevity we write $\Delta(\xi)$ instead of $\Delta(\xi, \varphi_1, \varphi_2)$.

Considering the circle D as a model of the Lobachevski plane, we denote by $\sigma(z_1, z_2)$ the non-Euclidean distance between the points z_1, z_2 in the circle D:

$$\sigma(z_1, z_2) = \frac{1}{2} \ln \frac{1+u}{1-u}, \quad \text{where} \ \ u = \left| \frac{z_1 - z_2}{1 - z_1 z_2} \right|.$$

Consider a real-valued function $f(z)$. Let $\xi \in \Gamma$ be a cluster point of a subset $S \subset D$. By $C(f, \xi, S)$ we denote the cluster set of the function $f(z)$ at point ξ with respect to the set S, i.e.

$$C(f, \xi, S) = \bigcap \overline{f(S \cap U(\xi))},$$

where the intersection is taken over all neighborhoods $U(\xi)$ of the point ξ, and the overline stands for the set closure with respect to the two-points compactification $\overline{R}$ of the set $R = (-\infty, +\infty)$ by adding two symbols $-\infty$ and $+\infty$.

We say that a point $\xi \in \Gamma$ belongs to the set $F(f)$, if $C(f, \xi, \Delta(\xi))$ consists of the unique value α. In this case α is called *the angular limit* of the function $f(z)$ at the point $\xi \in \Gamma$. The set $F(f)$ is called the set of the Fatou points for the function $f(z)$. We say that a point $\xi \in \Gamma$ belongs to the set $K(f)$ for a function $f(z)$, if for any pair of angles $\Delta(\xi, \varphi_1, \varphi_2)$ and $\Delta(\xi, \varphi_1', \varphi_2')$ with common vertex ξ we have

$$C(f, \xi, \Delta(\xi, \varphi_1, \varphi_2)) = C(f, \xi, \Delta(\xi, \varphi_1', \varphi_2')).$$

A meromorphic function is called *normal*, if it generates a normal family in the group T of all conform automorphisms of its domain of definition. This notion was generalized to harmonic and subharmonic functions (see [15]). In the case of the unit circle D the group T has the following form

$$T = \left\{ S(z) : S(z) = e^{i\alpha}(z + a)(1 + \overline{a}z)^{-1},\ a \in D,\ \alpha \text{ is arbitrary real number} \right\}.$$

Using the notation from the paper [7], we say that a real-valued function $f(z)$ belongs to the class $\Re$, if in the group T of all conformal automorphisms of the unit circle D the function $f(z)$ generates a family $\Phi : \{f(S(z)) : S(z) \in T\}$, which is normal in D in Montél's sense, i.e. for any sequence $\{f(S_n(z))\}$ from Φ, where $S_n(z) \in T$, there exists a subsequence $\{f(S_{n_k}(z))\}$, which uniformly converges in any compact K in D or uniformly diverges in K to $-\infty$ or $+\infty$.

By *meas* M we denote the Lebesgue measure of a set M. If a property holds everywhere on M, *meas* $M > 0$, except a set $E \subset M$, *meas* $E = 0$, then we say that this property holds almost everywhere on M. A non-negative subharmonic function $f(z)$ is called *logarithmic subharmonic*, if $\ln f(z)$ is subharmonic.

We need the following theorem from [4].

Theorem A. *Let $f(z)$ be a subharmonic function in the class $\Re$. Then the function $f(z)$ possesses an angular limit α at a point $\xi \in \Gamma$ exits if and only if the following two conditions are fulfilled:*

1) the cluster set $C(f, \xi, \Delta(\xi))$ is bounded from above by α,

2) there exists a Jordan curve L which completely lies in some angle $\Delta(\xi)$ with the vertex at the point ξ and is such that

$$\lim_{z \to \xi, z \in L} f(z) = \alpha.$$

2 Main result

The main result of the present paper is the following theorem.

