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# KRYLOV SUBSPACE METHODS OF APPROXIMATE SOLVING DIFFERENTIAL EQUATIONS FROM THE POINT OF VIEW OF FUNCTIONAL CALCULUS

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**Abstract.** The paper deals with projection methods of approximate solving the problem

$$Fx' = Gx + bu(t), \qquad y = \langle x, d\rangle$$

which consist in passage to the *reduced-order problem*

$$\widehat{F}\hat{x}' = \widehat{G}\hat{x} + \hat{b}u(t), \qquad \hat{y} = \langle \hat{x}, \hat{d}\rangle,$$

where

$$\widehat{F} = \Lambda F V, \qquad \widehat{G} = \Lambda G V, \qquad \hat{b} = \Lambda b, \qquad \hat{d} = V^* d.$$

It is shown that, if $V$ and $\Lambda$ are constructed on the basis of Krylov's subspaces, a projection method is equivalent to the replacement in the formula expressing the impulse response via the exponential function of the pencil $\lambda \mapsto \lambda F - G$, of the exponential function by its rational interpolation satisfying some interpolation conditions. Special attention is paid to the case when $F$ is not invertible.

## 1 Introduction

Differential equations of the form

$$Fx' = Gx + f(t), \tag{1.1}$$

where $F$ and $G$ are linear operators, arise in the theory of discrete linear electrical circuits [4, 6, 35, 42, 46] and in some other applications [25, 36, 43]. Equation (1.1) is not resolved for the derivative. Moreover, if the number of capacitors and inductors in a circuit is not too large, the operator $F$ is certainly not invertible. Usually the free term $f$ takes its values in a finite-dimensional subspace; this follows since the circuit has only a small number of voltage and current sources. Because of linearity of the circuit, the influence of different sources can be treated independently. Therefore the problem is reduced to finding a solution that corresponds to the free term of the form $f(t) = bu(t)$, where $u$ is a given scalar function and $b$ is a fixed vector. Furthermore, not all coordinates of $x$, but only a limited number of them, are usually of interest. These coordinates can be found separately. Hence it is enough to seek a function $y(t) = \langle x(t), d\rangle$, where $d$ is a fixed vector, instead of the whole solution $x$. Thus we arrive at our main problem

$$\begin{aligned} Fx' &= Gx + bu(t),\\ y &= \langle x, d\rangle. \end{aligned} \tag{1.2}$$

We assume that the input function $u$ is defined on the whole real axis and is equal to zero to the left of zero, and the solution $x$ of the equation $Fx' + Gx = bu$ is also defined on the whole axis and is equal to zero to the left of zero. For electrical circuits such a problem statement often has more physical sense than the traditional initial value problem.

If the dimension of the vector $x$ is large, then it may be convenient to turn (see, for example, [1, 2, 7, 14, 16, 18, 33, 39]) from problem (1.2) to the problem

$$\begin{aligned} \widehat{F}\hat{x}' &= \widehat{G}\hat{x} + \hat{b}u(t),\\ \hat{y}(t) &= \langle \hat{x}, \hat{d}\rangle, \end{aligned} \tag{1.3}$$

where the dimension of the vector $\hat{x}$ is essentially smaller. Problem (1.3) is called a *reduced-order* problem. Reduced-order problem (1.3) is usually constructed by *projection* methods (Section 6). These methods consist in determining the coefficients in (1.3) with

$$\widehat{F} = \Lambda F V, \qquad \widehat{G} = \Lambda G V, \qquad \hat{b} = \Lambda b, \qquad \hat{d} = V^* d,$$

where $\Lambda$ and $V$ are some operators. In this paper we discuss the case (see, for example, [1, 2, 8, 13, 14, 15, 18, 33, 44]) when the images of $V$ and $\Lambda^*$ contain the vectors $(\lambda_j F - G)^{-1}b$ and $\big[(\lambda_j F - G)^{-1}\big]^* d$ and their iterations, $j = 1, \dots, p$, respectively (for more accurate formulation see Theorem 7.1). The linear spans of the iterations of the vectors $(\lambda_j F - G)^{-1}b$ and $\big[(\lambda_j F - G)^{-1}\big]^* d$ are called [23, 34, 48] *Krylov's subspaces*. The name of the discussed class of methods derives from this term.

The main result of the paper (Theorem 7.1) reads as follows. The impulse response $H$ of equation (1.1) can be represented (Theorem 5.1) as a special analytic function EXP of the pencil $\lambda \mapsto \lambda F - G$. It turns out that the construction of the function EXP of the *reduced-order* pencil $\lambda \mapsto \lambda\widehat{F} - \widehat{G}$ is equivalent to the construction of some rational function $r$ of the *initial* pencil $\lambda \mapsto \lambda F - G$. The function $r$ can be defined as a result of interpolation of the function EXP on the spectrum of the pencil $\lambda \mapsto \lambda\widehat{F} - \widehat{G}$. Thus the role of the reduced-order pencil $\lambda \mapsto \lambda\widehat{F} - \widehat{G}$ may be interpreted as an implicit specification of interpolation points.

