ISSN 2077–9879

Eurasian Mathematical Journal

2013, Volume 4, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ will also publish survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Mathematical Reviews, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Honorarium. An honorarium will be paid to the authors (at 50 $ per page, but no more than 2000 $ for a paper) and to the reviewers (at 50 $ per review). All papers will be edited with regard to usage of English. If severe editing is required, the honorarium may be reduced to 40 $ per page. Papers may be submitted in Russian. They will be translated into English, and in this case the honorarium will be 25 $ per page.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors. Only original papers are accepted for consideration. In particular, translations into English of papers already published in another language are not accepted. No other forms of scientific misconduct are allowed, such as plagiarism, falsification, incorrect interpretation of other works, incorrect citations, etc.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

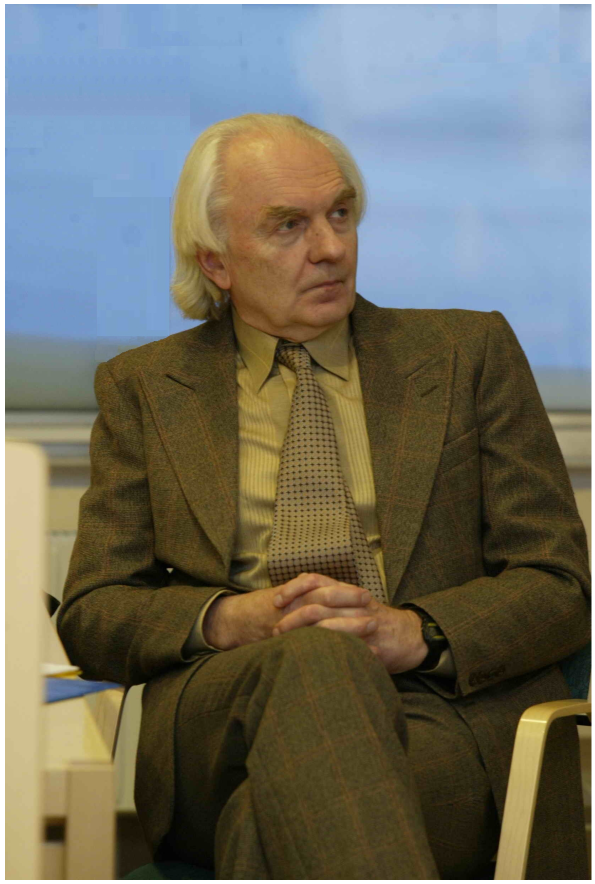
The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University,
Main Building
Room 355
Tel.: +7-7172-709500 extension 31313
2 Mirzoyan Street
010008 Astana
Kazakhstan

On the 80th birthday of Professor Oleg Vladimirovich Besov



This issue of the Eurasian Mathematical Journal is dedicated to the 80th birthday of Oleg Vladimirovich Besov, an outstanding mathematician, Doctor of Sciences in physics and mathematics, corresponding member of the Russian Academy of Sciences, academician of the European Academy of Sciences, Head of the Department of the theory of functions, V.A. Steklov Institute of Mathematics, honorary professor of the Department of Mathematics, Moscow Institute of Physics and Technology.

Oleg started scientific research while still a student of the Faculty of Mechanics and Mathematics of the M.V. Lomonosov Moscow State University. His research interests were formed under the influence of his scientific supervisor, the great Russian mathematician Sergei Mikhailovich Nikol'skii.

In the world mathematical community O.V. Besov is well known for introducing and studying the spaces $B^r_{p\theta}(\mathbb{R}^n)$, $1 \le p, \theta \le \infty$, of differentiable functions of several real variables, which are now named Besov spaces (or Nikol'skii–Besov spaces, because for $\theta = \infty$ they coincide with Nikol'skii spaces $H^r_p(\mathbb{R}^n)$).

The parameter r may be either an arbitrary positive number or a vector $r = (r_1, ..., r_n)$ with positive components r_j. These spaces consist of functions having common smoothness of order r in the isotropic case (not necessarily integer) and smoothness of orders r_j in variables x_j, $j = 1, ..., n$, in the anisotropic case, measured in L_p-metrics, and θ is an additional parameter allowing more refined classification in the smoothness property.

