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COMMENTS ON DEFINITIONS OF GENERAL LOCAL AND GLOBAL MORREY-TYPE SPACES

T.V. Tararykova

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Key words: general local and global Morrey-type spaces, regularization of the functional parameter.

AMS Mathematics Subject Classification: 42B35, 46E30.
Abstract. It is proved that in one of the popular definitions of general local and global Morrey-type spaces the functional parameter which enters these definitions can be replaced, without essential loss of generality, by another one, which has better regularity properties.

1 Introduction

In the last three decades there is a great interest in studying general Morrey-type spaces, operators acting in such spaces, and applications to real analysis and to the theory of partial differential equations. See, for example, recent survey papers [1, 2, 7, 8, 9, 10, 11, 12, 13].

One of popular definitions of such spaces is as follows. Let $B(x,r)$ denote the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ of radius $r > 0$.

Definition 1. Let $0 < p, \theta \leqslant \infty$ and let w be a non-negative Lebesgue measurable function on $(0,\infty)$. Then $LM_{p\theta,w(\cdot)} \equiv LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ is the local Morrey-type space, the space of all functions f Lebesgue measurable on $\mathbb{R}^n$ with finite quasi-norm

$$\|f\|_{LM_{p\theta,w(\cdot)}} = \left\| w(r) \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)}.$$

Furthermore, $GM_{p\theta,w(\cdot)} \equiv GM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ is the global Morrey-type space, the space of all functions f Lebesgue measurable on $\mathbb{R}^n$ with finite quasi-norm

$$\|f\|_{GM_{p\theta,w(\cdot)}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,w(\cdot)}} = \sup_{x\in\mathbb{R}^n} \left\| w(r) \|f\|_{L_p(B(x,r))} \right\|_{L_\theta(0,\infty)}.$$

The first natural question which arises is to find out for which functions w the spaces $LM_{p\theta,w(\cdot)}$ and $GM_{p\theta,w(\cdot)}$ are nontrivial, i. e. consist not only of functions equivalent to 0 on $\mathbb{R}^n$. In order to formulate the answer to this question the following definition is required.

Definition 2. Let $0 < p,\theta \leqslant \infty$. Then Ω_θ is the set of all functions w which are non-negative, Lebesgue measurable on $(0,\infty)$, not equivalent to 0 on (t,∞) for any $t>0$, and such that for some $t>0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty. \tag{1.1}$$

Furthermore, $\Omega_{p\theta}$ is the set of all functions w which are non-negative, Lebesgue measurable on $(0,\infty)$, not equivalent to 0 on (t,∞) for any $t>0$, and such that some $t>0$

$$\left\|w(r)r^{n/p}\right\|_{L_\theta(0,t)} < \infty\,, \qquad \|w(r)\|_{L_\theta(t,\infty)} < \infty\,, \tag{1.2}$$

or, which is equivalent,

$$\left\|w_2(r)\left(\frac{r}{t+r}\right)^{\frac{n}{p}}\right\|_{L_{\theta_2}(0,\infty)} < \infty\,. \tag{1.3}$$

Note that if condition (1.2) is satisfied for some $t>0$, then it is also satisfied for all $t>0$. (Hence condition (1.3) is also satisfied for all $t>0$.) Indeed, if $0<\tau\leqslant t$ then [1]

$$\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,\tau)} \leqslant \|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,t)} < \infty\,,$$

and

$$\begin{aligned}\|w(r)\|_{L_\theta(\tau,\infty)} &\leqslant 2^{\left(\frac{1}{\theta}-1\right)_+}\left(\|w(r)\|_{L_\theta(\tau,t)} + \|w(r)\|_{L_\theta(t,\infty)}\right)\\ &\leqslant 2^{\left(\frac{1}{\theta}-1\right)_+}\left(\tau^{-\frac{n}{p}}\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,t)} + \|w(r)\|_{L_\theta(t,\infty)}\right) < \infty\,.\end{aligned}$$

Also, if $t<\tau<\infty$ then

$$\|w(r)\|_{L_\theta(\tau,\infty)} \leqslant \|w(r)\|_{L_\theta(t,\infty)} < \infty\,,$$

and

$$\begin{aligned}\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,\tau)} &\leqslant 2^{\left(\frac{1}{\theta}-1\right)_+}\left(\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,t)} + \|w(r)r^{\frac{n}{p}}\|_{L_\theta(t,\tau)}\right)\\ &\leqslant 2^{\left(\frac{1}{\theta}-1\right)_+}\left(\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,t)} + \tau^{\frac{n}{p}}\|w(r)\|_{L_\theta(t,\infty)}\right) < \infty\,.\end{aligned}$$

Let, for a function $w\in\Omega_\theta$,

$$a = \inf\{t>0 : \|w\|_{L_\theta(t,\infty)} < \infty\}\,.$$

By the above it follows that if $w\in\Omega_{p\theta}$ then $a=0$.