Theorem 2.1. *Let $f(z)$ be a subharmonic function in the class $\Re$. The function $f(z)$ possesses an angular limit α at a point $\xi \in \Gamma$ if and only if:*

1) there exists a sequence $z_n \to \xi \in \Gamma$ belonging to an angle $\Delta(\xi, \varphi_1, \varphi_2)$, such that the limit $\lim\limits_{n \to \infty} f(z_n)$ exists and

$$\overline{\lim_{n \to \infty}} \sigma(z_n, z_{n+1}) \le M < +\infty; \tag{2.1}$$

and

2) the cluster set $C(f,\xi,\Delta(\xi))$ *is bounded from above by the number* $\alpha \in C(f,\xi,\Delta(\xi))$ *for any angle* $\Delta(\xi)$ *which contains the sequence* $\{z_n\}$.

Proof. The necessity is obvious. The sufficiency follows by the following lemma. □

Lemma 2.1. *Let* $f(z)$ *be a subharmonic function in the class* $\Re$ *defined in* D. *If there exists a sequence* $z_n \to \xi \in \Gamma$ *such that:*

$\{z_n\}$ *belongs to an angle* $\Delta(\xi)$,

$\{z_n\}$ *satisfies the condition* (2.1),

there exists the limit $\lim\limits_{n\to\infty} f(z_n)$

the cluster set $C(f,\xi,\Delta(\xi))$ *is bounded from above by some number* $\alpha \in C(f,\xi,\Delta(\xi))$

for any angle $\Delta(\xi)$ *which contains the sequence* $\{z_n\}$, *then the function* $f(z)$ *has the angular limit* α *at point* $\xi \in \Gamma$.

Proof. Without loss of generality, we can assume that $\xi = 1$. Connecting the points of the sequence $\{z_n\}$ by non-Euclidean segments, we obtain a curve L, which lies in some angle $\Delta(1,\varphi_1',\varphi_2')$.

First we prove that the number α is the least upper bound of the cluster set $C(f,\xi,\Delta(\xi))$ for every angle $\Delta(\xi)$, which contains the curve L. Indeed, assuming the contrary we obtain that there exist an angle $\Delta(\xi,\varphi_1,\varphi_2)$, which contains the curve L, and a number $\beta \neq \alpha$, which is the least upper bound of the cluster set $C(f,\xi,\Delta(\xi,\varphi_1,\varphi_2))$. Since for continuous function $f(z)$ the cluster set $C(f,\xi,\Delta(\xi))$ is closed and connected in any angle $\Delta(\xi)$, then the number β belongs to the cluster set $C(f,\xi,\Delta(\xi,\varphi_1,\varphi_2))$. There exist elements from this set, which are greater than α. But this the contradicts the conditions of Lemma 2.1.

In the case $\beta < \alpha$, since the cluster set $C(f,\xi,\Delta(\xi,\varphi_1,\varphi_2))$ is closed and connected in any angle, there exist the elements from this set, which are greater than β and less than or equal to α. But this contradicts the assumption that $\beta = \sup C(f,\xi,\Delta(\xi,\varphi_1,\varphi_2))$.

Now consider an angle $\Delta(\xi,\varphi_1',\varphi_2') \supset \Delta(\xi,\varphi_1,\varphi_2)$ and a sequence $l_n \in \Delta(\xi,\varphi_1,\varphi_2)$ such that $l_n \to \xi$ and $f(l_n) \to \alpha$ as $n \to \infty$. Through l_n we draw the non-Euclidean perpendiculars E_n to the radius through the point ξ. By q_n denote the intersection point $L \cap E_n$. By properties of the non-Euclidean geometry of the circle $\sigma(l_n,q_n) \le M_1$ for $n \in N$, where N is the set of natural numbers and M_1 is some finite number, the points q_k lie between z_{n_k} and z_{n_k+1} for $n \in N$. For any $n \in N$ consider the following map