O.V. Besov published more than 100 papers in leading mathematical journals most of which are dedicated to further development of the theory of the spaces $B^r_{p\theta}(\mathbb{R}^n)$. He considered the spaces $B^r_{p\theta}(\Omega)$ on regular and irregular domains $\Omega \subset \mathbb{R}^n$ and proved for them embedding, extension, trace, approximation and interpolation theorems. He also studied integral representations of functions, density of smooth functions, coercivity, multiplicative inequalities, error estimates in cubature formulas, spaces with variable smoothness, asymptotics of Kolmogorov widths, etc.

The theory of Besov spaces had a fundamental impact on the development of the theory of differentiable functions of several variables, the interpolation of linear operators, approximation theory, the theory of partial differential equations (especially boundary value problems), mathematical physics (Navier–Stokes equations in particular), the theory of cubature formulas, and other areas of mathematics.

Without exaggeration, one can say that Besov spaces have become a recognized and extensively applied tool in the world of mathematical analysis: they have been studied and used in thousands of articles and dozens of books. This is an outstanding achievement.

The first expositions of the basics of the theory of the spaces $B^r_{p\theta}(\mathbb{R}^n)$ were given by O.V. Besov in [2], [3].

Further developments of the theory of Besov spaces were discussed in a series of survey papers, e.g. [10], [5], [7]. The most detailed exposition of the theory of Besov spaces was given in the book by S.M. Nikol'skii [11] and in the book by O.V. Besov, V.P. Il'in, S.M. Nikol'skii [4], which in 1977 was awarded a State Prize of the USSR. Important further developments of the theory of Besov spaces were given in a series of books by Professor H. Triebel [13], [14], [15]. Many books on real analysis and the theory of partial differential equations contain chapters dedicated to various aspects of the theory of Besov spaces, e.g. [8], [1], [6]. Recently, in 2011, Professor Y. Sawano published the book "Theory of Besov spaces" [12] (in Japanese).

Besov spaces can be defined in several different equivalent ways: via best approximations, integrals involving differences, and Fourier transforms. In the isotropic case these have the following form.

In the first case – via best approximations – the norm of a function $f \in B^r_{p\theta}(\mathbb{R}^n)$ is expressed as

$$\|f\|_{B^r_{p\theta}(\mathbb{R}^n)} = \Big(\sum_{s=0}^{\infty} \big(2^{rs} E_{2^s}(f)_p\big)^{\theta} \Big)^{\frac{1}{\theta}},$$

where $E_{2^s}(f)_p$ is the best approximation of f by entire functions of exponential type 2^s in L_p-metrics.

In the second case – via integrals involving differences – the norm of a function $f \in B^r_{p\theta}(\mathbb{R}^n)$ is expressed as

$$\|f\|_{B^r_{p\theta}(\mathbb{R}^n)} = \|f\|_{L_p(\mathbb{R}^n)} + \Big(\int\limits_{\mathbb{R}^n} \Big(\frac{\|\Delta^k_h f\|_{L_p(\mathbb{R}^n)}}{|h|^r} \Big)^{\theta} \frac{dh}{|h|^n} \Big)^{\frac{1}{\theta}},$$

where $k > r$, $\Delta_h^k f(x)$ is the difference of order $k \in \mathbb{N}$ at the point $x \in \mathbb{R}^n$ of the function f with step $h \in \mathbb{R}^n$, and the L_p-norm is taken with respect to x.

To give the definition in terms of Fourier transforms, first let ψ be an infinitely continuously differentiable function on $\mathbb{R}^n$ such that $\psi(x) := 1$ if $|x| \le 1$ and $\psi(x) := 0$ if $|x| \ge \frac{3}{2}$. Moreover, let $\varphi(x) := \psi(x/2) - \psi(x)$, $\varphi_0(x) = \psi(x)$, $\varphi_s(x) := \varphi(2^{-s+1}x)$, $s \in \mathbb{N}$. (Clearly $\{\varphi_s\}_{s=0}^{\infty}$ is a partition of unity.) Then the norm of a function $f \in B^r_{p\theta}(\mathbb{R}^n)$ is expressed as

$$\|f\|_{B^r_{p\theta}(\mathbb{R}^n)} = \Big(\sum_{s=0}^{\infty} \big(2^{rs} \|F^{-1}(\varphi_s F f)\|_{L_p(\mathbb{R}^n)} \big)^{\theta} \Big)^{\frac{1}{\theta}},$$

where $Fg, F^{-1}g$ denote the Fourier transform, the inverse Fourier transform respectively, of a function g. (The last definition allows the range of the parameters to be extended to $0 < p, \theta \le \infty$ and $-\infty < r < \infty$, assuming that f belongs to the Schwartz space $S'(\mathbb{R}^n)$ of tempered distributions.)

