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# ON THE BOUNDEDNESS OF THE ANISOTROPIC FRACTIONAL MAXIMAL OPERATOR FROM ANISOTROPIC COMPLEMENTARY MORREY-TYPE SPACES TO ANISOTROPIC MORREY-TYPE SPACES

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Communicated by V.I. Burenkov

**Key words:** anisotropic fractional maximal operator, anisotropic local Morrey-type spaces, anisotropic complementary Morrey-type spaces, dual Hardy operator.

**AMS Mathematics Subject Classification:** 42B20, 42B25, 42B35.

**Abstract.** The problem of the boundedness of the anisotropic fractional maximal operator $M_\alpha^d$ from anisotropic complementary Morrey-type spaces to anisotropic Morrey-type spaces is reduced to the problem of boundedness of the dual Hardy operator in weighted $L_p$-spaces on the cone of non-negative non-increasing functions, which allows obtaining sharp sufficient conditions for the boundedness of $M_\alpha^d$.

## 1 Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at $x$ of radius $r$ and ${}^{\complement}B(x,r)$ denote its complement. Let $d = (d_1, \dots, d_n)$, $d_i \geq 1$, $i = 1, \dots, n$, $|d| = \sum_{i=1}^n d_i$ and $t^d x \equiv \left(t^{d_1} x_1, \dots, t^{d_n} x_n\right)$. By [2, 9], the function $F(x,\rho) = \sum_{i=1}^n x_i^2 \rho^{-2d_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x,\rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. It is a simple matter to check that $\rho(x-y)$ defines a distance between any two points $x$, $y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric $\rho$, defines a homogeneous metric space ([2, 9]). The balls with respect to $\rho$, centered at $x$ of radius $r$, are just the ellipsoids

$$\mathcal{E}_d(x,r) = \left\{ y \in \mathbb{R}^n : \frac{(y_1 - x_1)^2}{r^{2d_1}} + \cdots + \frac{(y_n - x_n)^2}{r^{2d_n}} < 1 \right\},$$

with the Lebesgue measure $|\mathcal{E}_d(x,r)| = v_n r^{|d|}$, where $v_n$ is the volume of the unit ball in $\mathbb{R}^n$. Let also ${}^{\complement}\mathcal{E}_d(x,r) = \mathbb{R}^n \setminus \mathcal{E}_d(x,r)$ be the complement of $\mathcal{E}_d(0,r)$. If $d = \mathbf{1} \equiv (1, \dots, 1)$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_{\mathbf{1}}(x,r) = B(x,r)$. Let $f \in L_1^{loc}(\mathbb{R}^n)$. The anisotropic fractional maximal operator $M_\alpha^d$ is defined by

$$\left(M_\alpha^d f\right)(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1+\frac{\alpha}{n}} \int_{\mathcal{E}(x,t)} |f(y)| dy,$$

where $0 \leq \alpha < n$ and $|\mathcal{E}(x,t)|$ is the Lebesgue measure of the ellipsoid $\mathcal{E}(x,t)$. If $\alpha = 0$, then $M^d \equiv M_0^d$ is the anisotropic Hardy-Littlewood maximal operator.

**Definition 1.** Let $0 < p, \theta \le \infty$ and let $w$ be a non-negative measurable function on $(0, \infty)$. We denote by $LM_{p\theta,w,d}$ the local Morrey-type space, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasi-norm

$$\|f\|_{LM_{p\theta,w,d}} \equiv \|f\|_{LM_{p\theta,w,d}(\mathbb{R}^n)} = \left\| w(r) \|f\|_{L_p(\mathcal{E}(0,r))} \right\|_{L_\theta(0,\infty)}.$$

In [1] the following statement was proved. (The isotropic case was considered in [5]).

**Lemma 1.1.** *Let $0 < p, \theta \le \infty$ and let $w$ be a non-negative measurable function on $(0, \infty)$. If for all $t > 0$*

$$\|w(r)\|_{L_\theta(t,\infty)} = \infty,$$

*then $LM_{p\theta,w,d} = \Theta$, where $\Theta$ is the set of all functions equivalent to $0$ on $\mathbb{R}^n$.*

**Definition 2.** Let $0 < p, \theta \le \infty$. We denote by $\Omega_\theta$ the set of all functions $w$ which are non-negative, measurable on $(0, \infty)$, not equivalent to 0 and such that for some $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty.$$

In the sequel, keeping in mind Lemma 1.1, when dealing with the spaces $LM_{p\theta,w,d}$ we always assume that $w \in \Omega_\theta$.

Various sufficient conditions for the boundedness of $M_\alpha^d$ from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$ were obtained in [1]. Moreover, in [1] for some values of the parameters also necessary and sufficient conditions for the boundedness of $M_\alpha^d$ were obtained. See also survey papers [3], Section 7; [4], Section 9.

We quote the main results of [1], which generalize the results for the isotropic case proved in [7].

**Lemma 1.2.** [1] *Let $1 < p_1 \le \infty$, $0 < p_2 \le \infty$, $0 \le \alpha < |d|$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$. Then the condition*

$$\alpha \le \frac{|d|}{p_1}$$

*is necessary for the boundedness of $M_\alpha^d$ from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$, in particular from $L_{p_1}$ to $LM_{p_2\theta_2,w_2,d}$.*

**Theorem 1.1.** [1] 1. *If $1 < p_1 \le \infty$, $0 < p_2 \le \infty$, $0 \le \alpha < |d|$, $0 < \theta_1$, $\theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, then the condition*

$$t^{\alpha - \frac{|d|}{p_1} + \min\{|d|-\alpha, \frac{|d|}{p_2}\}} \left\| w_2(r) \frac{r^{\frac{|d|}{p_2}}}{(t+r)^{\min\{|d|-\alpha, \frac{|d|}{p_2}\}}} \right\|_{L_{\theta_2}(0,\infty)} \le c_1 \|w_1\|_{L_{\theta_1}(t,\infty)} \tag{1.1}$$

*for all $t > 0$, where $c_1 > 0$ is independent of $t$, is necessary for the boundedness of $M_\alpha^d$ from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$.*

2. *If $1 < p_1 < \infty$, $0 < p_2 < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\theta_1 \le p_1$, $|d| \left( \frac{1}{p_1} - \frac{1}{p_2} \right)_+ \le \alpha < \frac{|d|}{p_1}$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, then the condition*

$$\left\| w_2(r) \frac{r^{\frac{|d|}{p_2}}}{(t+r)^{\frac{|d|}{p_1} - \alpha}} \right\|_{L_{\theta_2}(0,\infty)} \le c_2 \|w_1\|_{L_{\theta_1}(t,\infty)} \tag{1.2}$$

*for all* $t > 0$*, where* $c_2 > 0$ *is independent of* $t$*, is sufficient for the boundedness of* $M_\alpha^d$ *from* $LM_{p_1\theta_1,w_1,d}$ *to* $LM_{p_2\theta_2,w_2,d}$.