Special cases of Theorem 7.1 (up to small differences in the problem statement) were obtained earlier in [27, 38]. In [38] the case was considered when $F$ is the identity matrix and the images of $V$ and $\Lambda^*$ coincide with linear spans of the vectors $G^j b$ and $(G^j)^* d$, $j = 1, \dots, p$, respectively. In [27] the case was considered when the matrix $F$ is invertible, and the conditions were imposed only on $V$ and only in terms of $(\lambda_j F - G)^{-1}b$, $j = 1, \dots, p$.

The main problem arises already at the stage of the formulation of Theorem 7.1. Namely, the impulse response $H$ of equation (1.1) may contain (Theorem 4.3) the Dirac $\delta$-function and its derivatives, and thus the classical representation

$$H(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda F - G)^{-1}\,d\lambda$$

is surely unacceptable, because it is impossible to obtain the $\delta$-function as a result of calculation of such an integral. In order to overcome this problem we propose to employ the representation (Section 5)

$$H = \frac{1}{2\pi i}\int_\Gamma \mathrm{EXP}(\lambda)(\lambda F - G)^{-1}\,d\lambda, \tag{1.4}$$

where $\mathrm{EXP}(\lambda)$, for a fixed $\lambda$, is a function of the variable $t$; in particular, $\mathrm{EXP}(\lambda)$ may be a distribution.

In Section 2 we recall some preliminaries concerning pseudoresolvents [21], which we employ in proofs of Section 3; the main result is Theorem 2.2, it describes the partial fraction expansion of a pseudoresolvent. In Section 3 we recall the main definitions concerning the pencil resolvent $R_\lambda = (\lambda F - G)^{-1}$, and we present (Theorem 3.2) its partial fraction expansion. In Section 4 we discuss (Theorem 4.3) the representation of the impulse response $H$ of equation (1.2) in the form of a linear combination of the functions $t \mapsto t^{j-1}e^{\mu_i t}\eta(t)$ and $\delta^{(j)}$, where $\eta$ is the Heaviside function, and $\delta$ is the Dirac function. In Section 5 we present a version of functional calculus for analytic functions taking their values in the algebra $\mathscr{D}'_+(\alpha)$, which consists of distributions [40, 41, 47], and we establish (Theorem 5.1) representation (1.4). In Section 6 we give a description of projection methods based on the usage of Krylov subspaces. These methods are widely used in simulation of large electrical circuits and some other applications. Finally in Section 7 we prove Theorem 7.1, which is the main result of this paper.

In the main application that we keep in mind (related to linear electrical circuits), where the discussed methods of approximate solving of problem (1.2) are used, the operators $F$ and $G$ act in finite dimensional spaces (of a large dimension). In spite of this, keeping in mind a possibility of other applications, we consider a slightly more general case when $F$ and $G$ act in Banach spaces.

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## 2 Pseudoresolvents

Below (Proposition 3.2) we shall see that the pencil resolvent $\lambda \mapsto (\lambda F - G)^{-1}$ is an example of a pseudoresolvent. This fact enables us to obtain some pencil properties by referring to pseudoresolvent properties. Several facts and definition from [21, 30] related to pseudoresolvents are recalled in this Section. The main result of this Section is Theorem 2.2.

Let $\mathbf{B}$ be a complex algebra [10, 21, 37]. An algebra $\mathbf{B}$ is called *commutative* if $AB = BA$ for all $A$ and $B$.

If there exists an element $\mathbf{1} = \mathbf{1}_{\mathbf{B}} \in \mathbf{B}$ such that $A\mathbf{1} = \mathbf{1}A = A$ for all $A \in \mathbf{B}$, then the element $\mathbf{1}$ is called a *unit* and the algebra is called an *algebra with a unit* or *unital*. The unit is unique (provided the algebra has any).

Let $\mathbf{B}$ be an algebra with a unit. An element $A^{-1} \in \mathbf{B}$ is called the *inverse* of $A \in \mathbf{B}$ if $AA^{-1} = A^{-1}A = \mathbf{1}$.

If an algebra $\mathbf{B}$ is a Banach space and the property $\|AB\| \le \|A\| \cdot \|B\|$ takes place, then $\mathbf{B}$ is called a *Banach algebra*. If the algebra $\mathbf{B}$ has a unit and $\|\mathbf{1}\| = 1$, then $\mathbf{B}$ is called a *Banach algebra with a unit*. The simplest example of a Banach algebra with a unit is the algebra $\mathbf{B}(X)$ of all bounded linear operators acting in a Banach space $X$.

A subspace $\mathbf{R}$ of an algebra $\mathbf{B}$ is called a *subalgebra* if it is closed under the algebraic operations (addition, scalar multiplication and multiplication). Any subalgebra is an algebra. The closure of a subalgebra is a subalgebra as well. A closed subalgebra of a Banach algebra is also a Banach algebra.

Let $\mathbf{B}$ be an algebra with a unit and $A \in \mathbf{B}$. The set of all $\lambda \in \mathbb{C}$ such that the element $\lambda\mathbf{1} - A$ does not have an inverse is called the *spectrum* of $A$ and is denoted by $\sigma(A)$ or $\sigma_{\mathbf{B}}(A)$. The complement $\rho(A) = \mathbb{C} \setminus \sigma(A)$ is called the *resolvent set* of $A$. The function (family)

$$R_\lambda = (\lambda\mathbf{1} - A)^{-1}, \qquad \lambda \in \rho(A),$$

is called the *resolvent* of $A$. The spectrum of an element of a non-zero algebra with a unit is a nonempty compact subset of $\mathbb{C}$.