There are also several other equivalent definitions: via moduli of continuity, boundary values of solutions to the Laplace equation, initial values of solutions to the heat equation, interpolation, etc. Many of them are collected in books [10], [4], [1], [13] and survey paper [5].

Let $r \in \mathbb{N}$. If $p = \theta = 2$, then the space $B^r_{2,2}(\mathbb{R}^n)$ coincides with the Sobolev space $W^r_2(\mathbb{R}^n)$. If $p \ne 2$, then none of the spaces $B^r_{p\theta}(\mathbb{R}^n)$ coincides with $W^r_p(\mathbb{R}^n)$, and

$$B^r_{p\theta_1}(\mathbb{R}^n) \subset W^r_p(\mathbb{R}^n) \subset B^r_{p\theta_2}(\mathbb{R}^n),$$

where $1 < p < \infty$, $\theta_1 = \min\{p, 2\}$ and $\theta_2 = \max\{p, 2\}$.

The most fundamental facts about Besov spaces are the following:

1) the embedding theorem for Besov spaces

$$B^r_{p\theta}(\mathbb{R}^n) \subset B^{\varrho}_{q\theta}(\mathbb{R}^n), \quad 1 \le p < q \le \infty,\ 1 \le \theta \le \infty,\ \varrho = r - n\Big(\frac{1}{p} - \frac{1}{q}\Big) > 0;$$

2) the trace theorem for Besov spaces ($1 \le m < n$)

$$\mathrm{tr}_{\mathbb{R}^m} B^r_{p\theta}(\mathbb{R}^n) = B^{r - \frac{n-m}{p}}_{p\theta}(\mathbb{R}^m), \quad 1 \le p \le \infty, \quad 1 \le \theta \le \infty, \quad r - \frac{n-m}{p} > 0,$$

where $\mathrm{tr}_{\mathbb{R}^m} B^r_{p\theta}(\mathbb{R}^n)$ is the set of traces on $\mathbb{R}^m$ of all functions $f \in B^r_{p\theta}(\mathbb{R}^n)$;

3) the trace theorem for Sobolev spaces ($1 \le m < n$)

$$\mathrm{tr}_{\mathbb{R}^m} W^r_p(\mathbb{R}^n) = B^{r - \frac{n-m}{p}}_{pp}(\mathbb{R}^m), \quad 1 < p < \infty, \quad r \in \mathbb{N}, \quad r - \frac{n-m}{p} > 0;$$

4) the interpolation theorem for Besov spaces (real interpolation)

$$\big(B^{r_0}_{pq_0}(\mathbb{R}^n), B^{r_1}_{pq_1}(\mathbb{R}^n)\big)_{\theta,q} = B^r_{pq}(\mathbb{R}^n), \quad 1 \le p, q_0, q_1, q \le \infty, r_0, r_1 > 0, r_0 \ne r_1,$$

where $0 < \theta < 1$ and $r = (1-\theta)r_0 + \theta r_1$;

5) the interpolation theorem for Sobolev spaces (real interpolation)

$$\big(W^{r_0}_p(\mathbb{R}^n), W^{r_1}_p(\mathbb{R}^n)\big)_{\theta,q} = B^r_{pq}(\mathbb{R}^n), \quad 1 \le p, q \le \infty, r_0, r_1 \in \mathbb{N}, r_0 \ne r_1,$$

where $0 < \theta < 1$ and $r = (1-\theta)r_0 + \theta r_1$.

Statements 1) – 2) were proved by O.V. Besov [2], [3] for $\theta < \infty$; for $\theta = \infty$ they were earlier proved by S.M. Nikol'skii. Statement 3) was proved by O.V. Besov [2], [3] for $m \leq n-2$; for $m = n-1$ Statement 3) was earlier proved by N. Aronzajn, E. Gagliardo and L.N. Slobodetskii. Statements 4) and 5) were proved by J.L. Lions and J. Peetre [9].

These seminal results explain the role of Besov spaces in analysis and in numerous applications, especially to the boundary value problems for partial differential equations. However, this was only the start, and in order to develop the theory of Besov spaces and its applications to the top level a lot of further work was required. This work has been done by O.V. Besov and by many other mathematicians around the world.

In 1954 S.M. Nikol'skii organized the seminar "Differentiable functions of several variables and applications", which became the world recognized leading seminar on the theory of function spaces. Oleg participated in this seminar from the very beginning, first as the secretary and later, for more than 20 years, as the head of the seminar jointly with S.M. Nikol'skii and L.D. Kudryavtsev.