3. *In particular, if* $1 < p_1 \le p_2 < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\theta_1 \le p_1$, $\alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$*, then the condition*

$$\left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{|d|}{p_2}} \right\|_{L_{\theta_2}(0,\infty)} \le c_3 \, \|w_1\|_{L_{\theta_1}(t,\infty)} \tag{1.3}$$

*for all* $t > 0$*, where* $c_3 > 0$ *is independent of* $t$*, is necessary and sufficient for the boundedness of* $M_\alpha^d$ *from* $LM_{p_1\theta_1,w_1,d}$ *to* $LM_{p_2\theta_2,w_2,d}$.

Since $\alpha \le \frac{|d|}{p_1}$ is a necessary condition for the boundedness of $M_\alpha^d$ from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$ and from $L_{p_1}$ to $LM_{p_2\theta,w_2,d}$, a natural question arises whether, for $\frac{|d|}{p_1} < \alpha < |d|$, it is possible to find a space $Z$ such that $M_\alpha^d$ is bounded from $Z$ to the same target space $LM_{p_2\theta_2,w_2,d}$. In this paper we show that this is possible if $Z = {}^{\complement}LM_{p_1\theta_1,w_1,d} \cap L_{p_1}$, where ${}^{\complement}LM_{p_1\theta_1,w_1,d}$ is the local complementary Morrey-type space defined below, and we find necessary conditions and sufficient conditions close to necessary ones on $w_1$ and $w_2$ ensuring that $M_\alpha^d$ is bounded from ${}^{\complement}LM_{p_1\theta_1,w_1,d} \cap L_{p_1}$ to $LM_{p_2\theta_2,w_2,d}$.

## 2 Definitions and basic properties of complementary Morrey-type spaces

**Definition 3.** Let $0 < p, \theta \le \infty$ and let $w$ be a non-negative measurable function on $(0,\infty)$. By ${}^{\complement}LM_{p\theta,w,d}$ we denote the local complementary Morrey-type space (briefly the complementary Morrey-type or co-Morrey-type space), the space of all functions $f \in L_p({}^{\complement}\mathcal{E}(0,r))$ for all $r > 0$ with finite quasinorm

$$\|f\|_{{}^{\complement}LM_{p\theta,w,d}} \equiv \|f\|_{{}^{\complement}LM_{p\theta,w,d}(\mathbb{R}^n)} = \left\| w(r) \|f\|_{L_p({}^{\complement}\mathcal{E}(0,r))} \right\|_{L_\theta(0,\infty)}.$$

Along with the local Morrey-type spaces $LM_{p\theta,w,d}$ it makes sense to consider the global Morrey-type spaces $GM_{p\theta,w,d}$ of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasi-norm

$$\|f\|_{GM_{p\theta,w,d}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,w,d}} = \sup_{x\in\mathbb{R}^n} \left\| w(r) \|f\|_{L_p(\mathcal{E}(x,r))} \right\|_{L_\theta(0,\infty)}.$$

However, in the case of the complementary Morrey-type spaces ${}^{\complement}GM_{p\theta,w,d}$ the corresponding global variant of the spaces defined by the finiteness of the quasi-norm

$$\|f\|_{{}^{\complement}GM_{p\theta,w,d}} = \sup_{x\in\mathbb{R}^n} \left\| w(r) \|f\|_{L_p({}^{\complement}\mathcal{E}(x,r))} \right\|_{L_\theta(0,\infty)}$$

is of no particular interest because this expression is equal to the product $\|f\|_{L_p} \|w\|_{L_\theta(0,\infty)}$. Indeed, inequality $\|f\|_{{}^{\complement}GM_{p\theta,w,d}} \le \|f\|_{L_p} \|w\|_{L_\theta(0,\infty)}$ is obvious. On the other hand, given $R > 0$, $t > 0$, let $y = y(R,t) \in \mathbb{R}^n$ be such that $\rho(y) = R + t$, then for $0 < r \le t$, ${}^{\complement}\mathcal{E}(y,r) \supset \mathcal{E}(0,R)$, hence

$$\|f\|_{{}^{\complement}GM_{p\theta,w,d}} \ge \left\| w(r) \|f\|_{L_p({}^{\complement}\mathcal{E}(y,r))} \right\|_{L_\theta(0,t)} \ge \|f\|_{L_p(\mathcal{E}(0,R))} \|w(r)\|_{L_\theta(0,t)}.$$

Since this inequality holds for all $R>0$, $t>0$ it implies that

$$\|f\|_{{}^{\complement}GM_{p\theta,w,d}} \geq \|f\|_{L_p} \|w(r)\|_{L_\theta(0,\infty)}.$$

The condition $f \in LM_{p\theta,w,d}$ is aimed at describing the behaviour of $\|f\|_{L_p(\mathcal{E}(0,r))}$ for small $r>0$ hence of $f$ in a neighbourhood of the origin. If $f \notin L_p$, then it also imposes some restrictions on the behaviour of $f$ at infinity. However, if $f \in L_p$ it does not impose any further restrictions on the behaviour of $f$ at infinity. In contrast to this, the condition $f \in {}^{\complement}LM_{p\theta,w,d}$ is aimed at describing the behaviour of $\|f\|_{L_p({}^{\complement}\mathcal{E}(0,r))}$ for large $r>0$ hence of $f$ at infinity. If $f \notin L_p$, then it also imposes some restrictions on the behaviour of $f$ in a neighbourhood of the origin. If $f \in L_p$, then it does not impose any further restrictions on the behaviour of $f$ in a neighbourhood of the origin.

