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# THE DIRICHLET PROBLEM FOR THE GENERALIZED BI-AXIALLY SYMMETRIC HELMHOLTZ EQUATION

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**Abstract.** In [18], fundamental solutions for the generalized bi-axially symmetric Helmholtz equation were constructed in $R_2^+ = \{(x,y) : x > 0, y > 0\}$. They contain Kummer's confluent hypergeometric functions in three variables. In this paper, using one of the constructed fundamental solutions, the Dirichlet problem is solved in the domain $\Omega \subset R_2^+$. Using the method of Green's functions, solution of this problem is found in an explicit form.

## 1 Introduction

In the monograph of Gilbert [16], by applying methods of complex analysis, integral representations of solutions of the generalized bi-axially Helmholtz equation

$$H_{\alpha,\beta}^{\lambda}(u) \equiv u_{xx} + u_{yy} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y - \lambda^2 u = 0, \qquad \left(H_{\alpha,\beta}^{\lambda}\right)$$

were constructed via analytic functions. Here $0 < 2\alpha, 2\beta < 1$, $\alpha, \beta, \lambda$ are constants. When $\lambda = 0$ this equation is known as the equation of generalized axially symmetric potential theory. This terminology was used for the first time by Weinstein, who first considered fractional dimensional spaces in the potential theory [33, 34]. The special case with $\lambda = 0$ has also been investigated by Erdelyi [5, 6], Gilbert [9-15], Ranger [29], Henrici [21, 22]. There are many works [1-3, 8, 17, 23, 25-28, 30, 32] in which some problems for equation $\left(H_{\alpha,\beta}^{\lambda}\right)$ were studied. In the paper [18] for equation $\left(H_{\alpha,\beta}^{\lambda}\right)$ the following fundamental solutions on $R_2^+ = \{(x,y) : x > 0, y > 0\}$ have been constructed:

$$q_1(x,y;x_0,y_0) = k_1 \left(r^2\right)^{-\alpha-\beta} A_2^{(3)} (\alpha+\beta; \alpha, \beta; 2\alpha, 2\beta; \xi, \eta, \zeta), \tag{1.1}$$

$$\begin{aligned} q_2(x,y;x_0,y_0) &= k_2 \left(r^2\right)^{\alpha-\beta-1} x^{1-2\alpha} x_0^{1-2\alpha} A_2^{(3)} \\ &\times (1-\alpha+\beta; 1-\alpha, \beta; 2-2\alpha, 2\beta; \xi, \eta, \zeta), \end{aligned} \tag{1.2}$$

$$\begin{aligned} q_3(x,y;x_0,y_0) &= k_3 \left(r^2\right)^{-\alpha+\beta-1} y^{1-2\beta} y_0^{1-2\beta} A_2^{(3)} \\ &\times (1+\alpha-\beta; \alpha, 1-\beta; 2\alpha, 2-2\beta; \xi, \eta, \zeta), \end{aligned} \tag{1.3}$$

$$q_4(x,y;x_0,y_0) = k_4\left(r^2\right)^{\alpha+\beta-2} x^{1-2\alpha}y^{1-2\beta}x_0^{1-2\alpha}y_0^{1-2\beta}A_2^{(3)} \times (2-\alpha-\beta;1-\alpha,1-\beta;2-2\alpha,2-2\beta;\xi,\eta,\zeta), \tag{1.4}$$

where

$$k_1 = \frac{2^{2\alpha+2\beta}}{4\pi}\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta)}{\Gamma(2\alpha)\Gamma(2\beta)}, \tag{1.5}$$

$$k_2 = \frac{2^{2-2\alpha+2\beta}}{4\pi}\frac{\Gamma(1-\alpha)\Gamma(\beta)\Gamma(1-\alpha+\beta)}{\Gamma(2-2\alpha)\Gamma(2\beta)}, \tag{1.6}$$

$$k_3 = \frac{2^{2+2\alpha-2\beta}}{4\pi}\frac{\Gamma(\alpha)\Gamma(1-\beta)\Gamma(1+\alpha-\beta)}{\Gamma(2\alpha)\Gamma(2-2\beta)}, \tag{1.7}$$

$$k_4 = \frac{2^{4-2\alpha-2\beta}}{4\pi}\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(2-\alpha-\beta)}{\Gamma(2-2\alpha)\Gamma(2-2\beta)}, \tag{1.8}$$

$$r^2 = (x-x_0)^2+(y-y_0)^2,\ r_1^2 = (x+x_0)^2+(y-y_0)^2,\ r_2^2 = (x-x_0)^2+(y+y_0)^2,$$

$$\xi = \frac{r^2-r_1^2}{r^2}, \eta = \frac{r^2-r_2^2}{r^2}, \zeta = -\frac{\lambda^2}{4}r^2, \tag{1.9}$$

$$A_2^{(3)}(a;b_1,b_2;c_1,c_2;x,y,z) = \sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n-p}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n m!n!p!}x^m y^n z^p, \tag{1.10}$$

and $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.

