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The Editorial Office
The L.N. Gumilyov Eurasian National University,
Main Building
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ON DIMENSION-FREE INTEGRABILITY IMPROVEMENT FOR SOBOLEV EMBEDDINGS

M. Krbec, **H.-J. Schmeisser**

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Key words: Sobolev space, embedding theorem, uncertainty principle, best constants for embeddings, logarithmic Lebesgue space.

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Abstract. We survey recent dimension-invariant embedding theorems for Sobolev spaces.

1 Introduction

We are interested in the inequalities

$$\|f|L^p \log^\alpha(1+L)\| \leqslant c\|\nabla f|L^p\|, \qquad f \in W_0^{1,p}(\Omega), \tag{1.1}$$

and

$$\left(\int_\Omega |f(x)|^p V(x)\,dx\right)^{1/p} \leqslant c\left(\int_\Omega |\nabla f(x)|^p\,dx\right)^{1/p}, \qquad f \in W_0^{1,p}(\Omega), \tag{1.2}$$

where either Ω is a bounded domain in $\mathbb{R}^N$ (specifically, Ω will be the unit cube in $\mathbb{R}^N$) or $\Omega = \mathbb{R}^N$ and V is a weight on Ω, that is, a.e. non-negative and locally integrable function on Ω. In some cases both the inequalities overlap as we shall see later.

Our primary concern is behaviour of the constant c in the right hand sides with respect to N. Naturally c must be independent of f and it might be dependent on Ω and p as it is usual in embedding theorems but we will study the case when the dimension grows. Recall the Sobolev embedding theorem, which states that $W_0^{1,p}(\Omega)$ ($1 \leqslant p < N$, and if Ω has a sufficiently smooth boundary) is imbedded into $L^q(\Omega)$, where $1/q = 1/p - 1/N$. Hence if $N \to \infty$, then $q \to p_+$ and there is a natural question about some residual improvement of the integrability independent of N, that is, some proper subspace of L^p, into which $W_0^{1,p}(\Omega)$ is embedded and the norm of the embedding is independent of N. Note in passing that such a target space, if exists at all, must lie outside the range of Lebesgue spaces. Let us also observe that one should be very careful when working with norms—the usual concept of equivalent norms usually includes dependence on the dimension without explicit warning (see also [36] for some observations in this direction).

Inequality (1.2) has been studied in $\mathbb{R}^N$ or on domains in $\mathbb{R}^N$ using various methods, in particular, in connection with a boom of weighted inequalities during the 1970s and it is alternatively called the *trace inequality* or the *uncertainty principle*. It has found numerous applications in analysis.

As to (1.2) let us recall at least [1], [28], [29], [32], [18], [11], the special case $N = 2$ in [12], [24], [2].

The question concerning independence of the dimension has also importance for instance in theory of contraction semigroups and finds applications in quantum physics (see e.g. [25] for some of the references).

Notation. We shall use standard notations for the Sobolev, Lebesgue, Lorentz and Orlicz spaces, the respective domain will be sometimes omitted if no confusion can arise. Sometimes we shall write $\|f|L^p\|$ etc. instead of $\|f\|_p$ and the like. We shall work with Sobolev spaces of functions with zero traces, defined as a completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla f|L^p(\Omega)\|$. Note that this space does not generally coincide with the space of functions in $W^{1,p}(\mathbb{R}^N)$ whose support is contained in $\overline{\Omega}$. Both spaces coincide e.g. if Ω has a Lipschitz boundary. (See, for example, Triebel [35], Theorem 4.30, p. 121 for a more general result.)

Various constants independent of functions will be often denoted by the same symbol c and the like.

If V is a weight in a domain $\Omega \subset \mathbb{R}^N$, then the *weighted Lebesgue space* $L^p(V) = L^p(V)(\Omega)$ is defined as the space of all measurable f on Ω with the finite norm $\|f|L^p(V)\| = \left(\int_\Omega |f(x)|^p V(x)\,dx\right)^{1/p}$. If f is a measurable function in $\mathbb{R}^N$, then f^* will denote its *non-increasing rearrangement*.