Lemma. ([4], [5]) *Let $0<p,\theta\leqslant\infty$ and let w be a non-negative Lebesgue measurable function on $(0,\infty)$, which is not equivalent to 0 on (t,∞) for any $t>0$.*

Then the space $LM_{p\theta,w(\cdot)}$ is non-trivial if and only if $w\in\Omega_\theta$, and the space $GM_{p\theta,w(\cdot)}$ is non-trivial if and only if $w\in\Omega_{p\theta}$.

Moreover, if $w\in\Omega_\theta$, then the space $LM_{p\theta,w(\cdot)}$ contains all functions $f\in L_p(\mathbb{R}^n)$ such that $f=0$ on $B(0,t)$ for some $t>a$. If $w\in\Omega_{p\theta}$, then

$$L_p(\mathbb{R}^n)\cap L_\infty(\mathbb{R}^n) \subset GM_{p\theta,w(\cdot)}\,.$$

[1] As usual, $\alpha_+ = \max\{\alpha,0\}$ for $\alpha\in\mathbb{R}$.

2 Main result

If $a>0$ then $f\in LM_{p\theta,w(\cdot)}$ if and only if $f\in L_p^{loc}(\mathbb{R}^n)$, f is equivalent to 0 on $B(0,a)$, and

$$\|f\|_{LM_{p\theta,w(\cdot)}}=\big\|w(r)\|f\|_{L_p(B(0,r))}\big\|_{L_\theta(a,\infty)}<\infty\,.$$

If $w\in\Omega_\theta$ then it may happen that w is equivalent to zero on certain subintervals of (a,∞) which is not convenient for some applications. This drawback can be overcome if one replaces w by a function $\widetilde{w}$ which is positive on (a,∞) and is such that $\|f\|_{LM_{p\theta,w(\cdot)}}$ and $\|f\|_{LM_{p\theta,\widetilde{w}(\cdot)}}$ are sufficiently close. More precisely, the following statement holds.

Let Ω_θ^+ and $\Omega_{p\theta}^+$ be the sets of all positive on $(0,\infty)$ functions $w\in\Omega_\theta$, $w\in\Omega_{p\theta}$ respectively.

Theorem 2.1. *Let $0<p,\theta\leqslant\infty$.*

If $\theta<\infty$ and $w\in\Omega_\theta$, then for each $\varepsilon>0$ there exists a function $w_\varepsilon\in\Omega_\theta^+$ such that $w_\varepsilon\geqslant w$ on $(0,\infty)$, $LM_{p\theta,w_\varepsilon(\cdot)}=LM_{p\theta,w(\cdot)}$, and

$$\|f\|_{LM_{p\theta,w(\cdot)}}\leqslant\|f\|_{LM_{p\theta,w_\varepsilon(\cdot)}}\leqslant(1+\varepsilon)\|f\|_{LM_{p\theta,w(\cdot)}}\tag{2.1}$$

for all $f\in LM_{p\theta,w(\cdot)}$.

If $\theta=\infty$ and $w\in\Omega_\infty$, then there exists a function $\widetilde{w}\in\Omega_\infty^+$ such that $\widetilde{w}\geqslant w$ on $(0,\infty)$, $LM_{p\infty,\widetilde{w}(\cdot)}=LM_{p\infty,w(\cdot)}$, and

$$\|f\|_{LM_{p\infty,\widetilde{w}(\cdot)}}=\|f\|_{LM_{p\infty,w(\cdot)}}\tag{2.2}$$

for all $f\in LM_{p\infty,w(\cdot)}$. Also there exists a function $\bar{w}\in\Omega_\infty^+$ such that $\bar{w}\geqslant w$ almost everywhere on $(0,\infty)$, $\bar{w}$ is non-increasing and continuous on the right on (a,∞), $LM_{p\infty,\bar{w}(\cdot)}=LM_{p\infty,w(\cdot)}$, and equality (2.2) holds with $\widetilde{w}$ replaced by $\bar{w}$.