**Lemma 2.1.** *Let $0<p,\theta\leq\infty$ and $w$ be a non-negative measurable function on $(0,\infty)$. If for all $t>0$*

$$\|w(r)\|_{L_\theta(0,t)} = \infty, \tag{2.1}$$

*then ${}^{\complement}LM_{p\theta,w,d} = \Theta$.*

*Proof.* Let (2.1) be satisfied and $f$ be not equivalent to zero. Then, for some $t_0>0$, $\|f\|_{L_p({}^{\complement}\mathcal{E}(0,t_0))} > 0$. Hence

$$\|f\|_{{}^{\complement}LM_{p\theta,w,d}} \geq \Big\| w(r) \|f\|_{L_p({}^{\complement}\mathcal{E}(0,r))} \Big\|_{L_\theta(0,t_0)} \geq \|f\|_{L_p({}^{\complement}\mathcal{E}(0,t_0))} \|w(r)\|_{L_\theta(0,t_0)}.$$

Therefore $\|f\|_{{}^{\complement}LM_{p\theta,w,d}} = \infty$. □

**Definition 4.** Let $0<\theta\leq\infty$. We denote by ${}^{\complement}\Omega_\theta$ the set of all functions $w$ non-negative and measurable on $(0,\infty)$ such that for some $t>0$

$$\|w(r)\|_{L_\theta(0,t)} < \infty. \tag{2.2}$$

In the sequel, keeping in mind Lemma 2.1, when dealing with the spaces ${}^{\complement}LM_{p\theta,w,d}$ we always assume that $w \in {}^{\complement}\Omega_\theta$.

Note that if $w(r)\equiv 1$, then $LM_{p\infty,1,d} = {}^{\complement}LM_{p\infty,1,d} = L_p$.

For real-valued functions $\varphi,\psi$ defined on a set $I$ we shall write $\varphi \asymp \psi$ on $I$ if there exist $c,c'>0$ such that $c\varphi(t) \leq \psi(t) \leq c'\varphi(t)$ for all $t \in I$.

**Lemma 2.2.** *Let $0<p,\theta\leq\infty$ and $w_1$, $w_2 \in {}^{\complement}\Omega_\theta$. Then* [1]

$${}^{\complement}LM_{p\theta,w_1,d} = {}^{\complement}LM_{p\theta,w_2,d} \Longleftrightarrow \|w_1\|_{L_\theta(0,t)} \asymp \|w_2\|_{L_\theta(0,t)} \;\text{ on }\; (0,\infty).$$

*Proof.* The proof is similar to the proof of Lemma 2.4 in [7]. □

Recall that if $w_1$, $w_2 \in \Omega_\theta$, then $LM_{p\theta,w_1,d} = LM_{p\theta,w_2,d} \Longleftrightarrow \|w_1\|_{L_\theta(t,\infty)} \asymp \|w_2\|_{L_\theta(t,\infty)}$ on $(0,\infty)$ (see [1, 6]).

[1] For quasi-normed spaces $Z_1$ and $Z_2$ the notation $Z_1 = Z_2$ means that two continuous embeddings $Z_1 \subset Z_2$ and $Z_2 \subset Z_1$ hold.

**Corollary 2.1.** *Let $0 < p, \theta \leq \infty$ and $w_1, w_2 \in L_\theta(0,\infty)$, $w_1, w_2 > 0$. Then*

$$^{\complement}LM_{p\theta,w_1,d} = {}^{\complement}LM_{p\theta,w_2,d} \Longleftrightarrow \|w_1\|_{L_\theta(0,t)} \asymp \|w_2\|_{L_\theta(0,t)} \text{ on } (0,t_0) \text{ for some } t_0 > 0.$$

**Lemma 2.3.** *Let $1 < p_1 \leq \infty$, $0 < p_2 \leq \infty$, $0 \leq \alpha < |d|$, $0 < \theta_1, \theta_2 \leq \infty$, $w_1 \in {}^{\complement}\Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$. Then the condition*

$$\alpha \geq \frac{|d|}{p_1}$$

*is necessary for the boundedness of $M^d_\alpha$ from ${}^{\complement}LM_{p_1\theta_1,w_1,d} \cap L_{p_1}$ to $LM_{p_2\theta_2,w_2,d}$.*

*Proof.* Assume that $\alpha < \frac{|d|}{p_1}$ and $M^d_\alpha$ is bounded from ${}^{\complement}LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$. Let $f(x) = \rho(x)^{-\beta}$ if $\rho(x) \leq 1$, where $\alpha < \beta < \frac{|d|}{p_1}$, and $f(x) = 0$ if $\rho(x) > 1$. Then $f \in L_{p_1}$ and $f \in {}^{\complement}LM_{p_1\theta_1,w_1,d}$ since

$$\|f\|_{^{\complement}LM_{p_1\theta_1,w_1,d}} \leq \|w_1\|_{L_{\theta_1}(0,1)} \left\|\rho(x)^{-\beta}\right\|_{L_{p_1}(\mathcal{E}(0,1))} < \infty.$$

On the other hand for all $x \in \mathbb{R}^n$

$$M^d_\alpha f(x) \geq \lim_{t\to 0} |\mathcal{E}(x,t)|^{-1+\frac{\alpha}{n}} \int\limits_{\mathcal{E}(x,t)\setminus\mathcal{E}(x,\rho(x)+2)} \rho(y)^{-\beta} dy \geq c_4 \lim_{t\to 0} t^{\alpha-\beta} = \infty,$$

where $c_4$ depends only on $n$, $\alpha$ and $\beta$. □

## 3 $L_p$-estimates on the complements of balls

In order to obtain conditions on $w_1$ and $w_2$ ensuring the boundedness of $M^d_\alpha$ for other values of the parameters and to obtain simpler conditions for the case $p_1 = \theta_1$, $p_2 = \theta_2$ we shall reduce the problem of the boundedness of $M^d_\alpha$ from the complementary Morrey-type spaces to the local Morrey-type spaces to the problem of the boundedness of the dual Hardy operator in weighted $L_p$-spaces on the cone of non-negative non-increasing functions.

**Lemma 3.1.** [1, 6] *Let $1 < p_1 < \infty$, $0 < p_2 < \infty$ and $|d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \leq \alpha < |d|$. Then there exists $c_5 > 0$ such that*

$$\|M^d_\alpha f\|_{L_{p_2}(\mathcal{E}(0,r))} \leq c_5 r^{\frac{|d|}{p_2}} \left(\int_{\mathbb{R}^n} \frac{|f(x)|^{p_1}}{(\rho(x)+r)^{|d|-\alpha p_1}} dx\right)^{\frac{1}{p_1}} \tag{3.1}$$

*for all $r > 0$ and for all $f \in L^{loc}_1(\mathbb{R}^n)$.*

**Lemma 3.2.** *Let $\varphi$ be a function non-negative and measurable on $\mathbb{R}^n$. Then for all $r > 0$ and for $\beta > 0$*

$$\beta\, 2^{-\beta} \int_r^\infty \left(\int_{\mathcal{E}(0,t)} \varphi(x)dx\right) \frac{dt}{t^{1+\beta}} \leq \int_{\mathbb{R}^n} \frac{\varphi(x)dx}{(\rho(x)+r)^\beta} \leq$$

$$\leq \beta \int_r^\infty \left(\int_{\mathcal{E}(0,t)} \varphi(x)dx\right) \frac{dt}{t^{1+\beta}}$$