If Φ is a Young function, that is, if Φ is even, convex, $\Phi(0) = 0$, $\lim_{t\to\infty} \Phi(t)/t = \infty$, and $\Omega \subset \mathbb{R}^N$ is measurable, then $m(\Phi, f) = \int_\Omega \Phi(f(x))\,dx$ is the *modular* and the (quasi-)norm in the corresponding Orlicz space $L_\Phi = L_\Phi(\Omega)$ is the Minkowski functional of the modular unit ball, namely, $\|f|L_\Phi\| = \inf\{\lambda > 0 : m(\Phi, f/\lambda) \leqslant 1\}$ (the Luxemburg norm). Replacing dx by $V(x)dx$ where V is a weight function, we get the *weighted Orlicz space* $L_\Phi(V)$ (or $L_\Phi(\Omega, V)$) in a more detailed notation).

Assumptions on Φ can be weaker, e.g. Φ can be convex just on some interval (a, ∞) $(a > 0)$ or one can even consider Φ increasing rather than convex. We refer to [31] for the theory of classical Orlicz spaces and of general modular spaces.

We shall restrict ourselves to a characterization of weighted Orlicz spaces $L_\Phi(V) = L_\Phi(\Omega, V)$, generated by the modular $m(\Phi, V, f) = \int_\Omega \Phi(f(x))V(x)\,dx$ as special Musielak-Orlicz spaces. Generally, if $\Phi = \Phi(x,t) : \Omega \times \mathbb{R} \to [0, \infty)$ is a *generalized Young function* or the *Musielak-Orlicz function*, that is, Φ is a Young function of the variable t for each fixed $x \in \Omega$ and a measurable function of the variable x for each fixed $t \in \mathbb{R}$, then $m(\Phi, f) = \int_\Omega \Phi(x, f(x))\,dx$ is a modular and we can consider the corresponding Orlicz space, which is called the *Musielak-Orlicz space*. Hence with the modular $m(\Phi, V, f)$ the weighted Orlicz space becomes a Musielak-Orlicz space.

Special Orlicz spaces, with the generating Young function $t \mapsto |t|^p \log^\alpha(1 + |t|)$, $t \in \mathbb{R}$, will be denoted by $L^p \log^\alpha(1 + L)$ $(1 \leqslant p < \infty,\ \alpha > 0)$. The symbol $L_{\exp t^\alpha}$ for $\alpha > 0$ will stand for the space with the Young function $t \mapsto \exp(|t|^\alpha) - 1$, $t \in \mathbb{R}$. For $\alpha = 1$ we shall write $L^p \log(1 + L)$ and $L_{\exp}$. Note that the function $t \mapsto t^p \log^\alpha(1 + t)$ is not generally convex near the origin. It is, however, a purely technical problem to consider an equivalent Young function convex on the whole of $\mathbb{R}$.

2 The basic main result

Our first attempt to find the target space for the Sobolev dimension-free embedding goes back to [20], where we used the celebrated Gross logarithmic inequality [13], generalized later in various directions by several authors, see, e.g. [15], [14]. Recall that the Gross logarithmic inequality (see [25] for a detailed account and [3], [5], [10] for further interesting discussions of the topic),

$$\int_{\mathbb{R}^N} |f(x)|^2 \log\left(\frac{|f(x)|^2}{\|f\|_2^2}\right) dx + N\|f\|_2^2 \leqslant \frac{1}{\pi}\int_{\mathbb{R}^N} |\nabla f(x)|^2\, dx, \tag{2.1}$$

gives, for a function $f \in W^{1,2}(\mathbb{R}^N)$ and supported in a bounded domain $\Omega \subset \mathbb{R}^N$, $\|\nabla f|L^2(\Omega)\| \leqslant 1$, and sufficiently large N,

$$\int_{\Omega} |f(x)|^2 \log|f(x)|\, dx \leqslant \frac{1}{2\pi}\int_{\Omega} |\nabla f(x)|^2\, dx \tag{2.2}$$

(since for N large enough we have $\log\|f\|_2 \leqslant 0$; this follows from the claim on the best constant in the Sobolev embedding and simple application of Hölder's inequality—see (2.11)).

Note that one can formally put 0 in the integral on the left-hand side of (2.1) and (2.2) if $f(x) = 0$ (which corresponds well to $\lim_{t\to 0_+} t^\delta \log t = 0$ for any $\delta > 0$).

Inequalities (2.1) and/or (2.2) are *not* embedding inequalities since they express a fine balance of positive and negative terms.