## 2 Green's formulas

We consider an identity

$$x^{2\alpha}y^{2\beta}\left[uH_{\alpha,\beta}^{\lambda}(v) - vH_{\alpha,\beta}^{\lambda}(u)\right] = \frac{\partial}{\partial x}\left[x^{2\alpha}y^{2\beta}(v_x u - v u_x)\right] + \frac{\partial}{\partial y}\left[x^{2\alpha}y^{2\beta}(v_y u - v u_y)\right]. \tag{2.1}$$

Integrating both parts of identity (2.1) over $\Omega \subset R_2^+$, and using Green's formula we find

$$\int_{\Omega} x^{2\alpha}y^{2\beta}\left[uH_{\alpha,\beta}^{\lambda}(v) - vH_{\alpha,\beta}^{\lambda}(u)\right]dxdy = \int_{S} x^{2\alpha}y^{2\beta}u(v_x dy - v_y dx) - x^{2\alpha}y^{2\beta}v(u_x dy - u_y dx), \tag{2.2}$$

where $S = \partial\Omega$ is the boundary of the domain $\Omega$. Formula (2.2) named as Green's formula is deduced under the following assumptions:
- the functions $u$ and $v$ are continuous on the closure of the domain $\Omega$, i.e. on $\bar{\Omega}$,
- the partial derivatives of the first and second orders of $u$ and $v$ are continuous on $\Omega$,
- the integrals over $\Omega$, containing partial derivatives of the first and second orders of $u$ and $v$ have sense.

If $u$, $v$ are solutions of equation $\left(H_{\alpha,\beta}^{\lambda}\right)$, then by formula (2.2) we get

$$\int_{S} x^{2\alpha}y^{2\beta}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds = 0, \tag{2.3}$$

where
$$\frac{\partial}{\partial n}=\frac{dy}{ds}\frac{\partial}{\partial x}-\frac{dx}{ds}\frac{\partial}{\partial y},\ \frac{dy}{ds}=\cos\left(n,x\right),\ \ \frac{dx}{ds}=-\cos\left(n,y\right), \tag{2.4}$$
$n$ is the exterior normal to the curve $S$. The following identity also takes place:
$$\int\limits_{\Omega} x^{2\alpha}y^{2\beta}\left[u_x^2+u_y^2+\lambda^2u^2\right]dxdy=\int\limits_{S} x^{2\alpha}y^{2\beta}u\frac{\partial u}{\partial n}ds, \tag{2.5}$$
where $u$ is a solution of equation $\left(H_{\alpha,\beta}^{\lambda}\right)$.

## 3 The formulation and the uniqueness of the Dirichlet problem

Let $\Omega\subset R_2^+=\{(x,y):x>0,y>0\}$ be a domain limited by intervals $I_1=(0,a)\,,a=const>0,\ I_2=(0,b)\,,b=const>0$ of the axis $OX$, $OY$ respectively, and a curve $\Gamma$ with endpoints $A\left(a,0\right),\ B\left(0,b\right)$. The parametrical equation of the curve $\Gamma$ will be $x=x\left(s\right),\ y=y\left(s\right)$, where $s$ is the length of the arc counted from the point $A\left(a,0\right)$. Concerning the curve $\Gamma$ we shall assume that:
- the functions $x=x\left(s\right),y=y\left(s\right)$ have continuous derivatives $x'\left(s\right),y'\left(s\right)$ on the segment $[0,l]$, where $l$ is length of the curve $\Gamma$, not simultaneously equal to zero;
- the derivatives $x''\left(s\right),y''\left(s\right)$ satisfy to the Hölder condition on $[0,l]$;
- in neighborhoods of the points $A\left(a,0\right)$ and $B\left(0,b\right)$ the conditions
$$\left|\frac{dx}{ds}\right|\le Cy^{1+\varepsilon}\left(s\right),\ \left|\frac{dy}{ds}\right|\le Cx^{1+\varepsilon}\left(s\right),\ 0<\varepsilon<1,\ C=const,$$
are satisfied.

**Problem** $D$. Find a solution $u$ of equation $\left(H_{\alpha,\beta}^{\lambda}\right)$ belonging to the class $C\left(\bar{\Omega}\right)\cap C^2\left(\Omega\right)$, satisfying the conditions
$$u\left(x,y\right)|_{y=0}=\tau_1\left(x\right),\ x\in\bar{I}_1, \tag{3.1}$$
$$u\left(x,y\right)|_{x=0}=\tau_2\left(y\right),\ y\in\bar{I}_2, \tag{3.2}$$
$$u\left(x,y\right)|_{\Gamma}=\varphi\left(s\right),\ 0\le s\le l, \tag{3.3}$$
where $\tau_1,\tau_2,\varphi$ are given continuous functions and $\tau_1\left(0\right)=\tau_2\left(0\right),\tau_1\left(a\right)=\varphi\left(0\right),\tau_2\left(b\right)=\varphi\left(l\right)$.