*and for all* $r>0$ *and for* $\beta \leq 0$

$$r^{|\beta|} \int_{\mathbb{R}^n} \varphi(x)dx + |\beta| \int_r^\infty \left(\int_{\complement \mathcal{E}(0,t)} \varphi(x)dx\right) \frac{dt}{t^{\beta+1}} \leq \int_{\mathbb{R}^n} \frac{\varphi(x)dx}{(\rho(x)+r)^\beta} \leq$$

$$\leq 2^{|\beta|} \left(r^{|\beta|} \int_{\mathbb{R}^n} \varphi(x)dx + |\beta| \int_r^\infty \left(\int_{\complement \mathcal{E}(0,t)} \varphi(x)dx\right) \frac{dt}{t^{\beta+1}}\right).$$

*Proof.* The proof is similar to the proof of Lemma 4.3 in [7]. □

The proofs of main results in [1] were based on the following corollaries of Lemma 3.1 and the first part of Lemma 3.2.

**Corollary 3.1.** *Let* $1<p_1<\infty$, $0<p_2<\infty$ *and* $|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ \leq \alpha < \frac{|d|}{p_1}$. *Then there exists* $c_6>0$ *such that*

$$\|M_\alpha^d f\|_{L_{p_2}(\mathcal{E}(0,r))} \leq c_6 r^{\frac{|d|}{p_2}} \left(\int_r^\infty \left(\int_{\mathcal{E}(0,t)} |f(x)|^{p_1} dx\right) \frac{dt}{t^{|d|-\alpha p_1+1}}\right)^{\frac{1}{p_1}} \tag{3.2}$$

*for all* $r>0$ *and for all* $f \in L_{p_1}^{loc}(\mathbb{R}^n)$.

**Corollary 3.2.** *Let* $1<p_1<\infty$, $0<p_2<\infty$ *and* $|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ \leq \alpha \leq \frac{|d|}{p_1}$, *then there exists* $c_7>0$ *such that*

$$\|M_\alpha^d f\|_{L_{p_2}(\mathcal{E}(0,r))} \leq c_7 r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|f\|_{L_{p_1}} \tag{3.3}$$

*for all* $r>0$ *and for all* $f \in L_{p_1}$.

**Remark 1.** Let $1<p_1<\infty$, $0<p_2<\infty$ and $\frac{|d|}{p_1}<\alpha<|d|$. Then for any function $\psi$ non-negative and measurable on $(r,\infty)$ the inequality

$$\|M_\alpha^d f\|_{L_{p_2}(\mathcal{E}(0,r))} \leq c_8(r) \left(\int_r^\infty \left(\int_{\mathcal{E}(0,t)} |f(x)|^{p_1} dx\right) \psi(t)dt\right)^{\frac{1}{p_1}}, \tag{3.4}$$

where $c_8(r)>0$ is independent of $f$, is meaningless. Indeed, if for all $s>0$ $\int_s^\infty \psi(t)dt = \infty$, then

$$\int_r^\infty \left(\int_{\mathcal{E}(0,t)} |f(x)|^{p_1} dx\right) \psi(t)dt = \infty,$$

for all $f$ which are not equivalent to 0 on $\mathbb{R}^n$. If $\int_r^\infty \psi(t)dt < \infty$, then (3.4) implies that

$$\|M_\alpha^d f\|_{L_{p_2}(\mathcal{E}(0,r))} \leq c_8(r) \left(\int_r^\infty \psi(t)dt\right)^{\frac{1}{p_1}} \|f\|_{L_{p_1}}.$$

However, this inequality cannot hold because there exists a function $f \in L_{p_1}(\mathbb{R}^n)$, such that $(M^d_\alpha f)(x) = \infty$ for all $x \in \mathbb{R}^n$. For example, $f(x) = \rho(x)^{-\beta}\chi_{\complement_{\mathcal{E}(0,1)}}(x)$, where $\frac{|d|}{p_1} < \beta < \alpha$. To prove this it suffices to notice that

$$(M^d_\alpha f)(x) = v_n^{\frac{\alpha}{|d|}-1} \sup_{r>0} r^{\alpha-|d|} \int_{\rho(x-y)<r,\rho(y)\geq 1} \rho(y)^{-\beta} dy$$
$$\geq v_n^{\frac{\alpha}{|d|}-1} \sup_{r>\rho(x)} r^{\alpha-|d|} \int_{\rho(x-y)<r,\rho(y)\geq 1} \rho(y)^{-\beta} dy.$$

Hence, since $\rho(y) \leq \rho(x) + \rho(x-y) \leq 2r$,

$$(M^d_\alpha f)(x) \geq 2^{-\beta} v_n^{\frac{\alpha}{|d|}-1} \sup_{r>\rho(x)} r^{\alpha-\beta-|d|} \int_{\rho(x-y)<r,\rho(y)\geq 1} dy$$
$$\geq 2^{-\beta} v_n^{\frac{\alpha}{|d|}} \sup_{r>\rho(x)} r^{\alpha-\beta-|d|}(r^{|d|}-1) = \infty.$$

Further argument will be based on the following inequality which replaces inequality (3.2) for $\frac{|d|}{p_1} < \alpha < |d|$, which follows again by Lemma 3.1 and now by the second part of Lemma 3.2.

**Corollary 3.3.** *Let $1 < p_1 < \infty$, $0 < p_2 < \infty$ and $\frac{|d|}{p_1} < \alpha < |d|$. Then there exists $c_{10} > 0$ such that*

$$\|M^d_\alpha f\|_{L_{p_2}(\mathcal{E}(0,r))} \leq c_{10}\Big(r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|f\|_{L_{p_1}} + \tag{3.5}$$
$$+ r^{\frac{n}{p_2}}\left(\int_r^\infty \left(\int_{\complement_{\mathcal{E}(0,t)}} |f(x)|^{p_1}dx\right)\frac{dt}{t^{|d|-\alpha p_1+1}}\right)^{\frac{1}{p_1}}\Big) \tag{3.6}$$