In [20] we have employed the Gross theorem to show that

$$\int_{B} |f(x)|^2 \log(1 + |f(x)|/\|\nabla f\|_2)\, dx \leqslant c\|f|W_0^{1,2}(B)\|^2 \tag{2.3}$$

($W_0^{1,2}(B) = \overline{C_0^\infty(B)}^{W^{1,2}(B)}$, B being the unit ball in $\mathbb{R}^N$) with a constant c independent of f and N.

The next natural step was a generalization of (2.3), that is,

$$\int_{\Omega} |f(x)|^p \log\left(1 + \frac{|f(x)|}{\|\nabla f\|_p}\right) dx \leqslant c\|\nabla f|L^p(\Omega)\|^p,$$

and the weighted dimension-free embedding

$$\int_{\Omega} |f(x)|^p V(x)\, dx \leqslant c\|\nabla f|L^p(\Omega)\|^p, \tag{2.4}$$

for $f \in W_0^{1,p}(\Omega)$, where $2 \leqslant p < \infty$ when $\Omega = Q = (0,1)^N$, the unit cube in $\mathbb{R}^N$, and $1 < p < \infty$ when $\Omega = B$, c depending just on p and V. Note that the reason to consider $\Omega = Q$ is that the measure of B tends to 0 as $N \to \infty$ and this affects essentially the corresponding estimates. Conditions for V will be derived from a variant of Ishii's embedding theorem for generalized Orlicz-Musielak spaces [16] and [31], and will be expressed in terms of suitable exponential integrability of (a multiple of) V.

The generalized form of the Gross inequality for $1 < p < \infty$, see Gunson [14], has actually rather surprising form:

$$\int_{\mathbb{R}^N} |f(x)|^p \log(|f(x)|)\, dx + \gamma_{N,p} \leqslant \int_{\mathbb{R}^N} |\nabla f(x)|^p\, dx, \tag{2.5}$$

for all $f \in W^{1,p}(\mathbb{R}^N)$, $\|f\|_p = 1$, with

$$\begin{aligned}\gamma_{N,p} = \frac{N}{p} + \frac{N\log\pi}{2p} + \frac{N\log p}{p^2} - \frac{N(p-1)\log(p-1)}{p^2}\\ - \frac{1}{p}\log\left(\frac{\Gamma(1+N/2)}{\Gamma(1+N/p')}\right),\end{aligned} \tag{2.6}$$

where Γ is the Gamma function and $p' = p/(p-1)$.

Substituting $f(x)/\|f\|_p$ into (2.5) we get the more usual Lebesgue form of the above inequality, namely,

$$\int_{\mathbb{R}^N} |f(x)|^p \log\frac{|f(x)|}{\|f\|_p}\, dx + \gamma_{N,p}\, \|f\|_p^p \leqslant \int_{\mathbb{R}^N} |\nabla f(x)|^p\, dx. \tag{2.7}$$

First of all we shall make use of the (generalized) Gross logarithmic inequality to get the following

Theorem 2.1 ([22]). *Let* $2 \leqslant p \leqslant \infty$. *Then*

$$\int_Q |f(x)|^p \log\left(1 + \frac{|f(x)|}{\|\nabla f\|_p}\right) dx \leqslant c\|\nabla f|L^p(Q)\|^p \tag{2.8}$$

for all $f \in W_0^{1,p}(Q)$ *and some* c *independent of* f *and* N.

The following theorem is a weighted variant of the dimension-free estimate.

Theorem 2.2 ([22]). *Under the assumptions of the preceding Theorem, if* $V \in L_{\exp}$, *then*

$$\int_Q |f(x)|^p V(x)\, dx \leqslant c \int_Q |\nabla f(x)|^p\, dx, \qquad f \in W_0^{1,p}(Q). \tag{2.9}$$

Sketch of the proofs of Theorem 2.1 *and* 2.2. We shall restrict ourselves to the main steps and explaining the idea of the proofs.

Step 1. By a suitable manipulation with the constant $\gamma_{N,p}$ from (2.6) it is possible to get relation more similar to the original Gross inequality:

$$\int_{\mathbb{R}^N} |f(x)|^p \log\frac{|f(x)|}{\|f\|_p}\, dx \leqslant c(p) N\log N \|f\|_p^p + \int_{\mathbb{R}^N} |\nabla f(x)|^p\, dx, \tag{2.10}$$

true for all $f \in W^{1,p}(\mathbb{R}^N)$.