Moreover, a similar statement holds if everywhere Ω_θ and Ω_θ^+ are replaced by $\Omega_{p\theta}$ and $\Omega_{p\theta}^+$, and local Morrey-type spaces LM are replaced by global Morrey-type spaces GM.

Proof. 1. First, let $w\in\Omega_\theta$. Let $b_k=a+k-1, k\in\mathbb{N}$. We set

$$u_\theta=\begin{cases}1 & \text{if}\quad r\in(0,a],\\ 2^{-\frac{k}{\theta}}\,\|w\|_{L_\theta(b_k,\infty)} & \text{if}\quad r\in(b_{k-1},b_k],\ k\in\mathbb{N},\end{cases}\tag{2.3}$$

Furthermore, if $\theta<\infty$ for $\varepsilon>0$ we set

$$w_\varepsilon=w_{1,\varepsilon},\tag{2.4}$$

where

$$w_{1,\varepsilon}(r)=\left(w^\theta(r)+\delta u_\theta^\theta(r)\right)^{\frac{1}{\theta}},\quad r\in(0,\infty),\tag{2.5}$$

and $\delta=(1+\varepsilon)^\theta-1$.

Clearly $w_\varepsilon\geqslant w$ on $(0,\infty)$ and $w_\varepsilon>0$ on $(0,\infty)$.

Moreover, for all $t>a$

$$\|w_\varepsilon\|^\theta_{L_\theta(t,\infty)}=\|w_{1,\varepsilon}\|^\theta_{L_\theta(t,\infty)}\leqslant\int\limits_t^\infty w^\theta(r)dr+\delta\sum_{k:\ b_k>t}2^{-k}\int\limits_{b_k}^\infty w^\theta(r)dr$$

$$\leqslant \int_t^\infty w^\theta(r)dr + \delta\Big(\sum_{k:\ b_k>t} 2^{-k}\Big)\int_t^\infty w^\theta(r)dr \leqslant (1+\delta)\int_t^\infty w^\theta(r)dr\,,$$

therefore

$$\|w\|_{L_\theta(t,\infty)} \leqslant \|w_\varepsilon\|_{L_\theta(t,\infty)} \leqslant (1+\delta)^{\frac{1}{\theta}}\|w\|_{L_\theta(t,\infty)} = (1+\varepsilon)\|w\|_{L_\theta(t,\infty)} < \infty\,.$$

Hence $w_\varepsilon \in \Omega_\theta^+$.

2. Furthermore, for all $f \in LM_{p\theta,w(\cdot)}$, taking into account that f is equivalent to 0 on $B(0,a)$, we get

$$\|f\|^\theta_{LM_{p\theta,w_\varepsilon(\cdot)}} = \|f\|^\theta_{LM_{p\theta,w_{1,\varepsilon}(\cdot)}} = \int_a^\infty \big(w_{1,\varepsilon}(r)\|f\|_{L_p(B(0,r))}\big)^\theta dr$$

$$= \int_a^\infty \big(w(r)\|f\|_{L_p(B(0,r))}\big)^\theta dr + \delta\sum_{k=1}^\infty 2^{-k}\Big(\int_{b_k}^\infty w^\theta(r)dr\Big)\int_{b_{k-1}}^{b_k}\big(\|f\|_{L_p(B(0,r))}\big)^\theta dr$$

$$\leqslant \|f\|^\theta_{LM_{p\theta,w(\cdot)}} + \delta\sum_{k=1}^\infty 2^{-k}\Big(\|f\|^\theta_{L_p(B(0,b_k)}\int_{b_k}^\infty w^\theta(r)dr\Big)$$

$$\leqslant \|f\|^\theta_{LM_{p\theta,w(\cdot)}} + \delta\sum_{k=1}^\infty 2^{-k}\Big(\int_{b_k}^\infty w^\theta(r)\|f\|^\theta_{L_p(B(0,r))}dr\Big)$$

$$\leqslant \|f\|^\theta_{LM_{p\theta,w(\cdot)}} + \delta\Big(\sum_{k=1}^\infty 2^{-k}\Big)\|f\|^\theta_{LM_{p\theta,w(\cdot)}} = (1+\delta)\|f\|^\theta_{LM_{p\theta,w(\cdot)}}\,.$$

Therefore

$$\|f\|_{LM_{p\theta,w(\cdot)}} \leqslant \|f\|_{LM_{p\theta,w_\varepsilon(\cdot)}} \leqslant (1+\delta)^{\frac{1}{\theta}}\|f\|_{LM_{p\theta,w(\cdot)}} = (1+\varepsilon)\|f\|_{LM_{p\theta,w(\cdot)}}.$$

3. If $\theta=\infty$ we can, in the spirit of Step 1, set

$$\widetilde{w}(r) = \max\{w(r), u_\infty(r)\}\,, \quad r\in(0,\infty),$$

and prove that $\widetilde{w}\in\Omega_\infty^+$ and equality (2.2) holds. Also, clearly, $\widetilde{w}\geqslant w$. However, there is no guarantee that $\widetilde{w}$ is non-increasing and continuous on the right on (a,∞).