*for all $r > 0$ and for all $f \in L_{p_1}$.*

# 4 Fractional maximal operator and dual Hardy operator

Let $H^*$ be the dual Hardy operator, i.e.,

$$(H^*g)(r) = \int_r^\infty g(t)dt, \quad 0 < r < \infty.$$

**Lemma 4.1.** *Let $1 < p_1 < \infty$, $0 < p_2 < \infty$, $\frac{|d|}{p_1} < \alpha < |d|$, $0 < \theta \leq \infty$, $w \in \Omega_\theta$ and $w(r)r^{\alpha-\frac{|d|}{p_1}+\frac{|d|}{p_2}} \in L_\theta(0,\infty)$. Then there exists $c_{11} > 0$ such that*

$$\|M^d_\alpha f\|_{LM_{p_2\theta,w,d}} \leq c_{11}\Big(\big\|w(r)r^{\alpha-|d|(\frac{1}{p_1}-\frac{1}{p_2})}\big\|_{L_\theta(0,\infty)}\|f\|_{L_{p_1}} + \|H^*h\|^{\frac{1}{p_1}}_{L_{\frac{\theta}{p_1},v}(0,\infty)}\Big) \tag{4.1}$$

*for all $f \in L_{p_1}$, where*

$$h(t) = \int_{\complement_{\mathcal{E}\left(0,t^{\frac{1}{\alpha p_1-|d|}}\right)}} |f(y)|^{p_1}dt \tag{4.2}$$

*and*

$$v(r) = \big(w\big(r^{\frac{1}{\alpha p_1-|d|}}\big)r^{\frac{1}{\alpha p_1-|d|}(\frac{|d|}{p_2}+\frac{1}{\theta})-\frac{1}{\theta}}\big)^{p_1}. \tag{4.3}$$

*Proof.* By Corollary 3.3

$$\|M^d_\alpha f\|_{LM_{p_2\theta,w,d}} = \Big\| w(r) \|M^d_\alpha f\|_{L_{p_2}(\mathcal{E}(0,r))}\Big\|_{L_\theta(0,\infty)}$$
$$\leq c_{10} 2^{(\frac{1}{\theta}-1)_+}\bigg( \Big\| w(r) r^{\alpha-|d|(\frac{1}{p_1}-\frac{1}{p_2})}\Big\|_{L_\theta(0,\infty)} \|f\|_{L_{p_1}}$$
$$+ \bigg\| w(r) r^{\frac{|d|}{p_2}} \bigg( \int_r^\infty \bigg( \int_{\complement\mathcal{E}(0,t)} |f(x)|^{p_1} dx\bigg) \frac{dt}{t^{|d|-\alpha p_1+1}}\bigg)^{\frac{1}{p_1}} \bigg\|_{L_\theta(0,\infty)}\bigg).$$

Note that the second summand in the brackets is equal to

$$(\alpha p_1 - |d|)^{-\frac{1}{p_1}} \bigg\| w(r) r^{\frac{|d|}{p_2}} \bigg( \int_{r^{\alpha p_1-|d|}}^\infty \bigg( \int_{\complement\mathcal{E}(0,\tau^{\frac{1}{\alpha p_1-|d|}})} |f(x)|^{p_1} dx\bigg) d\tau\bigg)^{\frac{1}{p_1}} \bigg\|_{L_\theta(0,\infty)}$$
$$= (\alpha p_1 - |d|)^{-\frac{1}{p_1}} \bigg( \int_0^\infty \big(w(r) r^{\frac{|d|}{p_2}}\big)^\theta \bigg( \int_{r^{\alpha p_1-|d|}}^\infty h(\tau) d\tau\bigg)^{\frac{\theta}{p_1}} dr\bigg)^{\frac{1}{\theta}}$$
$$= (\alpha p_1 - |d|)^{-\frac{1}{p_1}-\frac{1}{\theta}} \bigg( \int_0^\infty \Big(w\big(\rho^{\frac{1}{\alpha p_1-|d|}}\big) \rho^{\frac{n}{p_2(\alpha p_1-|d|)}}\Big)^\theta \rho^{\frac{1}{\alpha p_1-|d|}-1} \bigg( \int_\rho^\infty h(\tau) d\tau\bigg)^{\frac{\theta}{p_1}} d\rho\bigg)^{\frac{1}{\theta}}$$
$$= (\alpha p_1 - |d|)^{-\frac{1}{p_1}-\frac{1}{\theta}} \|H^* h\|^{\frac{1}{p_1}}_{L_{\frac{\theta}{p_1},v}(0,\infty)}.$$

Hence inequality (4.1) follows. □

**Theorem 4.1.** *Let* $1 < p_1 < \infty$, $0 < p_2 < \infty$, $\frac{|d|}{p_1} < \alpha < |d|$, $0 < \theta_1, \theta_2 \leq \infty$, $w_1 \in {}^\complement\Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$ *and* $w_2(r) r^{\alpha-|d|(\frac{1}{p_1}-\frac{1}{p_2})} \in L_{\theta_2}(0,\infty)$. *Assume that* $H^*$ *is a bounded operator from* $L_{\frac{\theta_1}{p_1},v_1}(0,\infty)$ *to* $L_{\frac{\theta_2}{p_1},v_2}(0,\infty)$ *on the cone of all non-negative functions* $\varphi$ *non-increasing on* $(0,\infty)$ *and satisfying* $\lim_{t\to\infty}\varphi(t) = 0$, *where*

$$v_1(r) = \Big[ w_1\Big(r^{\frac{1}{\alpha p_1-|d|}}\Big) r^{\frac{1}{(\alpha p_1-|d|)\theta_1}-\frac{1}{\theta_1}}\Big]^{p_1}, \tag{4.4}$$

$$v_2(r) = \bigg[ w_2\Big(r^{\frac{1}{\alpha p_1-|d|}}\Big) r^{\frac{1}{\alpha p_1-|d|}\left(\frac{|d|}{p_2}+\frac{1}{\theta_2}\right)-\frac{1}{\theta_2}}\bigg]^{p_1}. \tag{4.5}$$

*Then* $M^d_\alpha$ *is bounded from* ${}^\complement LM_{p_1\theta_1,w_1,d} \cap L_{p_1}$ *to* $LM_{p_2\theta_2,w_2,d}$.

*Proof.* Lemma 4.1 applied to $LM_{p_2\theta_2,w_2,d}$

$$\|M^d_\alpha f\|_{LM_{p_2\theta_2,w_2,d}} \leq c_{13}\bigg( \Big\| w_2(r) r^{\alpha-|d|(\frac{1}{p_1}-\frac{1}{p_2})}\Big\|_{L_{\theta_2}(0,\infty)} \|f\|_{L_{p_1}} + \|H^* h\|^{\frac{1}{p_1}}_{L_{\frac{\theta_2}{p_1},v_2}(0,\infty)}\bigg),$$

where $c_{13} > 0$ is independent of $f$.