The best constant C in the Sobolev embedding $W_0^{1,p} \hookrightarrow L^q$, $1 \leqslant p < N$, $N \geqslant 3$,

$$\left(\int_{\mathbb{R}^N} |f(x)|^{Np/(N-p)}\, dx\right)^{(N-p)/Np} \leqslant C\|\nabla f|L^p\|, \qquad f \in W^{1,p}(\mathbb{R}^N),$$

is (see e.g. [34])

$$C = \sqrt{1/\pi}\frac{1}{N^{1/p}}\left(\frac{p-1}{N-p}\right)^{1-1/p}\left(\frac{\Gamma(N)\Gamma(1+N/2)}{\Gamma(N/p)\Gamma(1+N/p')}\right)^{1/N}. \tag{2.11}$$

Using asymptotic properties of the Gamma function (Stirling's formula) we obtain $C \sim 1/N^{1/2}$. Denoting again by f the extension of $f \in W_0^{1,p}(Q)$ by zero to the whole of $\mathbb{R}^N$, we have

$$\|f|L^p(Q)\|^p \leqslant \frac{c}{N^{p/2}}\|\nabla f|L^p(\mathbb{R}^N)\|^p.$$

Together,

$$c(p)N\log N\,\|f|L^p\|^p \leqslant \frac{c(p)\log N}{N^{(p/2)-1}}\|\nabla f|L^p\|^p.$$

If $p>2$, then the constant on the right hand side is uniformly bounded with respect to N.

Inserting this into (2.10) we get, for $2<p<N$,

$$\int_Q |f(x)|^p \log\frac{|f(x)|}{\|f\|_p}\,dx \leqslant c\|\nabla f|L^p\|^p, \tag{2.12}$$

for all $f \in W_0^{1,p}(Q)$, with a constant c independent of f and N. The case $p=2$ follows easily because $\gamma_{N,2}>0$ so that the term with this constant can be omitted directly.

Step 2. Now we want to show that for $N \geqslant 3$ and $2 \leqslant p < \infty$ there exists c independent of N such that

$$\int_Q |f(x)|^p \log\left(1+\frac{|f(x)|}{\|\nabla f\|_p}\right)dx \leqslant c\|\nabla f|L^p\|^p \tag{2.13}$$

for all $f \in W_0^{1,p}(Q)$, and the norm of the embedding of $W_0^{1,p}(Q)$ into $L^p\log(1+L)$ is independent of N. This part of the proof does not use particularly deep tools but it is rather technical and lengthy. The idea is to prove the auxiliary inequality

$$\begin{aligned}\int_{\mathbb{R}^N}(|f(x)|+\varepsilon^2 s_\varepsilon(x))^p\log(1+|f(x)|+\varepsilon^2 s_\varepsilon(x))\,dx\\ \leqslant c\|\nabla f|L^p\| + \varepsilon c(N) < \infty,\end{aligned} \tag{2.14}$$

where c is independent of the dimension ($c(N)$ might depend on N but it is independent of f) and of ε and with a suitable smooth function s_ε supported in $(1+\varepsilon)Q$. Letting ε tend to 0, inequality (2.13) follows from (2.14) by virtue of Fatou's lemma.

For a general $f \in W_0^{1,p}(Q)$, $f \neq 0$, note that (2.13) holds for $f(x)/(\|\nabla f|L^p\|)$, hence

$$\int_Q |f(x)|^p\log\left(1+\frac{|f(x)|}{\|\nabla f|L^p\|}\right)dx \leqslant c\int_Q |\nabla f(x)|^p\,dx.$$

Step 3. Theorem 2.2 follows from the embedding (2.13), nevertheless, after some longer effort. We shall just survey the main points of the procedure. Our weighted embedding will be the second part of the chain of $W_0^{1,p}(Q) \hookrightarrow L^p\log(1+L)(Q) \hookrightarrow L^p(V)(Q)$ and the proof borrows from the paper by Ishii [16], see also [31], which gives necessary and sufficient condition for an embeddings between general Musielak-Orlicz spaces. First one proves that

$$\int_Q |g(x)|^p V(x)\,dx \leqslant c \int_Q |g(x)|^p \log(1+|g(x)|)\,dx \tag{2.15}$$