**Proposition 2.1** ([21, Theorem 4.1.8])**.** *Let $\mathbf{B}$ be an algebra with a unit. The resolvent $R_{(\cdot)}$ of any element $A \in \mathbf{B}$, for all $\lambda, \mu \in \rho(A)$, satisfies the Hilbert identity*

$$R_\lambda - R_\mu = -(\lambda - \mu)R_\lambda R_\mu. \tag{2.1}$$

Let $\mathbf{B}$ be an algebra without a unit. The set $\widetilde{\mathbf{B}} = \mathbb{C} \oplus \mathbf{B}$ with the componentwise linear operations and the multiplication $(\alpha, A)(\beta, B) = (\alpha\beta, \alpha B + \beta A + AB)$ is, clearly, an algebra with the unit $\mathbf{1} = (1, 0)$. The element $(\alpha, A)$ is denoted by the symbol $\alpha\mathbf{1} + A$. The algebra $\widetilde{\mathbf{B}}$ is called the *algebra* $\mathbf{B}$ *with an adjoint unit*. If $\mathbf{B}$ is a Banach algebra, then we set $\|\alpha\mathbf{1} + A\| = |\alpha| + \|A\|$; obviously, in this case $\widetilde{\mathbf{B}}$ is also a Banach algebra. The *spectrum* (the *resolvent set*, the *resolvent*) of an element of an algebra without a unit is the spectrum (the resolvent set, the resolvent) of this element in the algebra with an adjoint unit. If $\mathbf{B}$ has a unit, then we mean by $\widetilde{\mathbf{B}}$ the algebra $\mathbf{B}$ itself.

Let $\mathbf{B}$ be a Banach algebra and $S \subseteq \mathbb{C}$ be a nonempty subset. A *pseudoresolvent* (on $S$ with values in $\mathbf{B}$) is [21, ch. 5, § 2, p. 184] a function (family) $\lambda \mapsto R_\lambda$ defined on $S$ with values in $\mathbf{B}$ and satisfying the Hilbert identity (2.1). The pseudoresolvent is called [5] *maximal* if it is not extendable to a wider set with preserving identity (2.1).

**Theorem 2.1** ([21, Theorem 5.8.6])**.** *Any pseudoresolvent has exactly one extension to a maximal pseudoresolvent. More precisely, if a pseudoresolvent is defined at a point $\lambda_0 \in S$, then the domain of its maximal extension is the set of all $\lambda \in \mathbb{C}$ such that the element $\mathbf{1} + (\lambda - \mu)R_{\lambda_0}$ is invertible in $\widetilde{\mathbf{B}}$. This extension has the form*

$$R_\lambda = R_{\lambda_0}\big(\mathbf{1} + (\lambda - \lambda_0)R_{\lambda_0}\big)^{-1}.$$

We call the *regular set* of a pseudoresolvent the domain $\rho(R_{(\cdot)})$ of its maximal extension, and we call the *singular set* of the pseudoresolvent the compliment $\sigma(R_{(\cdot)})$ of $\rho(R_{(\cdot)})$. Below we denote the maximal extension of the initial pseudoresolvent $R_{(\cdot)}$ by the same symbol $R_{(\cdot)}$.

**Corollary 2.1** ([21, Theorem 5.8.2])**.** *The domain of a maximal pseudoresolvent is an open set and a maximal pseudoresolvent is an analytic function with values in* $\mathbf{B}$*. Namely, the maximal pseudoresolvent has the power series expansion*

$$R_\lambda = \sum_{i=0}^{\infty} (\lambda_0 - \lambda)^i R_{\lambda_0}^{i+1}$$

*about any point* $\lambda_0 \in \rho(R_{(\cdot)})$*.*

Let a pseudoresolvent $R_{(\cdot)}$ be fixed. We denote by $\mathbf{B}_R$ the smallest closed subalgebra of the algebra $\mathbf{B}$ that contains all elements $R_\lambda$, $\lambda \in \rho(R_{(\cdot)})$, of the maximal extension of the pseudoresolvent.

**Proposition 2.2.** *The algebra* $\mathbf{B}_R$ *coincides with the closure of the linear span of all elements* $R_\lambda$*,* $\lambda \in \rho(R_{(\cdot)})$*, and is commutative.*

*Proof.* It is obvious that the closure of the linear span of the family $R_\lambda$, $\lambda \in \rho(R_{(\cdot)})$, is contained in $\mathbf{B}_R$. From Hilbert's identity (2.1) one can see that $R_\lambda R_\mu$ is in the linear span of the family $R_\lambda$, $\lambda \in \rho(R_{(\cdot)})$, provided $\lambda \neq \mu$. Hence, by continuity, the element $R_\lambda R_\mu$, where $\lambda = \mu$, is also in the closure of the linear span of the elements $R_\lambda$, $\lambda \in \rho(R_{(\cdot)})$. This implies that the closure of the linear span of the family $R_\lambda$, $\lambda \in \rho(R_{(\cdot)})$, forms a closed subalgebra.