Since $h$ is non-negative, non-increasing on $(0,\infty)$ and $\lim_{t\to\infty} h(t) = 0$ and $H^*$ is a bounded operator from $L_{\frac{\theta_1}{p_1},v_1}(0,\infty)$ to $L_{\frac{\theta_2}{p_1},v_2}(0,\infty)$ on the cone of functions containing $h$, we have

$$\|M^d_\alpha f\|_{LM_{p_2\theta_2,w_2,d}} \leq c_{14}\bigg( \Big\| w_2(r) r^{\alpha-|d|(\frac{1}{p_1}-\frac{1}{p_2})}\Big\|_{L_{\theta_2}(0,\infty)} \|f\|_{L_{p_1}} + \|h\|^{\frac{1}{p_1}}_{L_{\frac{\theta_1}{p_1},v_1}(0,\infty)}\bigg),$$

where $c_{14} > 0$ is independent of $f$.

Note that

$$\begin{aligned}
\|h\|^{\frac{1}{p_1}}_{L_{\frac{\theta_1}{p_1},v_1}(0,\infty)} &= \left(\int_0^\infty v_1(t)^{\frac{\theta_1}{p_1}} \|f\|^{\theta_1}_{L_{p_1}\left({}^{\complement}\mathcal{E}\left(0,t^{\frac{1}{\alpha p_1-|d|}}\right)\right)} dt\right)^{\frac{1}{\theta_1}} \\
&= (\alpha p_1 - |d|)^{\frac{1}{\theta_1}} \left(\int_0^\infty v_1(r^{\alpha p_1-|d|})^{\frac{\theta_1}{p_1}} r^{\alpha p_1-|d|-1} \|f\|^{\theta_1}_{L_{p_1}({}^{\complement}\mathcal{E}(0,r))} dr\right)^{\frac{1}{\theta_1}} \\
&= (\alpha p_1 - |d|)^{\frac{1}{\theta_1}} \left(\int_0^\infty \left(w_1(r)\|f\|_{L_{p_1}({}^{\complement}\mathcal{E}(0,r))}\right)^{\theta_1} dr\right)^{\frac{1}{\theta_1}} \\
&= (\alpha p_1 - |d|)^{\frac{1}{\theta_1}} \|f\|_{{}^{\complement}LM_{p_1\theta_1,w_1,d}}.
\end{aligned}$$

Hence

$$\|M^d_\alpha f\|_{LM_{p_2\theta_2,w_2,d}} \le c_{15}\left( \left\| w_2(r) r^{\alpha-|d|(\frac{1}{p_1}-\frac{1}{p_2})}\right\|_{L_{\theta_2}(0,\infty)} \|f\|_{L_{p_1}} + \|f\|_{{}^{\complement}LM_{p_1\theta_1,w_1,d}}\right),$$

where $c_{15} > 0$ is independent of $f$. □

## 5 Necessary conditions and sufficient conditions

For the majority of cases the necessary and sufficient conditions for the validity of

$$\|H^*\varphi\|_{L_{\frac{\theta_2}{p_1},v_2}(0,\infty)} \le c_{16}\, \|\varphi\|_{L_{\frac{\theta_1}{p_1},v_1}(0,\infty)}, \tag{5.1}$$

where $c_{16} > 0$ are independent of $\varphi$, for all non-negative non-increasing functions $\varphi$ are known, for detailed information see [10]. Application of any of those conditions gives sufficient conditions for the boundedness of the anisotropic fractional maximal operator $M^d_\alpha$ from ${}^{\complement}LM_{p_1\theta_1,w_1,d} \cap L_{p_1}$ to $LM_{p_2\theta_2,w_2,d}$.

In the case $0 < \theta_1 \le \theta_2 < \infty$ and $\theta_1 \le p_1$ the necessary conditions (coinciding with the sufficient ones) for the validity of inequality (5.1) for non-increasing functions are rather simple and can be obtained by taking $\varphi = \chi_{(0,t)}$ with an arbitrary $t > 0$.

Since in the proof of Theorem 4.1 inequality (5.1) is applied to the function $\varphi = g$, where $g$ is given by (4.2), it is natural, when proving the necessity, to choose, as test functions, functions $f_t$, $t > 0$, for which the integral $\int_{{}^{\complement}\mathcal{E}\left(0,u^{\frac{1}{(\alpha p_1-|d|)}}\right)} |f_t(y)|^{p_1} dy$ is equal or close to $A(t)\chi_{(0,t)}(u)$, $u > 0$, where $A(t)$ is independent of $u$. The simplest choice of $f$ satisfying this requirement is

$$f_t(y) = \chi_{\mathcal{E}(0,2t)\setminus\mathcal{E}(0,t)}(y), \quad y \in \mathbb{R}^n, \quad t > 0. \tag{5.2}$$

Note that,

$$\begin{aligned}
&\|f_t\|_{L_{p_1}({}^{\complement}\mathcal{E}(0,r))} = 0, \quad 2t \le r < \infty, \\
&\|f_t\|_{L_{p_1}({}^{\complement}\mathcal{E}(0,r))} \le c_{17} t^{\frac{|d|}{p_1}}, \quad 0 < r < 2t,
\end{aligned} \tag{5.3}$$

where $c_{17} > 0$ depends only on $n$ and $p_1$.

**Theorem 5.1.** 1. *If* $1 \le p_1 \le \infty$, $0 < p_2 \le \infty$, $\frac{|d|}{p_1} \le \alpha < |d|$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in {}^{\complement}\Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *then the condition*

$$t^{\alpha-\frac{|d|}{p_1}+\min\{|d|-\alpha,\frac{|d|}{p_2}\}}\Big\|\frac{w_2(r)r^{\frac{|d|}{p_2}}}{(t+r)^{\min\{|d|-\alpha,\frac{|d|}{p_2}\}}}\Big\|_{L_{\theta_2}(0,\infty)} \le c_{18}\big(1+\|w_1\|_{L_{\theta_1}(0,t)}\big), \tag{5.4}$$

*where* $c_{18}>0$ *is independent of* $t>0$, *is necessary for the boundedness of* $M^d_\alpha$ *from* ${}^{\complement}LM_{p_1\theta_1,w_1,d}\cap L_{p_1}$ *to* $LM_{p_2\theta_2,w_2,d}$.