$$S_n(z) = \frac{z + z_n}{1 + \overline{z_n}\, z}.$$

Obviously we have $S_n(0) = z_n$ for any n. By l_n' and q_n' we denote the pre-images: $S_n(l_n') = l_n$ and $S_n(q_n') = q_n$. Consider the compact

$$K = \big\{ z:\ |z| \le \tanh(M + M_1 + 1) \big\},$$

for which the set $S_n(K)$ lies in some angle $\Delta(\xi) \supseteq \Delta(\xi,\varphi_1',\varphi_2')$. By properties of the non-Euclidean geometry of the circle D, the set K is a closed non-Euclidean circle with the center at origin and the non-Euclidean radius $M + M_1 + 1$. The metric σ is invariant with respect to the maps $S_n(z)$, $n \in N$. Therefore, q'_n and l'_n with their cluster points lie within K. Since $f(z)$ is the subharmonic function from class R in D, there exists a subsequence $f(S_{n_k}(z))$, which uniformly converges in the compact K to subharmonic function $F(z)$ or uniformly diverges in K to $-\infty$ or to $+\infty$. Since $S_n(K) \subset \Delta(\xi)$ and the cluster set $C(f, \xi, \Delta(\xi))$ is bounded from above by the number α, then $F(z) \leq \alpha$. If $F(z_0) = \alpha$ for a cluster point z_0 of the sequence l'_n, then using the maximum principle for subharmonic functions we get $F(z) \equiv \alpha$ for $z \in K$. Therefore,

$$F(0) = \lim_{k\to\infty} f(S_{n_k}(0)) = \lim_{k\to\infty} f(z_{n_k}) = \alpha.$$

By the conditions of Lemma 2.1 we get

$$\lim_{n\to\infty} f(z_n) = \alpha. \tag{2.2}$$

In the case $\alpha = +\infty$, by definition of normality we have $F(z) \equiv +\infty$ for $z \in K$, hence (2.2) is valid. Let us prove that

$$\lim_{z\to\xi,\ z\in L} f(z) = \alpha. \tag{2.3}$$

Assuming the contrary we have a sequence $\{t_n\}$ such that $t_n \to \xi$, $t_n \in L$ and $\lim_{n\to\infty} f(t_n) = \beta \neq \alpha$. Without loss of generality, we can assume that the sequences $\{t_k\}_{k=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are not the same and the points t_k lie between z_{n_k} and z_{n_k+1} for any k. Therefore,

$$\sigma(t_k, z_{n_k}) \leq \sigma(z_{n_k}, z_{n_k+1}) \leq M < +\infty.$$

For arbitrary $k \in N$ consider the following map

$$S_{n_k}(z) = \frac{z + z_{n_k}}{1 + \overline{z_{n_k}}\, z}.$$

It is obvious, that for any k we have $S_{n_k}(0) = z_{n_k}$. Using the invariance of the metric σ we get that the points t'_k, where $S_{n_k}(t'_k) = t_k$, with their cluster points lie within the compact K. Repeating the above argument we obtain

$$\lim_{k\to\infty} F(t'_k) = \lim_{k\to\infty} f(S_{n_k}(t'_k)) = \lim_{k\to\infty} f(t_k) = \alpha,$$

which contradicts $\beta \neq \alpha$. Thus the conditions of Theorem A are fulfilled. By Theorem A the function $f(z)$ has the angular limit α. □

Remark 2.1. Stronger conditions for the existence of angular limits at an arbitrary point $\xi \in \Gamma$ for subharmonic functions in the class $\Re$ were obtained by J. Meek in [14].

3 Applications

The following theorem is an application of Theorem 2.1.