The commutative law follows from Hilbert's identity (2.1). □

**Proposition 2.3** ([21, Theorem 5.9.3])**.** *Let a pseudoresolvent* $R_{(\cdot)}$ *admit an analytic continuation in the annulus* $0 \le \gamma_1 < |\lambda| < \gamma_2 \le \infty$*. Then there exist elements* $P, A, B \in \mathbf{B}_R$ *such that*

$$P^2 = P, \qquad AP = PA = A, \qquad BP = PB = \mathbf{0},$$
$$R_\lambda = R_\lambda^1 + R_\lambda^0, \qquad \gamma_1 < |\lambda| < \gamma_2,$$

*where*

$$R_\lambda^1 = \frac{P}{\lambda} + \frac{A}{\lambda^2} + \frac{A^2}{\lambda^3} + \frac{A^3}{\lambda^4} + \dots, \qquad \gamma_1 < |\lambda|,$$
$$R_\lambda^0 = -B - \lambda B^2 - \lambda^2 B^3 - \lambda^3 B^4 - \dots, \qquad |\lambda| < \gamma_2.$$

**Corollary 2.2.** *Let a pseudoresolvent* $R_{(\cdot)}$ *admit an analytic continuation in a deleted* $\varepsilon$*-neighbourhood of a point* $\mu \in \mathbb{C}$ *and have a pole of order* $w$ *at* $\mu$*. Then there exist elements* $P, N, B \in \mathbf{B}_R$ *such that*

$$N^w = \mathbf{0}, \qquad P^2 = P, \qquad NP = PN = N, \qquad BP = PB = \mathbf{0},$$
$$R_\lambda = R_\lambda^1 + R_\lambda^0, \qquad 0 < |\lambda - \mu| < \varepsilon,$$

*where*

$$R^1_\lambda=\frac{P}{\lambda-\mu}+\frac{N}{(\lambda-\mu)^2}+\frac{N^2}{(\lambda-\mu)^3}+\dots+\frac{N^{w-1}}{(\lambda-\mu)^w},\qquad \lambda\neq\mu,$$
$$R^0_\lambda=-B-(\lambda-\mu)B^2-(\lambda-\mu)^2B^3-\dots,\qquad |\lambda-\mu|<\varepsilon.$$

*Proof.* Let us consider the shifted family $\widetilde{R}_\lambda=R_{\lambda+\mu}$; obviously, $R_\lambda=\widetilde{R}_{\lambda-\mu}$. One can easily see that $\widetilde{R}_{(\cdot)}$ is also a pseudoresolvent. If one applies Proposition 2.3 to $\widetilde{R}_{(\cdot)}$ and after that performs the inverse change of variables $\lambda\mapsto\lambda-\mu$, he obtains the statement of the Corollary. □

**Corollary 2.3.** *Let a pseudoresolvent $R_{(\cdot)}$ admit an analytic continuation in a deleted neighbourhood of the point $\infty$ and have a pole of order $w-1$*[5] *at $\infty$. Then there exist elements $P, A, N\in\mathbf{B}_R$ such that*

$$N^{w+1}=\mathbf{0},\qquad P^2=P,\qquad AP=PA=A,\qquad NP=PN=\mathbf{0},$$

*and the Laurent expansion about infinity has the form*

$$R_\lambda=-N-\lambda N^2-\lambda^2N^3-\lambda^3N^4-\dots-\lambda^{w-1}N^w+\frac{P}{\lambda}+\frac{A}{\lambda^2}+\frac{A^2}{\lambda^3}+\frac{A^3}{\lambda^4}+\dots.$$

*Proof.* The assertion follows from Proposition 2.3. □

**Theorem 2.2** ([30, Theorem 37])**.** *Let the singular set of a pseudoresolvent $R_{(\cdot)}$ consist of a finite number of points $\mu_1,\mu_2,\dots,\mu_q\in\mathbb{C}$. Let these points $\mu_1,\mu_2,\dots,\mu_q$, and the point $\mu_0=\infty$ be poles of the pseudoresolvent, and let their orders be equal to $w_1,w_2,\dots,w_q$ and $w_0-1$, respectively. Then*[6] *there exist elements $P_1,P_2,\dots,P_q$; $N_0,N_1,\dots,N_q\in\mathbf{B}_R$ such that*

$$P_i^2=P_i,\qquad N_iP_i=P_iN_i=N_i,\qquad i=0,1,\dots,q,$$
$$P_iP_j=\mathbf{0},\qquad N_iP_j=P_jN_i=\mathbf{0},\qquad i\neq j,$$
$$N_i^{w_i}=\mathbf{0},\qquad N_0^{w_0+1}=\mathbf{0},$$

*where $P_0=\mathbf{1}-\sum_{i=1}^q P_i$ and $\mathbf{1}$ is the (adjoint) unit, and the pseudoresolvent can be represented in the form*

$$R_\lambda=\sum_{i=1}^q\sum_{j=1}^{w_i}\frac{N_i^{j-1}}{(\lambda-\mu_i)^j}-\sum_{j=0}^{w_0-1}N_0^{j+1}\lambda^j,\tag{2.2}$$

*where $N_i^0$ means $P_i$.*

The numbers $w_1,w_2,\dots,w_q$ and $w_0$ will be called *multiplicities* of the corresponding spectrum points.

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[5]It is convenient to denote the order of a pole at infinity by $w-1$, but not by $w$, because it induces exactly $w$ terms in expansion (2.2).

[6]Theorem 2.2 remains valid for the case when the order $w_0-1$ of the pole $\mu_0=\infty$ equals to $-1$, i. e., $\mu_0=\infty$ is not a pole. The simplification of the statement for this case is obvious.