For this reason we shall use a different approach for constructing the functions $\bar{w}$ and $\widetilde{w}$. Let

$$\bar{w}(r) = \begin{cases} \max\{w(r),1\} & \text{if} \quad r\in(0,a], \\ \|w\|_{L_\infty(r,\infty)} & \text{if} \quad r\in(a,\infty). \end{cases} \tag{2.6}$$

Clearly, $0<\bar{w}(r)<\infty$ on $(0,\infty)$ and $\bar{w}$ is non-increasing on (a,∞). Also by the properties of essential supremums it follows that $\bar{w}$ is continuous on the right on (a,∞). Moreover, for all $t>a$

$$\|\bar{w}\|_{L_\infty(t,\infty)} = \|\|w\|_{L_\infty(r,\infty)}\|_{L_\infty(t,\infty)} \leqslant \|w\|_{L_\infty(t,\infty)} < \infty\,,$$

hence $\bar{w} \in \Omega_\theta^+$.

Note that $w(r) \leqslant \bar{w}(r)$ for almost all $r > 0$. Indeed, assume to the contrary that the Lebesgue measure $|A|$ of the set $A = \{r \in (a,\infty) : w(r) > \bar{w}(r)\}$ is positive. Let, for $\varepsilon > 0$, $A_\varepsilon = \{r \in (a,\infty) : w(r) \geqslant \bar{w}(r) + \varepsilon\}$. Since $A_{\varepsilon_1} \subset A_{\varepsilon_2}$ if $\varepsilon_1 > \varepsilon_2 > 0$ and $\bigcup_{\varepsilon > 0} A_\varepsilon = A$, it follows that $|A_\varepsilon| > 0$ for some $\varepsilon > 0$. Moreover, for this ε there exists $r \in A_\varepsilon$ such that $|A_\varepsilon \cap (r, r+\varepsilon)| > 0$ for all $\delta > 0$. Therefore for this r for all $\delta > 0$

$$\bar{w}(r) = \|w\|_{L_\infty(r,\infty)} \geqslant \|w\|_{L_\infty(A_\varepsilon \cap (r,r+\delta))} = \operatorname{ess\,sup}_{\varrho \in A_\varepsilon \cap (r,r+\delta)} w(\varrho)$$

$$\geqslant \operatorname{ess\,sup}_{\varrho \in A_\varepsilon \cap (r,r+\delta)} (\bar{w}(\varrho) + \varepsilon) \geqslant \bar{w}(r+\delta) + \varepsilon\,.$$

Since $\bar{w}$ is continuous on the right on (a,∞), by passing to the limit as $\delta \to 0^+$, we arrive at a contradiction.

Therefore for any $f \in LM_{p\infty,w(\cdot)}$

$$\|f\|_{LM_{p\infty,w(\cdot)}} = \big\|w(r)\|f\|_{L_p(B(0,r))}\big\|_{L_\infty(a,\infty)} \leqslant \big\|\bar{w}(r)\|f\|_{L_p(B(0,r))}\big\|_{L_\infty(a,\infty)}$$

$$= \|f\|_{LM_{p\infty,\bar{w}(\cdot)}} = \Big\|\|w(\varrho)\|_{L_\infty(r,\infty)}\|f\|_{L_p(B(0,r))}\Big\|_{L_\infty(a,\infty)}$$

$$\leqslant \Big\|\|w(\varrho)\|f\|_{L_p(B(0,\varrho))}\|_{L_\infty(r,\infty)}\Big\|_{L_\infty(a,\infty)}$$

$$\leqslant \|w(\varrho)\|f\|_{L_p(B(0,\varrho))}\|_{L_\infty(a,\infty)} = \|f\|_{LM_{p\infty,w(\cdot)}}\,,$$

hence equality (2.2) follows with $\widetilde{w}$ replaced by $\bar{w}$.