for all g such that $\|g|L^p\log(1+L)\| = 1$; the last equality is equivalent to

$$\int_Q |g(x)|^p \log(1+|g(x)|)\,dx = 1$$

(g belongs to the modular unit sphere). This can be proved by contradiction. Should there exists a sequence g_k, where g_k have the $L^p\log(1+L)$ norm equal to 1, and $B_k \to \infty$ such that

$$\int_Q |g_k(x)|^p V(x)\,dx \geqslant B_k \int_Q |g_k(x)|^p \log(1+|g_k(x)|)\,dx = B_k, \tag{2.16}$$

choose $\varkappa < 1$ and a sequence $A_k \searrow 0$. Then for large k,

$$\int_Q A_k^p |g_k(x)|^p \log(1+A_k|g_k(x)|)\,dx < \frac{\varkappa}{4K}.$$

Hence

$$\int_Q A_k^p |g_k(x)|^p V(x)\,dx \leqslant \varkappa$$

and putting $A_k = 1/B_k^{1/p}$ we get

$$\int_Q |g_k(x)|^p V(x)\,dx \leqslant \varkappa B_k, \tag{2.17}$$

which contradicts with (2.16). Hence the embedding (2.15) holds and according to Ishii's theorem [16] the embedding (2.15) is equivalent to

$$t^p V(x) \leqslant K t^p \log(1+t) + h(x), \qquad t > 0, \quad x \in Q, \tag{2.18}$$

where K is some constant, $K \geqslant 1$, and h is a suitable non-negative function, integrable over $(0,\infty)$. This condition can be reformulated as

$$\sup_{t>0} \left[tV(x)/K - t\log(1+t^{1/p}) \right] \leqslant h(x)/K, \qquad x \in Q \tag{2.19}$$

with some integrable function h. Hence the left hand side of (2.19) should be integrable over Q. Invoking Young's inequality for complementary Young functions (see e.g. [19]) this is guaranteed by integrability of $\Psi(V(x)/K)$, where Ψ is a Young function complementary to $t \mapsto |t|\log(1+|t|^{1/p})$. Plainly Ψ is equivalent to $t \mapsto \exp|t| - 1$. $\square$

3 An alternative approach and general α

Theorems of the preceding section suggest that we might look about an improvement in the sense of higher exponents at the logarithmic function. Moreover, the case $1 < p < 2$ has not been discussed until now. Consider again the space $W_0^{1,p}(Q)$, where Q is the unit cube in $\mathbb{R}^N$. Then relying on suitable expression for the (quasi-)norm in logarithmic Lebesgue spaces, using the claim on the best constant in the Sobolev embedding, one can give a surprisingly simple proof of a stronger dimension-free estimate. The weight estimate follows then as a corollary or, alternatively, with additional plugging of extrapolation characterization of exponential Orlicz spaces.

The same estimate for the logarithmic Lebesgue norms was independently proved by Martín and Milman [26], based on the isoperimetric inequality. It was quite surprising that both the complicated proofs from [26] and from our paper [22] are covered by the procedure described in the remainder of this section.

In [23] an interested reader can find detailed discussions on weight functions, some other variants of the proof, and a survey of equivalent (quasi-)norms in $L^p \log^\alpha(1+L)$.

First of all recall that an equivalent (quasi-)norm in $L^p \log^\alpha(1+L)$ is given by the formula

$$\left(\int_0^1 f^*(t)^p \log^\alpha \frac{e}{t}\, dt\right)^{1/p},$$

see, e.g. [6]. Here, the equivalence constants do not depend on the dimension ([23]).

Theorem 3.1 ([22], (3.1)). *If $1 < p < N$, then*

$$\|f|L^p \log^{p/2}(1+L)(Q)\| \leqslant c\|\nabla f|L^p(Q)\| \tag{3.1}$$

for all $f \in W_0^{1,p}(Q)$ and some c independent of f and N.