**Theorem 3.1.** *If the problem $D$ has a solution in the domain $\Omega$, then it is unique.*

*Proof.* Let $\tau_1\left(x\right)=\tau_2\left(y\right)=\varphi\left(s\right)=0$ then, by virtue of identity (2.5), we have
$$\int\limits_{\Omega} x^{2\alpha}y^{2\beta}\left[u_x^2+u_y^2+\lambda^2u^2\right]dxdy=0. \tag{3.4}$$
By (3.4) it follows that $u_x\left(x,y\right)=u_y\left(x,y\right)=u\left(x,y\right)=0$. Hence, we have $u\left(x,y\right)\equiv0$ in the domain $\Omega$. □

We note that the uniqueness of a solution of the problem $D$ in the domain $\Omega$ also follows by the extremum principle for elliptic differential equations.

## 4 The existence theorem

Let $a=b$ and $\Gamma =: \{(x,y)\in R_2^+ : x^2+y^2=a^2\}$. We denote this domain by $\Omega_0$. The function $G_4(x,y;x_0,y_0)$ satisfying the following conditions is called Green's function of the problem $D$ :
- inside the domain $\Omega_0$, except for the point $(x_0,y_0)$, this function is a regular solution of equation $\left(H_{\alpha,\beta}^{\lambda}\right)$;
- it satisfies the boundary condition

$$G_4(x,y;x_0,y_0)\big|_{\Gamma\cup I_1\cup I_2}=0; \tag{4.1}$$

- it can be represented in the form

$$G_4(x,y;x_0,y_0)=q_4(x,y;x_0,y_0)-\left(R_0^2\right)^{-\alpha-\beta}q_4(x,y;\bar{x}_0,\bar{y}_0), \tag{4.2}$$

where

$$R_0^2=x_0^2+y_0^2,\ \bar{x}_0=\frac{x_0}{R_0^2},\ \bar{y}_0=\frac{y_0}{R_0^2}, \tag{4.3}$$

$q_4(x,y;x_0,y_0)$ is a fundamental solution, $q_4(x,y;\bar{x}_0,\bar{y}_0)$ is a regular solution of equation $\left(H_{\alpha,\beta}^{\lambda}\right)$ in the domain $\Omega_0$.

Let $(x_0,y_0)\in\Omega_0$. We cut out from $\Omega_0$ a circle of small radius $\rho$ with the center at the point $(x_0,y_0)$ and the remaining part of $\Omega_0$, we denote by $\Omega_0^\rho$. $C_\rho$ is a boundary of the cutted out circle. Applying formula (2.3), we obtain

$$\begin{aligned}\int\limits_{C_\rho} x^{2\alpha}y^{2\beta}u\frac{\partial G_4(x,y;x_0,y_0)}{\partial n}ds &-\int\limits_{C_\rho} x^{2\alpha}y^{2\beta}G_4(x,y;x_0,y_0)\frac{\partial u}{\partial n}ds\\ &=\int\limits_0^a x^{2\alpha}y^{2\beta}\tau_1(x)\left.\frac{\partial}{\partial y}G_4(x,y;x_0,y_0)\right|_{y=0}dx\\ &+\int\limits_0^a x^{2\alpha}y^{2\beta}\tau_2(y)\left.\frac{\partial}{\partial x}G_4(x,y;x_0,y_0)\right|_{x=0}dy\\ &-\int\limits_{\Gamma} x^{2\alpha}y^{2\beta}\varphi(s)\frac{\partial G_4(x,y;x_0,y_0)}{\partial n}ds.\end{aligned} \tag{4.4}$$

Using the derivation formula

$$\begin{aligned}&\frac{\partial^{i+j+k}}{\partial x^i\partial y^j\partial z^k}A_2^{(3)}(a;b_1,b_2;c_1,c_2;x,y,z)\\ &=\frac{(a)_{i+j-k}(b_1)_i(b_2)_j}{(c_1)_i(c_2)_j}A_2^{(3)}(a+i+j-k;b_1+i,b_2+j;c_1+i,c_2+j;x,y,z),\end{aligned} \tag{4.5}$$

and considering the adjacent relation

$$\begin{aligned}&\frac{ab_1}{c_1}xA_2^{(3)}(1+a;1+b_1,b_2;1+c_1,c_2;x,y,z)\\ &+\frac{ab_2}{c_2}yA_2^{(3)}(1+a;b_1,1+b_2;c_1,1+c_2;x,y,z)\\ &-\frac{1}{a-1}zA_2^{(3)}(a-1;b_1,b_2;c_1,c_2;x,y,z)\\ &=aA_2^{(3)}(1+a;b_1,b_2;c_1,c_2;x,y,z)-aA_2^{(3)}(a;b_1,b_2;c_1,c_2;x,y,z),\end{aligned} \tag{4.6}$$