*Proof.* For every $i = 1, 2, \ldots, q$, we define the idempotent $P_i = P$ and the nilpotent $N_i = N$ according to Corollary 2.2 (with the notation $P$ and $N$ from Corollary 2.2). Next we form the sums $\sum_{j=1}^{w_i} \frac{N_i^{j-1}}{(\lambda-\mu_i)^j}$ according to Corollary 2.2 again. We determine the idempotent $P_0 = \mathbf{1} - P$ and the nilpotent $N_0 = N$ according to Corollary 2.3 (with the notation $P$ and $N$ from Corollary 2.3) and we form the sum $\sum_{j=0}^{w_0-1} N_0^{j+1}\lambda^j$ as well. We combine these sums into the function (we do not know in advance whether $\widetilde{R}_\lambda$ coincides with the pseudoresolvent $R_\lambda$)

$$\widetilde{R}_\lambda = \sum_{i=1}^{q}\sum_{j=1}^{w_i} \frac{N_i^{j-1}}{(\lambda-\mu_i)^j} - \sum_{j=0}^{w_0-1} N_0^{j+1}\lambda^j.$$

We note that the difference $\lambda \mapsto R_\lambda - \widetilde{R}_\lambda$, where $R_\lambda$ is the pseudoresolvent, has no singular points (the poles $\mu_0, \mu_1, \ldots, \mu_q$ are removable singular points). Therefore by the Liouville theorem this difference is a constant function. This constant may be calculated as the coefficient of $\lambda^0$ in the Laurent expansion about infinity. The coefficients of $\lambda^0$ in the Laurent expansions of $\lambda \mapsto R_\lambda$ and $\lambda \mapsto \widetilde{R}_\lambda$ about infinity coincide with $N_0$ by Corollary 2.3 and the definition of $N_0$. Thus the constant is equal to zero and $R_\lambda = \widetilde{R}_\lambda$.

The identities $P_i^2 = P_i$ and $N_iP_i = P_iN_i = N_i$, where $i = 0, 1, 2, \ldots, q$, follow from Corollaries 2.3 and 2.2.

Next we show, for example, that $P_iP_1 = \mathbf{0}$ for $i = 2, \ldots, q$, and $N_iP_1 = \mathbf{0}$ for $i = 2, \ldots, q$ and $i = 0$. We recall that in a deleted neighborhood of the point $\mu_1$, the pseudoresolvent $R_{(\cdot)}$ has the form

$$R_\lambda = \frac{P_1}{\lambda-\mu_1} + \frac{N_1}{(\lambda-\mu_1)^2} + \ldots + \frac{N_1^{w_1-1}}{(\lambda-\mu_1)^{w_1}} - B - (\lambda-\mu_1)B^2 - \ldots,$$

where $BP_1 = P_1B = \mathbf{0}$ (and $P_1^2 = P_1$ and $N_1P_1 = P_1N_1 = N_1$, as it was noted above), by Corollary 2.2. This implies that,

$$R_\lambda P_1 = P_1R_\lambda = \frac{P_1}{\lambda-\mu_1} + \frac{N_1}{(\lambda-\mu_1)^2} + \ldots + \frac{N_1^{w_1-1}}{(\lambda-\mu_1)^{w_1}}. \tag{2.3}$$

Identity (2.3) takes place at all points $\lambda \neq \mu_1$ by the uniqueness of analytic continuation. But on the other hand, by (2.2) it follows that,

$$R_\lambda P_1 = \sum_{i=1}^{q}\sum_{j=1}^{w_i} \frac{N_i^{j-1}P_1}{(\lambda-\mu_i)^j} - \sum_{j=0}^{w_0-1} N_0^{j+1}P_1\lambda^j. \tag{2.4}$$

If one compares the coefficients in the Laurent expansions (2.3) and (2.4) about the points $\mu_2, \ldots, \mu_q$ and $\mu_0 = \infty$, (taking into account the equality $N_i^0 = P_i$) he obtains that $P_iP_1 = \mathbf{0}$ for $i = 2, \ldots, q$, and $N_iP_1 = \mathbf{0}$ for $i = 2, \ldots, q$ and $i = 0$.

Finally we show that $P_iP_0 = \mathbf{0}$ and $N_iP_0 = \mathbf{0}$ for $i = 1, \ldots, q$. We note that in a deleted neighbourhood of the point $\mu_0 = \infty$ the pseudoresolvent $R_\lambda$ has the form

$$R_\lambda = -N_0 - \lambda N_0^2 - \lambda^2 N_0^3 - \ldots - \lambda^{w_0-1}N^{w_0} + \frac{\mathbf{1}-P_0}{\lambda} + \frac{A}{\lambda^2} + \frac{A^2}{\lambda^3} + \frac{A^3}{\lambda^4} + \ldots, \tag{2.5}$$

and $A(\mathbf{1}-P_0)=(\mathbf{1}-P_0)A=A$, and consequently, $AP_0=P_0A=\mathbf{0}$ (and as it was noted above, $P_0^2=P_0$ and $N_0P_0=P_0N_0=N_0$) by Corollary 2.3. Hence

$$R_\lambda P_0=P_0R_\lambda=-N_0-\lambda N_0^2-\lambda^2N_0^3-\ldots-\lambda^{w_0-1}N^{w_0}. \tag{2.6}$$