2. *Let* $1<p_1<\infty$, $0<p_2<\infty$, $0<\theta_1,\theta_2\le\infty$, $\frac{|d|}{p_1}<\alpha<|d|$, $w_1\in {}^{\complement}\Omega_{\theta_1}$ *and* $w_2\in\Omega_{\theta_2}$. *If* $\theta_1\ge p_1$, *then the condition*

$$\begin{cases} \|w_2(r)r^{\alpha-|d|(\frac{1}{p_1}-\frac{1}{p_2})}\|_{L_{\theta_2}(0,\infty)}<\infty, \\ \Big\|w_2(r)r^{\frac{|d|}{p_2}}\|w_1(t)^{-1}t^{\frac{\alpha p_1-|d|-1}{p_1}}\|_{L_s(r,\infty)}\Big\|_{L_{\theta_2}(0,\infty)}<\infty\,, \end{cases} \tag{5.5}$$

*where* $s=\frac{\theta_1 p_1}{\theta_1-p_1}$ *(if* $\theta_1=p_1$, *then* $s=\infty$*), and if* $\theta_1\le\min\{p_1,\theta_2\}$, *then the condition*

$$\begin{cases} \|w_2(r)r^{\alpha-|d|(\frac{1}{p_1}-\frac{1}{p_2})}\|_{L_{\theta_2}(0,\infty)}<\infty, \\ \Big\|w_2(r)r^{\frac{|d|}{p_2}}\left(t^{\alpha p_1-|d|}-r^{\alpha p_1-|d|}\right)^{\frac{1}{p_1}}\Big\|_{L_{\theta_2}(0,t)}\le c_{19}\,\|w_1\|_{L_{\theta_1}(0,t)}\,,\ 0<t<\infty\,, \end{cases} \tag{5.6}$$

*where* $c_{19}>0$ *is independent of* $t$, *are sufficient for the boundedness of* $M^d_\alpha$ *from* ${}^{\complement}LM_{p_1\theta_1,w_1,d}\cap L_{p_1}$ *to* $LM_{p_2\theta_2,w_2,d}$.

*Proof. Sufficiency.* First, let $\theta_1\ge p_1$, then the statement follows by applying Theorem 5 and the following simple sufficient condition for the validity of (5.1)

$$\Big\|v_2(r)^{\frac{1}{p_1}}\|v_1(t)^{-\frac{1}{p_1}}\|_{L_s(r,\infty)}\Big\|_{L_{\theta_2}(0,\infty)}<\infty\,,$$

which follows by applying Hölder's inequality, where $v_1$ and $v_2$ are defined by (4.4) and (4.5), and replacing $t$ by $t^{\alpha p_1-|d|}$ and then $r$ by $r^{\alpha p_1-|d|}$.

Next, let $\theta\le\min\{p_1,\theta_2\}$. It is known [10] that the necessary and sufficient conditions for the validity of (5.1), where $0<\theta_1\le\theta_2\le\infty$, $\theta_1\le p_1$, for all non-negative decreasing on $(0,\infty)$ functions $\varphi$ has the form: for some $c_{20}>0$

$$\|v_2(r)(t-r)\|_{L_{\frac{\theta_2}{p_1}}(0,t)}\le c_{20}\,\|v_1(r)\|_{L_{\frac{\theta_1}{p_1}}(0,t)}$$

for all $t>0$. Applying this condition to the functions $v_1$ and $v_2$ defined by (4.4) and (4.5) and replacing $r$ by $r^{\alpha p_1-|d|}$ and then $t$ by $t^{\alpha p_1-|d|}$, we arrive at the second inequality in (5.6). Now it suffices to apply Theorem 4.1.

*Necessity.* Assume that, for some $c_{21}>0$ and for all $f\in {}^{\complement}LM_{p_1\theta_1,w_1,d}\cap L_{p_1}$

$$\|M^d_\alpha f\|_{LM_{p_2\theta_2,w_2,d}}\le c_{21}\left(\|f\|_{L_{p_1}}+\|f\|_{{}^{\complement}LM_{p_1\theta_1,w_1,d}}\right). \tag{5.7}$$

In (5.7) take $f = f_t$, where $f_t$ is defined by (5.2). Then by (5.3) the right hand side of (5.7) does not exceed a constant multiplied by

$$t^{\frac{|d|}{p_1}}\left(1 + \|w_1\|_{L_{\theta_1}(0,2t)}\right).$$

Furthermore, in the proof of the necessity in Theorem 11 in [6] it is shown that the left-hand side of inequality (5.7) is greater than or equal to a constant multiplied by

$$t^{\alpha+\min\{|d|-\alpha,\frac{|d|}{p_2}\}}\left\|w_2(r)\frac{r^{\frac{|d|}{p_2}}}{(t+r)^{\min\{|d|-\alpha,\frac{|d|}{p_2}\}}}\right\|_{L_{\theta_2}(0,\infty)}.$$

Replacing $2t$ by $t$ we arrive at (5.4). □

**Remark 2.** Condition (5.4) implies that $w_2(r)r^{\alpha-|d|(\frac{1}{p_1}-\frac{1}{p_2})} \in L_{\theta_2}(0,t)$ for all $t > 0$, because the left-hand side of (5.4) is greater than or equal to

$$t^{\alpha-\frac{|d|}{p_1}+\min\{|d|-\alpha,\frac{|d|}{p_2}\}}\left\|\frac{w_2(r)r^{\alpha-|d|(\frac{1}{p_1}-\frac{1}{p_2})}}{r^{\alpha-\frac{|d|}{p_1}}(t+r)^{\min\{|d|-\alpha,\frac{|d|}{p_2}\}}}\right\|_{L_{\theta_2}(0,t)}$$
$$\geq 2^{-\min\{|d|-\alpha,\frac{|d|}{p_2}\}}\left\|w_2(r)r^{\alpha-|d|(\frac{1}{p_1}-\frac{1}{p_2})}\right\|_{L_{\theta_2}(0,t)}$$

since $\frac{|d|}{p_1} < \alpha < |d|$.

If $w_1 \in L_{\theta_1(0,\infty)}$, this inequality, together with inequality (5.4), also implies that the condition

$$\|w_2(r)r^{\alpha-|d|(\frac{1}{p_1}-\frac{1}{p_2})}\|_{L_{\theta_2}(0,\infty)} < \infty$$

is a necessary one.

**Remark 3.** According to [1] the first part of conditions (5.5) and (5.6) is a sufficient condition for the boundedness of $M_\alpha^d$ from $L_{p_1}$ to $LM_{p_2\theta_2,w_2,d}$ for $|d|(\frac{1}{p_1}-\frac{1}{p_2})_+ \leq \alpha \leq \frac{|d|}{p_1}$. Moreover, the second part of condition (5.5) is a sufficient condition for the boundedness of $M_\alpha^d$ from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$ for $|d|(\frac{1}{p_1}-\frac{1}{p_2})_+ \leq \alpha < \frac{|d|}{p_1}$.

## 6 The case of weak Morrey-type spaces

Next we consider anisotropic local weak complementary Morrey-type spaces and formulate the results for the boundedness of $M_\alpha^d$ in these space, which follow by the estimates of the previous sections.