we find that

$$\begin{aligned} x^{2\alpha}\frac{\partial}{\partial x}q_4(x,y;x_0,y_0) &= k_4(1-2\alpha)(r^2)^{\alpha+\beta-2}x_0^{1-2\alpha}(yy_0)^{1-2\beta}A_2^{(3)} \\ &\times(2-\alpha-\beta;1-\alpha,1-\beta;2-2\alpha,2-2\beta;\xi,\eta,\zeta) \\ &-2k_4(2-\alpha-\beta)x(r^2)^{\alpha+\beta-3}x_0^{2-2\alpha}(yy_0)^{1-2\beta}A_2^{(3)} \\ &\times(3-\alpha-\beta;2-\alpha,1-\beta;3-2\alpha,2-2\beta;\xi,\eta,\zeta) \\ &-2k_4(2-\alpha-\beta)x(r^2)^{\alpha+\beta-3}x_0^{1-2\alpha}(yy_0)^{1-2\beta}(x-x_0)A_2 \\ &\times(3-\alpha-\beta;1-\alpha,1-\beta;2-2\alpha,2-2\beta;\xi,\eta,\zeta) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} y^{2\beta}\frac{\partial}{\partial y}q_4(x,y;x_0,y_0) &= k_4(1-2\beta)(r^2)^{\alpha+\beta-2}(xx_0)^{1-2\alpha}y_0^{1-2\beta} \\ &\times A_2^{(3)}(2-\alpha-\beta;\ 1-\alpha,1-\beta;2-2\alpha,2-2\beta;\xi,\eta,\zeta) \\ &-2k_4(2-\alpha-\beta)y(r^2)^{\alpha+\beta-3}(xx_0)^{1-2\alpha}y_0^{2-2\beta} \\ &\times A_2^{(3)}(3-\alpha-\beta;1-\alpha,2-\beta;2-2\alpha,3-2\beta;\xi,\eta,\zeta) \\ &-2k_4(2-\alpha-\beta)y(r^2)^{\alpha+\beta-3}(xx_0)^{1-2\alpha}y_0^{1-2\beta}(y-y_0) \\ &\times A_2^{(3)}(3-\alpha-\beta;1-\alpha,1-\beta;2-2\alpha,2-2\beta;\xi,\eta,\zeta). \end{aligned} \tag{4.8}$$

It is easy to prove that the following formulas are true:

$$A_2^{(3)}(a;b_1,b_2;c_1,c_2;0,y,z)=\sum_{n,p=0}^{\infty}\frac{(a)_{n-p}(b_2)_n}{(c_2)_n n!p!}y^n z^p=H_3(a,b_2;c_2;y,z), \tag{4.9}$$

$$A_2(a;b_1,b_2;c_1,c_2;x,0,z)=\sum_{m,p=0}^{\infty}\frac{(a)_{m-p}(b_1)_m}{(c_1)_m m!p!}x^m z^p=H_3(a,b_1;c_1;x,z), \tag{4.10}$$

where $H_3(a,b;c;x,y)$ is Kummer's hypergeometric function in two arguments ([7], p. 221, formula (31)). By virtue of equalities (4.7),(4.8),(4.9), (4.10) and taking into account that $\xi|_{x=0}=0,\ \eta|_{y=0}=0$, we get

$$\begin{aligned} y^{2\beta}\frac{\partial}{\partial y}G_4(x,y;x_0,y_0)\Big|_{y=0} &= k_4(1-2\beta)x_0^{1-2\alpha}y_0^{1-2\beta}x^{1-2\alpha} \\ \times\left\{\frac{H_3(2-\alpha-\beta,1-\alpha;2-2\alpha;\rho_1,\rho_1^*)}{[(x-x_0)^2+y_0^2]^{2-\alpha-\beta}}\right. &- \left.\frac{H_3(2-\alpha-\beta,1-\alpha;2-2\alpha;\rho_2,\rho_2^*)}{\left[\left(a-\dfrac{xx_0}{a}\right)^2+\dfrac{1}{a^2}x^2y_0^2\right]^{2-\alpha-\beta}}\right\} \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} x^{2\alpha}\frac{\partial}{\partial x}G_4(x,y;x_0,y_0)\Big|_{x=0} &= k_4(1-2\alpha)x_0^{1-2\alpha}y_0^{1-2\beta}y^{1-2\beta} \\ \times\left\{\frac{H_3(2-\alpha-\beta,1-\beta;2-2\beta;\rho_3,\rho_3^*)}{[x_0^2+(y-y_0^2)^2]^{2-\alpha-\beta}}\right. &- \left.\frac{H_3(2-\alpha-\beta,1-\beta;2-2\beta;\rho_4,\rho_4^*)}{\left[\left(a-\dfrac{yy_0}{a}\right)^2+\dfrac{1}{a^2}x_0^2y^2\right]^{2-\alpha-\beta}}\right\}, \end{aligned} \tag{4.12}$$