Theorem 3.1. *Let $f(z)$ be a subharmonic function in the class $\Re$. The function $f(z)$ possesses the angular limit at a point $\xi \in K(f)$ if and only if there exists a sequence $z_n \to \xi \in \Gamma$ such that $\{z_n\}$ belongs to an angle $\Delta(\xi, \varphi_1, \varphi_2)$, there exists the limit $\lim_{n\to\infty} f(z_n)$ and*

$$\overline{\lim_{n\to\infty}} \sigma(z_n, z_{n+1}) \leq M < +\infty.$$

Proof. Since $\xi \in K(f)$, then for any angle $\Delta(\xi)$ the cluster set $C(f, \xi, \Delta(\xi))$ is bounded from above by some number $\alpha \in C(f, \xi, \Delta(\xi))$. It remains to apply Theorem 2.1. □

Remark 3.1. Note that for an arbitrary function $f(z)$ defined in D, the Lebesgue measure of the set $K(f)$ is equal to 2π, and almost everywhere in Γ the cluster set of a normal subharmonic function $|\mu(z)|$ is $[0, +\infty]$, where $\mu(z)$ is a modular function. Hence in Theorems 2.1 and 3.1 we can not change the sequence $\{z_n\}$ satisfying the condition (2.1) by an arbitrary sequence $\{z_n\} \to \xi$, $z_n \in \Delta(\xi)$. The example $f(z) = \arg(1 - z)$ shows that in Theorems 2.1 and 3.1 we cannot omit the condition $\alpha \in C(f, \xi, \Delta(\xi))$ for any angle $\Delta(\xi)$.

Let $f(z)$ be a bounded subharmonic function in D, $h(\xi, \varphi)$ be the chord of D with endpoint $\xi = e^{i\theta} \in \Gamma$, $\xi \in K(f)$ and assume there exist the coinciding chordal limits $\lim_{z\in h(\xi,\varphi)} f(z) = 0$ for almost all $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Let us show that in this case the existence of the angular limit is important. To this end consider a subharmonic function constructed by M. Tsuji [16], p. 175. This function possesses the following properties: $-1 \leq f(z) \leq 0$ and almost everywhere in Γ except a set E_1, *meas* $E_1 = 0$ the function $f(z)$ has chordal limits

$$\lim_{z\to\xi\in\Gamma,\, z\in h(\xi,\varphi)} f(z) = 0$$

for almost all $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. However, at each point $\xi \in \Gamma \setminus E_1$, hence in the set

$$E_2 = K(f) \cap (\Gamma \setminus E_1), \quad meas\ E_2 = 2\pi,$$

the function $f(z)$ has the vanishing chordal limits for almost all $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore, at almost each point $\xi \in K(f)$ the function $f(z)$ has the vanishing chordal limits for almost all $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. However, at each such point the function $f(z)$ has not the angular limit, because in any angle $\Delta(\xi)$ there exists a sequence $z_n \to \xi$ such that $\lim_{n\to\infty} f(z_n) = -1$.

From Lemma 2.1 and Theorem 3.1 we can obtain the following theorem.

Theorem 3.2. *Let $f(z)$ be a subharmonic function defined in D in the class $\Re$. If $\xi \in K(f)$ and there exists a sequence $z_n \to \xi \in \Gamma$ from some angle $\Delta(\xi, \varphi_1, \varphi_2)$ such that $\lim_{n\to\infty} f(z_n) = \alpha$ and condition* (2.1) *holds, then α is the angular limit of the function $f(z)$ at the point ξ.*

Remark 3.2. Note that in [5] an analog of Theorem 3.2 was obtained, under the condition that a non-tangent curve L non-tangent to Γ is considered instead of the sequence $\{z_n\}$.

Remark 3.3. In [15] D. Rung studied the problem of the existence of the vanishing angular limits for logarithmic subharmonic functions from the class $\Re$ at arbitrary point $\xi \in \Gamma$. Later on the authors in [6] reduced the curve L non-tangential to Γ to a sequence $z_n \to \xi \in \Gamma$ from some angle $\Delta(\xi, \varphi_1, \varphi_2)$ satisfying the condition $\lim_{n\to\infty} \sigma(z_n, z_{n+1}) = 0$.