Since the function $\widetilde{w}$ defined by

$$\widetilde{w}(r) = \max\{w(r), \bar{w}(r)\}, \quad r > 0, \tag{2.7}$$

is equivalent to $\bar{w}$ on $(0,\infty)$, it satisfies the requirements of the theorem.

4. Next, let $w \in \Omega_{p\theta}$ with $\theta < \infty$. In this case $a = 0$. Let $\tau > 0$ be such that w is not equivalent to 0 on $(0,\tau)$, let $b_k^* = \max\{b_k, \tau\}$, and let

$$v_\theta = 2^{-\frac{k}{\theta}}\,\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,b_{k-1}^*)} \quad \text{if} \quad r \in (b_{k-1}, b_k], \quad k \in \mathbb{N}. \tag{2.8}$$

Furthermore, if $\theta < \infty$ we set for $\varepsilon > 0$

$$w_\varepsilon = \min\{w_{1,\varepsilon}, w_{2,\varepsilon}\} \tag{2.9}$$

where $w_{1,\varepsilon}$ is the same as in Step 1 and

$$w_{2,\varepsilon}(r) = \big(w^\theta(r) + \delta v_\theta^\theta(r)\big)^{\frac{1}{\theta}}\,, \quad r \in (0,\infty). \tag{2.10}$$

Then by Step 1 for all $t > 0$

$$\|w_\varepsilon\|_{L_\theta(t,\infty)} \leqslant \|w_{1,\varepsilon}\|_{L_\theta(t,\infty)} \leqslant (1+\varepsilon)\|w\|_{L_\theta(t,\infty)} < \infty\,.$$

Moreover,

$$\|w_\varepsilon(r)r^{\frac{n}{p}}\|^\theta_{L_\theta(0,t)} \leqslant \|w_{2,\varepsilon}(r)r^{\frac{n}{p}}\|^\theta_{L_\theta(0,t)}$$

$$\leqslant \int_0^t \big(w(r)r^{\frac{n}{p}}\big)^\theta dr + \delta \sum_{k:\ b_{k-1}^* < t} 2^{-k} \int_0^{b_{k-1}^*} \big(w(r)r^{\frac{n}{p}}\big)^\theta dr$$

$$\leqslant \int_0^t \big(w(r)r^{\frac{n}{p}}\big)^\theta dr + \delta\Big(\sum_{k=1}^\infty 2^{-k}\Big) \int_0^{\max\{t,\tau\}} \big(w(r)r^{\frac{n}{p}}\big)^\theta dr$$

$$= (1+\delta) \int_0^{\max\{t,\tau\}} \big(w(r)r^{\frac{n}{p}}\big)^\theta dr\,,$$

therefore

$$\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,t)} \leqslant \|w_\varepsilon(r)r^{\frac{n}{p}}\|_{L_\theta(0,t)} \leqslant (1+\varepsilon)\|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,\max\{t,\tau\})} < \infty\,.$$

Hence $w_\varepsilon \in \Omega^+_{p\theta}$.

5. If $\theta = \infty$, $\in \Omega_{p\infty}$ and τ is the same as in Step 4, we set

$$\bar w = \begin{cases} r^{-\frac{n}{p}}\|w(\varrho)\varrho^{\frac{n}{p}}\|_{L_\infty(r,2\tau)} & \text{if} \quad r\in(0,\tau), \\ \|w(\varrho)\|_{L_\infty(r,\infty)} & \text{if} \quad r\in[\tau,\infty), \end{cases} \tag{2.11}$$

where

$$c = \tau^{\frac{n}{p}}\|w(\varrho)\varrho^{\frac{n}{p}}\|^{-1}_{L_\infty(\tau,2\tau)}\|w(\varrho)\|_{L_\infty(\tau,\infty)}\,.$$

Clearly, $\bar w$ is positive and non-increasing on $(0,\infty)$. Moreover, by the properties of essential supremums it follows that $\bar w$ is continuous on the right on $(0,\infty)$.

Similarly to Step 3

$$\|\bar w\|_{L_\infty(\tau,\infty)} \leqslant \|w\|_{L_\infty(\tau,\infty)} < \infty\,.$$

Also

$$\|\bar w(r)r^{\frac{n}{p}}\|_{L_\infty(0,\tau)} = c\,\big\|\|w(\varrho)\varrho^{\frac{n}{p}}\|_{L_\infty(r,2\tau)}\big\|_{L_\infty(0,\tau)} \leqslant \|w(\varrho)\varrho^{\frac{n}{p}}\|_{L_\infty(0,2\tau)} < \infty\,.$$

Hence $\bar w \in \Omega^+_{p\theta}$.