where

$$\rho_1 = \frac{-4xx_0}{(x-x_0)^2+y_0^2}, \qquad \rho_1^* = -\frac{\lambda^2}{4}\left[(x-x_0)^2+y_0^2\right],$$

$$\rho_2 = \frac{-4xx_0}{\left(a-\frac{xx_0}{a}\right)^2+\frac{1}{a^2}x^2y_0^2}, \qquad \rho_2^* = -\frac{a^2\lambda^2}{4R_0^2}\left[\left(a-\frac{xx_0}{a}\right)^2+\frac{1}{a^2}x^2y_0^2\right],$$

$$\rho_3 = \frac{-4yy_0}{x_0^2+(y-y_0)^2}, \qquad \rho_3^* = -\frac{\lambda^2}{4}\left[x_0^2+(y-y_0)^2\right],$$

$$\rho_4 = \frac{-4yy_0}{\left(a-\frac{yy_0}{a}\right)^2+\frac{1}{a^2}x_0^2y^2}, \qquad \rho_4^* = -\frac{a^2\lambda^2}{4R_0^2}\left[\left(a-\frac{yy_0}{a}\right)^2+\frac{1}{a^2}x_0^2y^2\right].$$

Now we shall consider the right-hand side of identity (4.4). Taking into account (4.7) and (4.8), we find that

$$\begin{aligned}
&\frac{\partial q_4}{\partial n} = -k_4\left(2-\alpha-\beta\right)\left(r^2\right)^{\alpha+\beta-2}\left(xx_0\right)^{1-2\alpha}\left(yy_0\right)^{1-2\beta}\\
&\times A_2^{(3)}\left(3-\alpha-\beta;1-\alpha,1-\beta;2-2\alpha,2-2\beta;\xi,\eta,\zeta\right)\frac{\partial}{\partial n}\left[\ln r^2\right]\\
&+k_4\left(r^2\right)^{\alpha+\beta-2}x_0^{1-2\alpha}y_0^{1-2\beta}x^{-2\alpha}y^{-2\beta}\left[\left(1-2\alpha\right)y\frac{dy}{ds}-\left(1-2\beta\right)x\frac{dx}{ds}\right]\\
&\times A_2^{(3)}\left(2-\alpha-\beta;1-\alpha,1-\beta;2-2\alpha,2-2\beta;\xi,\eta,\zeta\right)\\
&-2k_4\left(2-\alpha-\beta\right)\left(r^2\right)^{\alpha+\beta-3}\left(xx_0\right)^{1-2\alpha}\left(yy_0\right)^{1-2\beta}A_2^{(3)}\\
&\times\left(3-\alpha-\beta;2-\alpha,1-\beta;3-2\alpha,2-2\beta;\xi,\eta,\zeta\right)\tfrac{dy}{ds}\\
&+2k_4\left(2-\alpha-\beta\right)\left(r^2\right)^{\alpha+\beta-3}\left(xx_0\right)^{1-2\alpha}\left(yy_0\right)^{1-2\beta}A_2^{(3)}\\
&\times\left(3-\alpha-\beta;1-\alpha,2-\beta;2-2\alpha,3-2\beta;\xi,\eta,\zeta\right)\tfrac{dx}{ds}.
\end{aligned} \tag{4.13}$$

Further we have

$$\begin{aligned}
\int\limits_{C_\rho} x^{2\alpha}y^{2\beta}u\frac{\partial G_4\left(x,y;x_0,y_0\right)}{\partial n}ds &= \int\limits_{C_\rho} x^{2\alpha}y^{2\beta}u\frac{\partial q_4\left(x,y;x_0,y_0\right)}{\partial n}ds\\
&-\left(R_0^2\right)^{-\alpha-\beta}\int\limits_{C_\rho} x^{2\alpha}y^{2\beta}u\tfrac{\partial q_4\left(x,y;\bar{x}_0,\bar{y}_0\right)}{\partial n}ds\\
&= J_1+J_2.
\end{aligned} \tag{4.14}$$

Substituting (4.13) in (4.14) and passing to the polar coordinates $x = x_0 + \rho\cos\varphi$,