This equality holds at all points $\lambda\neq\mu_i$ by the uniqueness of analytic continuation. On the other hand, we have

$$R_\lambda P_0=\sum_{i=1}^{q}\sum_{j=1}^{w_i}\frac{N_i^{j-1}P_0}{(\lambda-\mu_i)^j}-\sum_{j=0}^{w_0-1}N_0^{j+1}P_0\lambda^j, \tag{2.7}$$

by equation (2.2). If one compares the coefficients in the Laurent expansions (2.6) and (2.7) about the points $\mu_1,\ldots,\mu_q$ (keeping in mind that $N_i^0=P_i$) he arrives at $P_iP_0=\mathbf{0}$ and $N_iP_0=\mathbf{0}$, where $i=1,\ldots,q$.

Finally, if one compares the residues at infinity of expressions (2.5) and (2.2), he obtains the identity $\mathbf{1}-P_0=\sum_{i=1}^q P_i$. □

## 3 Resolvent of a pencil

The resolvent of a pencil is a natural spectral tool for investigating of equation (1.1). The main result of this Section is Theorem 3.2, which describes a partial fractions expansion for the resolvent of a finite-dimensional pencil.

Let $X$ and $Y$ be complex Banach spaces. We denote by $\mathbf{B}(X,Y)$ the set of all bounded linear operators acting from $X$ to $Y$. We use the shorthand $\mathbf{B}(X)$ for $\mathbf{B}(X,X)$. We denote the identity operator by $\mathbf{1}\in\mathbf{B}(X)$.

Let $F,G\in\mathbf{B}(X,Y)$. An (*operator*) *pencil* [17, 22, 32] is the function (family)

$$\lambda\mapsto\lambda F-G\colon X\to Y,\qquad \lambda\in\mathbb{C}.$$

The *resolvent set* of the pencil is the set $\rho(F,G)$ that consists of all points $\lambda\in\mathbb{C}$ such that the operator $\lambda F-G\colon X\to Y$ is invertible, and the *resolvent* of the pencil is the function (family)

$$R_\lambda=(\lambda F-G)^{-1}\colon Y\to X,\qquad \lambda\in\rho(F,G).$$

The compliment $\sigma(F,G)$ of $\rho(F,G)$ is called the *spectrum* of the pencil. We assume that all pencils under consideration are *regular* [17, 22], i. e., their resolvent sets are nonempty.

**Proposition 3.1** (see., for example, [29]). *The resolvent of the pencil satisfies the $F$-Hilbert identity*

$$R_\lambda-R_\mu=-(\lambda-\mu)R_\lambda FR_\mu,\qquad \lambda,\mu\in\rho(F,G).$$

We denote by $\mathbf{B}_{(F,G)}(Y,X)$ the closure of the linear span of all operators $R_\lambda$, $\lambda\in\rho(F,G)$, with respect to the norm of $\mathbf{B}(Y,X)$. We define the operation of $F$-*multiplication* [29] on $\mathbf{B}_{(F,G)}(Y,X)$ by the formula

$$A\odot B=AFB.$$

We denote powers and inverses with respect to the $F$-multiplication by the symbols of the kind $A^{n\odot}$ and $A^{-1\odot}$.

**Theorem 3.1 ([29]).** *The Banach space $\mathbf{B}_{(F,G)}(Y,X)$ is a commutative Banach algebra with $F$-multiplication as the operation of multiplication (to within a replacement of the norm to an equivalent one). This algebra has a unit if and only if the operator $F\colon X \to Y$ is invertible; in this case the unit $\mathbf{1}_\odot$ is $F^{-1}$.*

**Proposition 3.2 ([29]).** *The resolvent of a pencil is a maximal $F$-pseudoresolvent, i. e., it can not be extended with the preservation of the $F$-Hilbert identity to a wider set than $\rho(F,G)$.*

**Corollary 3.1.** *The domain of the resolvent of a pencil is an open set and the resolvent itself is an analytic function with values in $\mathbf{B}_{(F,G)}(Y,X)$. Namely, the Taylor expansion of the pencil resolvent $R_\lambda = (\lambda F - G)^{-1}$ about any point $\lambda_0 \in \rho(F,G)$ has the form*

$$R_\lambda = \sum_{i=0}^{\infty} (\lambda_0 - \lambda)^i R_{\lambda_0}^{(i+1)\odot}.$$

*Proof.* The assertion follows from Theorem 3.1, Proposition 3.2 and Corollary 2.1. □

**Proposition 3.3.** *Let the operator $F$ be invertible. Then the power series expansion of the pencil resolvent $R_\lambda = (\lambda F - G)^{-1}$ about infinity has the form*

$$R_\lambda = \sum_{i=1}^{\infty} \frac{1}{\lambda^i} (F^{-1}GF^{-1})^{(i-1)\odot},$$

*where $(F^{-1}GF^{-1})^{0\odot}$ means $\mathbf{1}_\odot = F^{-1}$.*

*Proof.* We note that

$$(\lambda F - G)^{-1} = F^{-1}(\lambda F^{-1} - F^{-1}GF^{-1})^{-1}F^{-1} = (\lambda \mathbf{1}_\odot - F^{-1}GF^{-1})^{-1\odot}.$$