6. By Step 2 it follows that for $\theta < \infty$

$$\|f\|_{GM_{p\theta,w(\cdot)}} \leqslant \|f\|_{GM_{p\theta,w_\varepsilon(\cdot)}} \leqslant \|f\|_{GM_{p\theta,w_{1,\varepsilon}(\cdot)}} = \sup_{x\in\mathbb{R}^n}\big\|w_{1,\varepsilon}(r)\|f\|_{L_p(B(x,r))}\big\|_{L_\theta(0,\infty)}$$

$$\leqslant (1+\varepsilon)\sup_{x\in\mathbb{R}^n}\big\|w(r)\|f\|_{L_p(B(x,r))}\big\|_{L_\theta(0,\infty)} = (1+\varepsilon)\|f\|_{GM_{p\theta,w(\cdot)}}\,,$$

because the argument of Step 2 does not change if the ball $B(0,r)$ is replaced by the ball $B(x,r)$. If $\theta=\infty$ then similarly

$$\|f\|_{GM_{p\infty,w(\cdot)}} \leqslant \|f\|_{GM_{p\infty,\bar w(\cdot)}} = \sup_{x\in\mathbb{R}^n}\big\|\bar w(r)\|f\|_{L_p(B(x,r))}\big\|_{L_\infty(0,\infty)}$$

$$\leqslant \sup_{x\in\mathbb{R}^n}\big\|w(r)\|f\|_{L_p(B(x,r))}\big\|_{L_\infty(0,\infty)} = \|f\|_{GM_{p\infty,w(\cdot)}}\,,$$

hence

$$\|f\|_{GM_{p\infty,\bar w(\cdot)}} = \|f\|_{GM_{p\infty,w(\cdot)}}\,.$$

$\square$

3 Applications

The meaning of Theorem 2.1 is that without essential loss of generality one may assume that in Definition 2 the function w belongs to Ω^+_θ for the case of local Morrey-type spaces and w belongs to $\Omega^+_{p\theta}$ for the case of global Morrey-type spaces.

Clearly Theorem 2.1 allows reducing the problem of boundedness of a certain operator A from one local Morrey-type space $LM_{p_1\theta_1,w_1(\cdot)}$ to another one $LM_{p_2\theta_2,w_2(\cdot)}$ for $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$ to the case in which $w_1 \in \Omega^+_{\theta_1}$ and $w_2 \in \Omega^+_{\theta_2}$ or from one global Morrey-type space $GM_{p_1\theta_1,w_1(\cdot)}$ to another one $GM_{p_2\theta_2,w_2(\cdot)}$ for $w_1 \in \Omega_{p_1\theta_1}$ and $w_2 \in \Omega_{p_2\theta_2}$ to the case in which $w_1 \in \Omega^+_{p_1\theta_1}$ and $w_2 \in \Omega^+_{p_2\theta_2}$.

Indeed, assume, for example, that for a certain class $F(p_1, \theta_1, p_2, \theta_2)$ of pairs $w_1 \in \Omega^+_{\theta_1}$ and $w_2 \in \Omega^+_{\theta_2}$ the inequality

$$\|Af\|_{LM_{p_2\theta_2,w_2(\cdot)}} \leqslant c(w_1, w_2)\,\|f\|_{LM_{p_1\theta_1,w_1(\cdot)}} \tag{3.1}$$

holds, where $c(w_1, w_2) > 0$ is independent of $f \in LM_{p_1\theta_1,w_1(\cdot)}$.

Next, let $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$. Consider the functions $w_{1,\varepsilon} \in \Omega^+_{\theta_1}$ and $w_{2,\varepsilon} \in \Omega^+_{\theta_2}$ constructed in the proof of Theorem 2.1 for all sufficiently small $\varepsilon > 0$. Assume that the class $F(p_1, \theta_1, p_2, \theta_2)$ is such that the pairs $w_{1,\varepsilon}, w_{2,\varepsilon}$ belong to it for all such ε. Then by (3.1)

$$\|Af\|_{LM_{p_2\theta_2,w_2(\cdot)}} \leqslant \|Af\|_{LM_{p_2\theta_2,w_{2,\varepsilon}(\cdot)}}$$
$$\leqslant c(w_{1,\varepsilon}, w_{2,\varepsilon})\,\|f\|_{LM_{p_1\theta_1,w_{1,\varepsilon}(\cdot)}} \leqslant c(w_{1,\varepsilon}, w_{2,\varepsilon})(1+\varepsilon)\,\|f\|_{LM_{p_1\theta_1,w_1(\cdot)}}\,,$$

hence A is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

Moreover, it may happen that $\lim\limits_{\varepsilon\to 0^+} c(w_{1,\varepsilon}, w_{2,\varepsilon}) = c(w_1, w_2)$ in which case we arrive at inequality (3.1).