$y = y_0 + \rho \sin \varphi$, we have

$$
\begin{aligned}
J_1 &= 2k_4 (2-\alpha-\beta)\, x_0^{1-2\alpha} y_0^{1-2\beta} \\
&\times \int_0^{2\pi} (x_0+\rho\cos\varphi)(y_0+\rho\sin\varphi)\, u(x_0+\rho\cos\varphi, y_0+\rho\sin\varphi) \\
&\times (\rho^2)^{\alpha+\beta-2} A_2^{(3)} (3-\alpha-\beta; 1-\alpha, 1-\beta; 2-2\alpha, 2-2\beta; \xi, \eta, \zeta)\, d\varphi \\
&+ k_4 x_0^{1-2\alpha} y_0^{1-2\beta} \int_0^{2\pi} u(x_0+\rho\cos\varphi, y_0+\rho\sin\varphi) \\
&\times [(1-2\alpha) y_0 \cos\varphi + (1-2\beta) x_0 \sin\varphi + (1-\alpha-\beta)\rho\sin 2\varphi] \\
&\times \left(\rho^2\right)^{\alpha+\beta-1} A_2^{(3)} (2-\alpha-\beta; 1-\alpha, 1-\beta; 2-2\alpha, 2-2\beta; \xi, \eta, \zeta)\, d\varphi \\
&- 2k_4 (2-\alpha-\beta)\, x_0^{1-2\alpha} y_0^{1-2\beta} \\
&\times \int_0^{2\pi} (x_0+\rho\cos\varphi)(y_0+\rho\sin\varphi)\, u(x_0+\rho\cos\varphi, y_0+\rho\sin\varphi) \\
&\times (\rho^2)^{\alpha+\beta-2} A_2^{(3)} (3-\alpha-\beta; 2-\alpha, 1-\beta; 3-2\alpha, 2-2\beta; \xi, \eta, \zeta) \cos\varphi d\varphi \\
&- 2k_4 (2-\alpha-\beta)\, x_0^{1-2\alpha} y_0^{1-2\beta} \\
&\times \int_0^{2\pi} (x_0+\rho\cos\varphi)(y_0+\rho\sin\varphi)\, u(x_0+\rho\cos\varphi, y_0+\rho\sin\varphi) \\
&\times (\rho^2)^{\alpha+\beta-2} A_2^{(3)} (3-\alpha-\beta; 1-\alpha, 2-\beta; 2-2\alpha, 3-2\beta; \xi, \eta, \zeta) \sin\varphi d\varphi \\
&= J_{11} + J_{12} + J_{13} + J_{14}.
\end{aligned} \tag{4.15}
$$

For evaluation of (4.15) we use the expansion formula ((2.27) in [18], p. 678)

$$
\begin{aligned}
&A_2^{(3)} (a; b_1, b_2; c_1, c_2; x, y, z) \\
&= (1-x)^{-b_1} (1-y)^{-b_2} \sum_{i,j=0}^{\infty} \frac{(a)_{i-j} (b_1)_i (b_2)_i}{(c_1)_i (c_2)_i\, i! j!} \left(\frac{x}{1-x}\right)^i \left(\frac{y}{1-y}\right)^i z^j \\
&\times F\left(c_1 - a + j, b_1 + i; c_1 + i; \frac{x}{x-1}\right) F\left(c_2 - a + j, b_2 + i; c_2 + i; \frac{y}{y-1}\right),
\end{aligned} \tag{4.16}
$$

where $F(a, b; c; x)$ is the Gauss hypergeometric function ([7], p. 69, formula (2)). Hence we obtain

$$
\begin{aligned}
&A_2^{(3)} (3-\alpha-\beta; 1-\alpha, 1-\beta; 2-2\alpha, 2-2\beta; \xi, \eta, \zeta) \\
&= \left(\rho^2\right)^{2-\alpha-\beta} \left(\rho^2 + 4x_0^2 + 4x_0\rho\cos\varphi\right)^{\alpha-1} \left(\rho^2 + 4y_0^2 + 4y_0\rho\sin\varphi\right)^{\beta-1} P_{11},
\end{aligned} \tag{4.17}
$$

where

$$\begin{aligned}P_{11}=&\sum_{i,j=0}^{\infty}\frac{(3-\alpha-\beta)_{i-j}\,(1-\alpha)_i\,(1-\beta)_i}{(2-2\alpha)_i\,(2-2\beta)_i\,i!j!}\\ &\times\left(\frac{4x_0^2+4x_0\rho\cos\,\varphi}{\rho^2+4x_0^2+4x_0\rho\cos\,\varphi}\right)^i\left(\frac{4y_0^2+4y_0\rho\sin\,\varphi}{\rho^2+4y_0^2+4y_0\rho\sin\,\varphi}\right)^i\left(-\frac{\lambda^2}{4}\rho^2\right)^j\\ &\times F\left(-\alpha+\beta-1+j,1-\alpha+i;2-2\alpha+i;\frac{4x_0^2+4x_0\rho\cos\,\varphi}{\rho^2+4x_0^2+4x_0\rho\cos\,\varphi}\right)\\ &\times F\left(\alpha-\beta-1+j,1-\beta+i;2-2\beta+i;\frac{4y_0^2+4y_0\rho\sin\,\varphi}{\rho^2+4y_0^2+4y_0\rho\sin\,\varphi}\right),\end{aligned}\tag{4.18}$$