A justification of the last transformation is reduced to the direct verification of the equalities

$$\begin{gathered} \big[F^{-1}(\lambda F^{-1} - F^{-1}GF^{-1})^{-1}F^{-1}\big] \odot (\lambda \mathbf{1}_\odot - F^{-1}GF^{-1}) = \mathbf{1}_\odot, \\ (\lambda \mathbf{1}_\odot - F^{-1}GF^{-1}) \odot \big[F^{-1}(\lambda F^{-1} - F^{-1}GF^{-1})^{-1}F^{-1}\big] = \mathbf{1}_\odot. \end{gathered}$$

It remains to employ the Neumann series in the algebra $\mathbf{B}_{(F,G)}(Y,X)$. □

**Corollary 3.2.** *Let infinity be a pole of order $w_0 - 1$ of the pencil resolvent $R_{(\cdot)}$. Then there exist elements $N, \Pi, A \in \mathbf{B}_{(F,G)}(Y,X)$ such that*

$$N^{w_0+1\odot} = \mathbf{0}, \qquad \Pi^{2\odot} = \Pi, \qquad N \odot \Pi = \Pi \odot N = \mathbf{0}, \qquad A \odot \Pi = \Pi \odot A = A$$

*and the Laurent expansion of the pencil resolvent about infinity has the form*

$$R_\lambda = R_{r,\lambda} + R_{s,\lambda},$$

*where*

$$\begin{aligned} R_{r,\lambda} &= \frac{\Pi}{\lambda} + \frac{A}{\lambda^2} + \frac{A^{2\odot}}{\lambda^3} + \frac{A^{3\odot}}{\lambda^4} + \dots, \\ R_{s,\lambda} &= -N - \lambda N^{2\odot} - \lambda^2 N^{3\odot} - \dots - \lambda^{w_0-1} N^{w_0\odot}. \end{aligned} \tag{3.1}$$

*Proof.* The assertion follows from Theorem 3.1 and Corollary 2.3. □

The *augmented resolvent set* of a pencil $\lambda \mapsto \lambda F - G$ is [5, p. 31] the subset $\bar{\rho}(F,G)$ of the extended complex plane $\overline{\mathbb{C}}$ that consists of $\rho(F,G)$ and may by of the point $\infty$. The point $\infty$ belongs to $\bar{\rho}(F,G)$ if the operator $F$ is invertible. Otherwise, the point $\infty$ belongs to the *augmented spectrum* $\bar{\sigma}(F,G)$. If $\infty \in \bar{\rho}(F,G)$, then by Proposition 3.3 the resolvent $\lambda \mapsto (\lambda F - G)^{-1}$ is defined on a deleted neighbourhood of infinity and $\lim_{\lambda\to\infty} \lambda R_\lambda = F^{-1}$ in the norm of $\mathbf{B}(Y,X)$.

In the most interesting case, when $F$ is not invertible, the algebra $\mathbf{B}_{(F,G)}(Y,X)$ has no unit by Theorem 3.1. In this case we denote by $\widetilde{\mathbf{B}}_{(F,G)}(Y,X)$ the algebra $\mathbf{B}_{(F,G)}(Y,X)$ with the adjoint unit $\mathbf{1}_\odot$. If $F$ is invertible, we mean by $\widetilde{\mathbf{B}}_{(F,G)}(Y,X)$ the algebra $\mathbf{B}_{(F,G)}(Y,X)$ itself and we mean by $\mathbf{1}_\odot$ the operator $F^{-1}$.

**Theorem 3.2.** *Let the spectrum of a pencil consist of finitely many points $\mu_1,\mu_2,\dots,\mu_q \in \mathbb{C}$. Let the points $\mu_1,\mu_2,\dots,\mu_q$ as well as the point $\mu_0 = \infty$ be poles of orders $w_1,w_2,\dots,w_q$ and $w_0-1$ respectively. Then there exist the operators $\Pi_1,\Pi_2,\dots,\Pi_q; N_0,N_1,\dots,N_q \in \mathbf{B}_{(F,G)}(Y,X)$ such that*

$$\begin{aligned}
&\Pi_i^{2\odot} = \Pi_i, && N_i \odot \Pi_i = \Pi_i \odot N_i = N_i, && i = 0,1,\dots,q,\\
&\Pi_i \odot \Pi_j = \mathbf{0}, && N_i \odot \Pi_j = \Pi_j \odot N_i = \mathbf{0}, && i \neq j,\\
& && N_i^{w_i\odot} = \mathbf{0}, && N_0^{(w_0+1)\odot} = \mathbf{0},
\end{aligned}$$

*where $\Pi_0 = \mathbf{1}_\odot - \sum_{i=1}^q \Pi_i$, and the resolvent of a pencil can be represented in the form*

$$R_\lambda = \sum_{i=1}^{q}\sum_{j=1}^{w_i} \frac{N_i^{(j-1)\odot}}{(\lambda-\mu_i)^j} - \sum_{j=0}^{w_0-1} N_0^{(j+1)\odot}\lambda^j, \tag{3.2}$$

*where $N_i^{0\odot} = \Pi_i$. If $\dim X = \dim Y = N$, then $\sum_{i=0}^q w_i \le N$.*

*Proof.* Everything except the inequality $\sum_{i=0}^q w_i \le N$ follows from Theorem 2.2, Proposition 3.2, and Theorem 3.1.