In many cases for a proof of inequality (3.1) or of more complicated inequalities of such type it is not important whether $w_1 \in \Omega^+_{\theta_1}$, $w_2 \in \Omega^+_{\theta_2}$ or $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$. However, it may happen that there are difficulties in giving direct proof of such inequalities for all $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$. This is the case in paper [3] where the following interpolation theorem is stated.

Theorem 3.1. *Let* $0 < p, q_0, q_1, q < \infty, q_0 \neq q_1, 0 < \theta < 1,$

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\,,$$

and $w \in \Omega^+_1$. *Then*

$$\Big(LM_{pq_0,w^{\frac{1}{q_0}}(\cdot)},\ LM_{pq_1,w^{\frac{1}{q_1}}(\cdot)}\Big)_{\theta,q} = LM_{pq,w^{\frac{1}{q}}(\cdot)}\,. \tag{3.2}$$

Moreover, there exist $c_1, c_2 > 0$ *depending only on* p, q_0, q_1 *and* θ *such that*

$$c_1\,\|f\|_{LM_{pq,w^{\frac{1}{q}}(\cdot)}} \leqslant \|f\|_{\big(LM_{pq_0,w^{\frac{1}{q_0}}(\cdot)},\ LM_{pq_1,w^{\frac{1}{q_1}}(\cdot)}\big)_{\theta,q}} \leqslant c_2\,\|f\|_{LM_{pq,w^{\frac{1}{q}}(\cdot)}} \tag{3.3}$$

for all $f \in LM_{pq,w^{\frac{1}{q}}(\cdot)}$.

The proof outlined in [3] is based on the equality

$$\|f\|_{LM_{p\sigma,u(\cdot)}} = \|f\|_{LM_{p\sigma}^{v(\cdot)}} \equiv \left(\int_{\alpha}^{\infty} \left(\frac{\|f\|_{L_p(B(0,r))}}{v(r)} \right)^{\sigma} \frac{dv(r)}{v(r)} \right)^{\frac{1}{\sigma}},$$

where $0 < \sigma < \infty$,

$$v(r) = \sigma^{-\frac{1}{\sigma}} \|u\|^{-1}_{L_\sigma(r,\infty)}, \quad a < r < \infty, \quad \alpha = \lim_{r\to a^+} v(r),$$

which holds only if $u \in \Omega_\sigma^+$. (For such u the function v is positive locally absolutely continuous and strictly increasing on (a,∞) which allows changing variables in order to obtain the above equality.)

Theorem 3.2. *Theorem* 3.1 *holds for any* $w \in \Omega_1$*. Moreover, inequality* (3.3) *holds for* $w \in \Omega_1$ *with the same* c_1, c_2 *as in Theorem* 3.1*.*

Proof. Consider the functions u_θ defined by equality (2.3) for $\theta = 1, q_0, q_1$. Then it follows that

$$u_{q_m} = (u_1)^{\frac{1}{q_m}}, \quad m = 1,2.$$

Let

$$\nu_\varepsilon(r) = w(r) + \gamma u_1(r), \quad r \in (0,\infty),$$

where $\gamma = \min\{\delta_0, \delta_1\}, \delta_m = (1+\varepsilon)^{q_m} - 1, m = 1,2.$

Hence by formulas (2.3)–(2.5) with $\theta = q_m$ and $\delta = \delta_m$

$$\left(w^{\frac{1}{q_m}}\right)_\varepsilon = \left(\left(w^{\frac{1}{q_m}}\right)^{q_m}(r) + \delta_m (u_{q_m})^{q_m} \right)^{\frac{1}{q_m}}$$

$$= (w(r) + \delta_m u_1(r))^{\frac{1}{q_m}} \geqslant (w(r) + \gamma u_1(r))^{\frac{1}{q_m}} = (\nu_\varepsilon)^{\frac{1}{q_m}}.$$