Using equality (46) in [7], p. 112,

$$F\,(a,b;c;1)=\frac{\Gamma\,(c)\,\Gamma\,(c-a-b)}{\Gamma\,(c-a)\,\Gamma\,(c-b)},\ \ c\neq 0,-1,-2,...,\ \ Re\,(c-a-b)>0,$$

it is not complicated to calculate

$$\lim_{\rho\to 0}P_{11}=\frac{\Gamma\,(2-2\alpha)\,\Gamma\,(2-2\beta)}{\Gamma\,(1-\alpha)\,\Gamma\,(1-\beta)\,\Gamma\,(3-\alpha-\beta)}.\tag{4.19}$$

By virtue of (4.17) we calculate $J_{11}$:

$$\begin{aligned}J_{11}=&\,2k_4\,(2-\alpha-\beta)\,x_0^{1-2\alpha}y_0^{1-2\beta}\\ &\times\int_0^{2\pi}(x_0+\rho\cos\,\varphi)\,(y_0+\rho\sin\,\varphi)\,u\,(x_0+\rho\cos\,\varphi,y_0+\rho\sin\,\varphi)\\ &\times\left(\rho^2+4x_0^2+4x_0\rho\cos\,\varphi\right)^{\alpha-1}\left(\rho^2+4y_0^2+4y_0\rho\sin\,\varphi\right)^{\beta-1}P_{11}d\varphi.\end{aligned}$$

Passing to the limit as $\rho\to 0^+$ and taking into account (1.8), we have

$$\lim_{\rho\to 0}J_{11}=u\,(x_0,y_0)\,.\tag{4.20}$$

Similarly it can be proved that

$$\begin{aligned}&\lim_{\rho\to 0}J_{12}=\lim_{\rho\to 0}J_{13}=\lim_{\rho\to 0}J_{14}=\lim_{\rho\to 0}J_2=0,\\ &\lim_{\rho\to 0}\int_{C_\rho}x^{2\alpha}y^{2\beta}G_4\,(x,y;x_0,y_0)\,\frac{\partial u}{\partial n}ds=0.\end{aligned}\tag{4.21}$$

Thus, using equalities (4.13), (4.14), (4.20) and (4.21), by (4.4) we deduce that

$$
\begin{aligned}
u(x_0,y_0) &= k_4(1-2\beta)\,x_0^{1-2\alpha}y_0^{1-2\beta}\\
&\times\int_0^a x\tau_1(x)\left\{\frac{H_3\left(2-\alpha-\beta,1-\alpha;2-2\alpha;\rho_1,\rho_1^*\right)}{\left[(x-x_0)^2+y_0^2\right]^{2-\alpha-\beta}}-\frac{H_3\left(2-\alpha-\beta,1-\alpha;2-2\alpha;\rho_2,\rho_2^*\right)}{\left[\left(a-\dfrac{xx_0}{a}\right)^2+\dfrac{1}{a^2}x^2y_0^2\right]^{2-\alpha-\beta}}\right\}dx\\
&+k_4(1-2\alpha)\,x_0^{1-2\alpha}y_0^{1-2\beta}\\
&\times\int_0^a y\tau_2(y)\left\{\frac{H_3\left(2-\alpha-\beta,1-\beta;2-2\beta;\rho_3,\rho_3^*\right)}{\left[x_0^2+\left(y-y_0^2\right)^2\right]^{2-\alpha-\beta}}-\frac{H_3\left(2-\alpha-\beta,1-\beta;2-2\beta;\rho_4,\rho_4^*\right)}{\left[\left(a-\dfrac{yy_0}{a}\right)^2+\dfrac{1}{a^2}x_0^2y^2\right]^{2-\alpha-\beta}}\right\}dy\\
&-\int_\Gamma x^{2\alpha}y^{2\beta}\varphi(s)\,\frac{\partial G_4(x,y;x_0,y_0)}{\partial n}ds.
\end{aligned}
\tag{4.22}
$$

If we use the formula

$$
H_3(a,b;c;x,y)=(1-x)^{-b}\,F^{1:1;0}_{0:1;2}\left[\begin{matrix} c-a: & b; & -; \\ -: & c; & 1-a,c-a;\end{matrix}\ \frac{x}{x-1},-y\right],
$$

which connects Kummer's function with the hypergeometric function of Kampe de Feriet ([4], p. 150, formula (29))

$$
F^{p:q;k;}_{l:m;n;}\left[\begin{matrix}(a_p): & (b_q); & (c_k); \\ (\alpha_l): & (\beta_m); & (\gamma_n);\end{matrix}\ x,y\right]=\sum_{r,s=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_{r+s}\prod\limits_{j=1}^{q}(b_j)_r\prod\limits_{j=1}^{k}(c_j)_s}{\prod\limits_{j=1}^{l}(\alpha_j)_{r+s}\prod\limits_{j=1}^{m}(\beta_j)_r\prod\limits_{j=1}^{n}(\gamma_j)_s\,r!s!}x^ry^s,
$$