O.V. Besov participated in numerous research projects supported by grants of several countries, and led many of them.

He takes active part in the international mathematical life, participates in and contributes to organizing many international conferences. He has given more than 100 invited talks at conferences and has been invited to universities in more than 20 countries.

For more than 40 years O.V. Besov has been a professor at the Department of Mathematics of the Moscow Institute of Physics and Technology. He is a celebrated and sought-after lecturer who is able to develop the student's independent thinking. On the basis of his lectures he wrote a popular text-book on mathematical analysis in two volumes.

He spends a lot of time on supervising post-graduate students. One of his former post-graduate students H. Ghazaryan, now a distinguished professor, plays an active role in the mathematical life of Armenia and has many post-graduate students of his own.

Professor Besov has close academic ties with Kazakhstan mathematicians. He has many times visited Kazakhstan and is an honorary professor of the Shakarim Semipalatinsk State University and a member of the editorial board of the Eurasian Mathematical Journal. He has been awarded a medal for his meritorious role in the development of science of the Republic of Kazakhstan.

Oleg is in great physical and mental shape and leads an active life.

The Editorial Board of the Eurasian Mathematical Journal is happy to congratulate Oleg Vladimirovich Besov on occasion of his 80th birthday, wishes him good health and further productive work in mathematics and mathematical education.

On behalf of the Editorial Board

V.I. Burenkov and T.V. Tararykova

References

[1] J. Bergh and J. Löfström, *Interpolation spaces. An introduction.* Springer, Berlin, 1976.

[2] O.V. Besov, *On some families of functional spaces. Embedding and extension theorems.* Dokl. Akad. Nauk SSSR, 126 (1959), no. 6, 1163-1165 (in Russian).

[3] O.V. Besov, *Investigation of a class of function spaces in connection with embedding and extension theorems.* Trudy Mat. Inst. Steklov., 60 (1961), 42-81 (in Russian).

[4] O.V. Besov, V.P. Il'in, S.M. Nikolskii, Integral representations of functions and embedding theorems, Nauka, Moscow, 1975 (in Russian); English transl. Halsted Press [John Wiley & Sons], v. I, 1978, v. II, 1979; 2nd edition (updated and complemented), Nauka, Moscow, 1996.

[5] V.I. Burenkov, *Embedding and extension theorems for classes of differentiable functions of several variables defined on the entire space* Itogi Nauki. Ser. Matematika. Mat. Anal. 1965, VINITI, Moscow (1966), 71-155 (in Russian); English transl. in Progress in Mathematics 2. Mathematical Analysis. Plenum Press, New York (1968), 73-161.

[6] V.I. Burenkov, *Sobolev spaces on domains.* B.G. Teubner, Stuttgart-Leipzig, 1998.

[7] L.D. Kudryavtsev, S.M. Nikol'skii, *Spaces of differentiable functions of several variables and embedding theorems.* Analysis - 3. Itogi Nauki i Tech. Ser. Sovrem. probl. mat. Fundament. napravleniya 26, VINITI, Moscow (1988), 5-157; English transl. in Encyclopaedia Math. Sci. 26, Springer, Berlin (1991), 1-140.

[8] J.-L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications.* Vol. 1, Dunod, Paris, 1968.

[9] J.-L. Lions, J. Peetre, *Sur une classe d'espaces d'interpolation.* Inst. Hautes Etudes Sci. Publ. Math. 19 (1964), 5-68.

[10] S.M. Nikol'skii, *On embedding, extension and approximation theorems for differentiable functions of several variables.* Uspekhi Mat. Nauk 16 (101) (1961), no. 5, 63-114 (in Russian); English transl. in Russian Math. Surveys 16 (1961), no. 5, 55-104.

[11] S.M. Nikol'skii, *Approximation of functions of several variables and embedding theorems.* Nauka, Moscow (in Russian); English transl. Grundl. Math. Wissensch., 205, Springer-Verlag, New York, 1975.

[12] Y. Sawano, *Besofu kuukanron* (*Theory of Besov spaces*) (in Japanese). Nihon Hyoronsha, 2011.

[13] H. Triebel, *Interpolation theory, function spaces, differential operators.* North Holland, Amsterdam, 1978.

[14] H. Triebel, *Theory of function spaces.* Birkhäuser, Basel, 1983.

[15] H. Triebel, *Theory of function spaces II.* Birkhäuser, Basel, 1992.