Theorem B. *Let $f(z)$ be a logarithmic subharmonic function in the class $\Re$. If there exists a sequence $z_n \to \xi \in \Gamma$ in some angle $\Delta(\xi, \varphi_1, \varphi_2)$ such that $\lim_{n\to\infty} \sigma(z_n, z_{n+1}) = 0$ and $\lim_{n\to\infty} f(z_n) = 0$, then the function $f(z)$ has the angular limit at the point ξ and this limit is equal to 0.*

The assertion of Theorem B is not valid for the weaker condition (2.1). Indeed, F. Bagemihl and W. Seidel constructed in [1] a Blaschke product $B(z)$ with zeros $\{z_n\}$, which lie on the radius at the point $\xi = 1$ and satisfy condition (2.1). The function $B(z)$ is bounded in D, and hence $B(z)$ is the normal analytic function. Therefore, the module of $B(z)$ is the normal logarithmic subharmonic function, which vanishes on the sequence $\{z_n\}$, but at the point $\xi = 1$ has not even a radial limit.

As a corollary of Theorem 3.2 we obtain the following theorem.

Theorem 3.3. *Let $f(z)$ be a logarithmic subharmonic function defined in D in the class $\Re$. If $\xi \in K(f)$ and there exists a sequence $z_n \to \xi \in \Gamma$ from some angle $\Delta(\xi, \varphi_1, \varphi_2)$ such that $\lim_{n\to\infty} f(z_n) = 0$ and condition (2.1) holds, then the function $f(z)$ has the angular limit at the point ξ and this limit is equal to 0.*

Let us prove that in Theorems 3.2 and 3.3 the condition $f(z) \in \Re$ is important. To this end we need the following lemma.

Lemma 3.1. *Let $\ln p(r)$, $0 \le r < 1$ be a positive monotonically increasing function, $\ln p(r) \to +\infty$ as $r \to 1$. Then there exists a logarithmic subharmonic function $f(z) \neq 0$ in D such that*

1) $f(z) < p(|z|)$,

2) $\lim_{r\to 1} f(re^{i\theta}) = 0$ for almost all $\theta \in [0, 2\pi]$.

Proof. Lemma 3.1 and Theorem D13 of [6] imply that there exists an analytic function $g(z) = u(z) + iv(z)$, such that

1. $g(z) < \ln p(|z|)$,
2. $\lim_{r\to 1} u(re^{i\theta}) = \lim_{r\to 1} Re\ g(re^{i\theta}) = -\infty$ for almost all $\theta \in [0, 2\pi]$.

Consider a logarithmic subharmonic function $f(z) = \exp[u(z)]$. We have

$$u(z) < |g(z)| < \ln p(|z|), \quad f(z) < \exp[\ln p(|z|)] = p(|z|),$$

i.e. property 1) holds. On the other hand

$$\lim_{r\to 1} f(re^{i\theta}) = \lim_{r\to 1} \exp[u(re^{i\theta})] = 0$$

for almost all $\theta \in [0, 2\pi]$, i.e. the property 2) holds. □

Now we can prove that in Theorems 3.2 and 3.3 the condition $f(z) \in \Re$ is important, and it can not be omitted even in the case, where $\xi \in K(f)$ and the radial limit of the function $f(z)$ at the point ξ is equal to 0. By $E_1 \subset \Gamma$ we denote a set of points, at which the property 2) of Lemma 3.1 does not hold. Obviously $meas\ E_1 = 0$. Consider the set $E_2 = K(f) \cap (\Gamma \setminus E_1)$, we have $meas\ E_2 = 2\pi$. Property 2) of Lemma 3.1 holds at any point of set E_2. Therefore, at almost every point $\xi \in K(f)$ the radial limit of function $f(z)$ exists and equals 0. However, at almost every point $\xi \in K(f)$ the function $f(z)$ have not a vanishing angular limit, because otherwise by uniqueness theorem for logarithmic subharmonic functions [12] we get $f(z) \equiv 0$, which contradicts our assumptions.

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