we find that solution (4.22) of the problem $D$ may be represented as

$$
\begin{aligned}
u(x_0,y_0) &= k_4(1-2\beta)\,x_0^{1-2\alpha}y_0^{1-2\beta}\\
&\times\int_0^a x\tau_1(x)\left\{\frac{F^{1:1;0}_{0:1;2}\left[\begin{matrix}\beta-\alpha: & 1-\alpha; & -; \\ -: & 2-2\alpha; & \alpha+\beta-1,\beta-\alpha;\end{matrix}\ \sigma_1,\sigma_1^*\right]}{\left[(x-x_0)^2+y_0^2\right]^{1-\beta}\left[(x+x_0)^2+y_0^2\right]^{1-\alpha}}\right.\\
&\left.-\frac{F^{1:1;0}_{0:1;2}\left[\begin{matrix}\beta-\alpha: & 1-\alpha; & -; \\ -: & 2-2\alpha; & \alpha+\beta-1,\beta-\alpha;\end{matrix}\ \sigma_2,\sigma_2^*\right]}{\left[\left(a-\dfrac{xx_0}{a}\right)^2+\dfrac{1}{a^2}x^2y_0^2\right]^{1-\beta}\left[\left(a+\dfrac{xx_0}{a}\right)^2+\dfrac{1}{a^2}x^2y_0^2\right]^{1-\alpha}}\right\}dx
\end{aligned}
$$

$$
\begin{aligned}
&+k_4\left(1-2\alpha\right)x_0^{1-2\alpha}y_0^{1-2\beta}\\
&\times\int\limits_0^a y\tau_2\left(y\right)\left\{\frac{F^{1:1;0}_{0:1;2}\left[\begin{matrix}\alpha-\beta: & 1-\beta; & -; \\ -: & 2-2\beta; & \alpha+\beta-1,\alpha-\beta;\end{matrix}\ \sigma_3,\sigma_3^*\right]}{\left[x_0^2+\left(y-y_0^2\right)^2\right]^{1-\alpha}\left[x_0^2+\left(y+y_0\right)^2\right]^{1-\beta}}\right.\\
&-\left.\frac{F^{1:1;0}_{0:1;2}\left[\begin{matrix}\alpha-\beta: & 1-\beta; & -; \\ -: & 2-2\beta; & \alpha+\beta-1,\alpha-\beta;\end{matrix}\ \sigma_4,\sigma_4^*\right]}{\left[\left(a-\frac{yy_0}{a}\right)^2+\frac{1}{a^2}x_0^2y^2\right]^{1-\alpha}\left[\left(a+\frac{yy_0}{a}\right)^2+\frac{1}{a^2}x_0^2y^2\right]^{1-\beta}}\right\}dy\\
&-\int\limits_\Gamma x^{2\alpha}y^{2\beta}\varphi\left(s\right)\frac{\partial G_4\left(x,y;x_0,y_0\right)}{\partial n}ds,
\end{aligned}
\tag{4.23}
$$

where

$$
\begin{aligned}
\sigma_1&=\frac{4xx_0}{\left(x+x_0\right)^2+y_0^2}, & \sigma_1^*&=\frac{\lambda^2}{4}\left[\left(x-x_0\right)^2+y_0^2\right],\\
\sigma_2&=\frac{4xx_0}{\left(a+\frac{xx_0}{a}\right)^2+\frac{1}{a^2}x^2y_0^2}, & \sigma_2^*&=\frac{a^2\lambda^2}{4R_0^2}\left[\left(a-\frac{xx_0}{a}\right)^2+\frac{1}{a^2}x^2y_0^2\right],\\
\sigma_3&=\frac{4yy_0}{x_0^2+\left(y+y_0\right)^2}, & \sigma_3^*&=\frac{\lambda^2}{4}\left[x_0^2+\left(y-y_0\right)^2\right],\\
\sigma_4&=\frac{4yy_0}{\left(a+\frac{yy_0}{a}\right)^2+\frac{1}{a^2}x_0^2y^2}, & \sigma_4^*&=\frac{a^2\lambda^2}{4R_0^2}\left[\left(a-\frac{yy_0}{a}\right)^2+\frac{1}{a^2}x_0^2y^2\right].
\end{aligned}
$$

Now we can formulate the main result.

**Theorem 4.1.** *The problem D has the unique solution defined by formula* (4.23).

We note that expansions for the hypergeometric functions of Lauricella $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$, $F_D^{(n)}$ are found in [19, 20] and applied in [17] for finding fundamental solutions and later for investigating boundary value problems for 3-D singular elliptic equations [24].


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