So

$$(\nu_\varepsilon)^{\frac{1}{q_m}} \leqslant \left(w^{\frac{1}{q_m}}\right)_\varepsilon, \quad m = 1,2.$$

Therefore by the left-hand-side inequality in (3.3) and inequality (2.1)

$$c_1 \|f\|_{LM_{pq,w^{\frac{1}{q}}(\cdot)}} \leqslant c_1 \|f\|_{LM_{pq,(\nu_\varepsilon)^{\frac{1}{q}}(\cdot)}} \leqslant \|f\|_{\left(LM_{pq_0,(\nu_\varepsilon)^{\frac{1}{q_0}}(\cdot)},\, LM_{pq_1,(\nu_\varepsilon)^{\frac{1}{q_1}}(\cdot)}\right)_{\theta,q}}$$

$$= \left\| \inf_{f=f_0+f_1} \left(\|f_0\|_{LM_{pq_0,(\nu_\varepsilon)^{\frac{1}{q_0}}(\cdot)}} + t\,\|f_1\|_{LM_{pq_1,(\nu_\varepsilon)^{\frac{1}{q_1}}(\cdot)}} \right) \right\|_{\Phi_{\theta,q}}$$

$$\leqslant \left\| \inf_{f=f_0+f_1} \left(\|f_0\|_{LM_{pq_0,\left(w^{\frac{1}{q_0}}\right)_\varepsilon(\cdot)}} + t\,\|f_1\|_{LM_{pq_1,\left(w^{\frac{1}{q_1}}\right)_\varepsilon(\cdot)}} \right) \right\|_{\Phi_{\theta,q}}$$

$$\leqslant (1+\varepsilon) \left\| \inf_{f=f_0+f_1} \left(\|f_0\|_{LM_{pq_0,w^{\frac{1}{q_0}}(\cdot)}} + t\,\|f_1\|_{LM_{pq_1,w^{\frac{1}{q_1}}(\cdot)}} \right) \right\|_{\Phi_{\theta,q}}$$

$$= (1+\varepsilon)\|f\|_{\left(LM_{pq_0,w^{\frac{1}{q_0}}(\cdot)},\, LM_{pq_1,w^{\frac{1}{q_1}}(\cdot)}\right)_{\theta,q}}. \tag{3.4}$$

Here the infimum is taken over all representations $f = f_0 + f_1$ where

$$f_0 \in LM_{pq_0, w_\varepsilon^{\frac{1}{q_0}}(\cdot)} = LM_{pq_0, w^{\frac{1}{q_0}}(\cdot)} \quad \text{and} \quad f_1 \in LM_{pq_0, w_\varepsilon^{\frac{1}{q_1}}(\cdot)} = LM_{pq_0, w^{\frac{1}{q_1}}(\cdot)}.$$

Furthermore, let $\delta = (1+\varepsilon)^{\frac{1}{q}} - 1$. Since q lies between q_0 and q_1 we have $\delta \geqslant \gamma$ and by formulas (2.3)–(2.5)

$$\left(w^{\frac{1}{q}}\right)_\varepsilon = (w(r) + \delta u(r))^{\frac{1}{q}} \geqslant (w(r) + \gamma u_1(r))^{\frac{1}{q}} = (\nu_\varepsilon)^{\frac{1}{q}}.$$

Hence by the right-hand-side inequality in (3.3) and inequality (2.1)

$$\|f\|_{\left(LM_{pq_0, w^{\frac{1}{q_0}}(\cdot)},\, LM_{pq_1, w^{\frac{1}{q_1}}(\cdot)}\right)_{\theta,q}} \leqslant \|f\|_{\left(LM_{pq_0, (\nu_\varepsilon)^{\frac{1}{q_0}}(\cdot)},\, LM_{pq_1, (\nu_\varepsilon)^{\frac{1}{q_1}}(\cdot)}\right)_{\theta,q}}$$

$$\leqslant c_2 \|f\|_{LM_{pq, (\nu_\varepsilon)^{\frac{1}{q}}(\cdot)}} \leqslant c_2 \|f\|_{LM_{pq, \left(w^{\frac{1}{q}}\right)_\varepsilon(\cdot)}} \leqslant (1+\varepsilon)\, c_2 \|f\|_{LM_{pq, w^{\frac{1}{q}}(\cdot)}}. \qquad (3.5)$$

Since c_1 and c_2 are independent of ε, by passing to the limit in (3.4) and (3.5) as $\varepsilon \to 0^+$, we get inequality (3.3). □

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