**Definition 5.** Let $0 < p, \theta \leq \infty$ and let $w$ be a non-negative measurable function on $(0,\infty)$. Denote by $LWM_{p\theta,w,d}$, the local weak Morrey-type space, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LWM_{p\theta,w,d}} \equiv \|f\|_{LWM_{p\theta,w,d}(\mathbb{R}^n)} = \left\|w(r)\|f\|_{WL_p(\mathcal{E}(0,r))}\right\|_{L_\theta(0,\infty)},$$

where

$$\|f\|_{WL_p(\mathcal{E}(0,r))} = \sup_{t>0} t\left(\mathrm{meas}\{x \in \mathcal{E}(0,r) : |f(x)| > t\}\right)^{\frac{1}{p}}.$$

If $p = \infty$, then $WL_\infty \equiv L_\infty$ and $LWM_{\infty\theta,w,d} \equiv LM_{\infty\theta,w,d}$.

Below we formulate the corresponding analogue of Theorem 5.1.

**Theorem 6.1.** 1. *If* $1 \le p_1 \le \infty$, $0 < p_2 \le \infty$, $\frac{|d|}{p_1} < \alpha < |d|$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in {}^{\complement}\Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *then the condition* (5.4) *is necessary for the boundedness of* $M^d_\alpha$ *from* ${}^{\complement}LM_{p_1\theta_1,w_1,d} \cap L_{p_1}$ *to* $LWM_{p_2\theta_2,w_2,d}$.

2. *Let* $1 \le p_1 < \infty$, $0 < p_2 < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $\frac{|d|}{p_1} < \alpha < |d|$, $w_1 \in {}^{\complement}\Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$. *If* $\theta_1 \ge p_1$ *then condition* (5.5) *and if* $\theta_1 \le \min\{p_1, \theta_2\}$ *then condition* (5.6) *are sufficient for the boundedness of* $M^d_\alpha$ *from* ${}^{\complement}LM_{p_1\theta_1,w_1,d} \cap L_{p_1}$ *to* $LWM_{p_2\theta_2,w_2,d}$.

## 7 Concluding remarks

The assumption made at the beginning of the paper $d_i \ge 1$, $i = 1, \dots, n$, is not essential. One may assume that $d_i > 0$, $i = 1, \dots, n$. However, under this assumption the function $\rho(x-y)$, $x, y \in \mathbb{R}^n$ is in general a quasi-distance, which does note cause any problem.

Also note that if $\nu > 0$ then for all $\nu > 0$

$$M^{\nu d}_{\nu\alpha} = M^d_\alpha, \quad \|f\|_{L_p(\mathcal{E}_d(0,r))} = \|f\|_{L_p(\mathcal{E}_{\nu d}(0,r^{1/\nu}))}, \quad \|f\|_{L_p({}^{\complement}\mathcal{E}_d(0,r))} = \|f\|_{L_p({}^{\complement}\mathcal{E}_{\nu d}(0,r^{1/\nu}))}.$$

**Lemma 7.1.** *Let* $1 < p_1 \le p_2 < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$. *Then for* $\nu > 0$

$$\|M^d_\alpha f\|_{{}^{\complement}LM_{p_1\theta_1,w_1,d} \cap L_{p_1} \to LM_{p_2\theta_2,w_2,d}} = \|M^{\nu d}_{\nu\alpha} f\|_{{}^{\complement}LM_{p_1\theta_1,w_1(\rho^\nu)\rho^{\frac{\nu-1}{\theta_1}},\nu d} \cap L_{p_1} \to LM_{p_2\theta_2,w_2(\rho^\nu)\rho^{\frac{\nu-1}{\theta_2}},\nu d}}.$$

*Proof.*

$$\|M^d_\alpha f\|_{{}^{\complement}LM_{p_1\theta_1,w_1,d} \cap L_{p_1} \to LM_{p_2\theta_2,w_2,d}} = \sup_{f \nsim 0, f \in {}^{\complement}LM_{p_1\theta_1,w_1,d} \cap L_{p_1}} \frac{\|M^d_\alpha f\|_{LM_{p_2\theta_2,w_2,d}}}{\|f\|_{{}^{\complement}LM_{p_1\theta_1,w_1,d} \cap L_{p_1}}}$$

$$= \sup_{f \nsim 0, f \in {}^{\complement}LM_{p_1\theta_1,w_1,d} \cap L_{p_1}} \frac{\|w_2(r)\|M^d_\alpha f\|_{L_p(\mathcal{E}_d(0,r))}\|_{L_{\theta_2}(0,\infty)}}{\max\left\{\|w_1(r)\|f\|_{L_p({}^{\complement}\mathcal{E}_d(0,r))}\|_{L_{\theta_1}(0,\infty)}, \|f\|_{L_p}\right\}}$$

$$= \sup_{f \nsim 0, f \in {}^{\complement}LM_{p_1\theta_1,w_1,d} \cap L_{p_1}} \frac{\|w_2(r)\|M^{\nu d}_{\nu\alpha} f\|_{L_p(\mathcal{E}_{\nu d}(0,r^{1/\nu}))}\|_{L_{\theta_2}(0,\infty)}}{\max\left\{\|w_1(r)\|f\|_{L_p({}^{\complement}\mathcal{E}_{\nu d}(0,r^{1/\nu}))}\|_{L_{\theta_1}(0,\infty)}, \|f\|_{L_p}\right\}}$$

$$= \nu^{1/\theta_2 - 1/\theta_1} \sup_{f \nsim 0, f \in {}^{\complement}LM_{p_1\theta_1,w_1,d} \cap L_{p_1}} \frac{\|w_2(\rho^\nu)\rho^{\frac{\nu-1}{\theta_2}}\|M^{\nu d}_{\nu\alpha} f\|_{L_p(\mathcal{E}_{\nu d}(0,\rho))}\|_{L_{\theta_2}(0,\infty)}}{\max\left\{\|w_1(\rho^\nu)\rho^{\frac{\nu-1}{\theta_1}}\|f\|_{L_p({}^{\complement}\mathcal{E}_{\nu d}(0,\rho))}\|_{L_{\theta_1}(0,\infty)}, \|f\|_{L_p}\right\}}$$

$$= \|M^{\nu d}_{\nu\alpha} f\|_{{}^{\complement}LM_{p_1\theta_1,w_1(\rho^\nu)\rho^{\frac{\nu-1}{\theta_1}},\nu d} \cap L_{p_1} \to LM_{p_2\theta_2,w_2(\rho^\nu)\rho^{\frac{\nu-1}{\theta_2}},\nu d}}.$$

□

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