Proof. The embedding inequality follows from the following chain of estimates:

$$\begin{aligned}
&\int_0^1 f^*(t)^p \left(\log \frac{e}{t}\right)^\alpha dt \\
&\qquad \leqslant \left(\int_0^1 f^*(t)^{Np/(N-p)}\, dt\right)^{(N-p)/N} \left(\int_0^1 \left(\log\frac{e}{t}\right)^{N\alpha/p} dt\right)^{p/N} \\
&\qquad \leqslant \frac{c}{N^{p/2}}\|\nabla f|L^p\|^p \left[\Gamma\left(1+\frac{N\alpha}{p}\right)\right]^{p/N} \\
&\qquad \leqslant \frac{c}{N^{p/2}}\|\nabla f|L^p\|^p \left[\left(\frac{N\alpha}{p}\right)^{p/(N\alpha)}\right]^\alpha \left(\left[\Gamma\left(\frac{N\alpha}{p}\right)\right]^{p/(N\alpha)}\right)^\alpha \\
&\qquad \leqslant \frac{c}{N^{p/2}}\|\nabla f|L^p\|^p \left(\frac{N\alpha}{p}\right)^\alpha .
\end{aligned}$$

We have used claim on the best constant for the Sobolev embedding and properties of the Gamma function ($\Gamma(\xi)^{1/\xi} \sim \xi$ as $\xi \to \infty$). It is clear that $\alpha \leqslant p/2$ guarantees independence of N. ☐

Remark 18. As to the weighted inequality (1.2) one can proceed similarly. We use [4] as to the best constant for the improved Sobolev embedding of $W_0^{1,p}$ $(1 < p < N)$ into the Lorentz space $L^{q,p}$, where q is the Sobolev exponent corresponding to p; note in passing that it has the same value as for the embedding into L^q. We have

$$\begin{aligned}\int_Q |f(x)|^p V(x)\,dx &\leqslant c\frac{\|V|L^{N/p,\infty}\|}{N^{p/2}}\,\|\nabla f|L^p\|^p\\ &\leqslant c\frac{\|V|L^{N/p}\|}{N^{p/2}}\,\|\nabla f|L^p\|^p\\ &\leqslant cc(p)\frac{\|V|L^{N/p}\|}{(N/p)^{p/2}}\,\|\nabla f|L^p\|^p\\ &\leqslant cc(p)\sup_{q\geqslant 1}\frac{\|V|L^q\|}{q^{p/2}}\,\|\nabla f|L^p\|^p.\end{aligned} \tag{3.2}$$

Now it suffices to use the standard extrapolation fact that

$$\|V|L_{\exp t^\beta}(Q)\| \sim \sup_{q\geqslant 1}\frac{\|V|L^q(Q)\|}{q^{1/\beta}} < \infty. \tag{3.3}$$

On the other hand one can use the equivalence

$$\|V|L_{\exp t^\beta}(Q)\| \sim \sup_{0<t<1}\frac{V^*(t)}{(1+|\log t|)^{1/\beta}} < \infty. \tag{3.4}$$

to obtain (1.2) *as a consequence of Theorem* 3.1.

Theorem 3.2 ([22], Theorem (3.1)). *If* $1 < p < N$*, then*

$$\left(\int_Q |f(x)|^p\, V(x)\,dx\right)^{1/p} \leqslant c\Big(\|V|L_{\exp t^{2/p}}(Q)\|\Big)^{1/p}\,\|\nabla f|L^p(Q)\| \tag{3.5}$$

for all $f \in W_0^{1,p}(Q)$ *and some* c *independent of* f *and* N.

Proof. Let $V \in L_{\exp t^{2/p}}(Q)$. It follows from (3.4) that

$$V^*(t) \leqslant c\,\|V|L_{\exp t^{2/p}}(Q)\|\,(1+|\log t|)^{p/2}\,.$$

Using the estimate in the proof of Theorem 3.1 with $\alpha = p/2$ we find

$$\begin{aligned}\int_Q |f(x)|^p\, V(x)\,dx &\leqslant \int_0^1 f^*(t)^p\, V^*(t)\,dt\\ &\leqslant \|V|L_{\exp t^{2/p}}(Q)\| \int_0^1 (1+|\log t|)^{p/2}\, f^*(t)^p\,dt\\ &\leqslant c\,\|V|L_{\exp t^{2/p}}(Q)\|\,\|\nabla f|L^p(Q)\|^p\,.\end{aligned}$$

□

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Miroslav Krbec
Institute of Mathematics, Academy of Sciences
Žitná 25, 115 67 Praha 1
Czech Republic

Mathematisches Institut
Fakultät für Mathematik und Informatik
Friedrich-Schiller-Universiät
Ernst-Abbe-Platz 1-2, 07743 Jena
Germany
E-mail: mhj@uni-jena.de

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