Let the spaces $X$ and $Y$ have a finite dimension $N$. Then the inverse $(\lambda F - G)^{-1}$ can be found (using a passage to matrix representation) by Cramer's rule, i. e., dividing the cofactors by the determinant. It is obvious that the determinant is a polynomial of degree $m \le N$; and if $F$ is not invertible, the degree is strictly less than $N$. The determinant does not vanish identically, because the resolvent set is not empty. The degrees of the cofactors do not exceed $N-1$. Thus the resolvent is a rational function with the denominator degree less than or equal to $N$ and the numerator degree less than or equal to $N-1$. Thus [26, ch. 1, § 3.5] the resolvent can be represented in the form

$$R_\lambda = p_0(\lambda) + \sum_{i=1}^{q} \frac{\tilde{p}_i(\lambda)}{(\lambda-\mu_i)^{k_i}},$$

where $\mu_1,\dots,\mu_q$ are the roots of the denominator and $k_1,\dots,k_q$ are their multiplicities, $k_1+\dots+k_q = m$, the degrees of the polynomials $\tilde{p}_1,\dots,\tilde{p}_q$ (their coefficients belong to $\mathbf{B}(Y,X)$) are strictly less than $k_1,\dots,k_q$, and the degree $k_0$ of the polynomial $p_0$ is less than or equal to $N-1-m$. Thus, $\sum_{i=0}^q k_i \le N-1$. If $\mu_i$ is a root of the polynomial $\tilde{p}_i$, then the numerator and the denominator of the fraction $\dfrac{\tilde{p}_i(\lambda)}{(\lambda-\mu_i)^{k_i}}$ can be cancelled by some power of the difference $\lambda-\mu_i$. As a result we arrive at the representation

$$R_\lambda = p_0(\lambda) + \sum_{i=1}^q \frac{p_i(\lambda)}{(\lambda-\mu_i)^{w_i}},$$

where the fractions $\dfrac{p_i(\lambda)}{(\lambda-\mu_i)^{w_i}}$ are irreducible. Obviously, the degree of $p_i$ is strictly less than $w_i$, $i=1,\dots,q$. We denote the degree $k_0$ of the polynomial $p_0$ by $w_0-1$. We note that the last formula must coincide with (3.2). This implies the inequality $\sum_{i=0}^q w_i \le N$. □

## 4 Impulse response

The main results of this Section are Theorems 4.2 and 4.3, where representations for the impulse response of equation (1.1) are described.

Let $X$ and $Y$ be complex Banach spaces. We denote by $X^*$ the conjugate space of $X$, and by $\langle x, x^*\rangle$ the action of the functional $x^*\in X^*$ on the vector $x\in X$. Let $F,G\in \mathbf{B}(X,Y)$, $b\in Y$, $d\in X^*$. We recall that our main object is problem (1.2). We also recall that the *resolvent set of the pencil* $\lambda\mapsto\lambda F-G$ *is assumed to be nonempty.*

We denote by $\mathscr{D}=\mathscr{D}(\mathbb{R},\mathbb{C})$ the linear space of all infinitely many times differentiable functions $\psi\colon\mathbb{R}\to\mathbb{C}$ with compact support equipped with the pointwise operations of addition and scalar multiplication. A sequence $\psi_k\in\mathscr{D}$ is called *convergent* to a function $\psi\in\mathscr{D}$ if

(a) the supports of the functions $\psi_k$ are *uniformly bounded*, i. e., there exists a segment $[a,b]$ that contains all the supports;

(b) the sequence $\psi_k^{(n)}$ uniformly converges to $\psi^{(n)}$ for all $n=0,1,2,\dots$; here $\psi^{(n)}$ is the $n$-th derivative of $\psi$, particularly, $\psi^{(0)}=\psi$.

Let $\mathbb{E}$ be an arbitrary complex Banach space with the norm $|\cdot|$. Let $f\colon\mathscr{D}\to\mathbb{E}$ be a linear operator; we interpret such an operator as a vector-valued functional. We denote the value of the functional $f$ at $\psi\in\mathscr{D}$ by the symbol $\langle\psi,f\rangle$. The functional $f$ is called *continuous* if $\langle\psi_k,f\rangle$ converges to $\langle\psi,f\rangle$ whenever $\psi_k$ converges to $\psi$ in $\mathscr{D}$. Every continuous linear functional $f\colon\mathscr{D}\to\mathbb{E}$ is called (see [41] and [47] for details) a *distribution* on $\mathbb{R}$ with values in $\mathbb{E}$. We denote the linear space of all distributions with the natural operations of addition and scalar multiplication by the symbol $\mathscr{D}'=\mathscr{D}'(\mathbb{R},\mathbb{E})$. By misuse of language, the elements of the space $\mathscr{D}'$ are usually called simply functions.

A *derivative* of the functional $f\in\mathscr{D}'$ is the functional $f'$ defined by the rule

$$\langle\psi,f'\rangle=-\langle\psi',f\rangle.$$

It is easy to see that $f'$ is actually a linear continuous functional and hence it belongs to $\mathscr{D}'$.

Let $f, g \in \mathscr{D}'$. We say that $f$ and $g$ *coincide* on an open set $M \subseteq \mathbb{R}$ if $\langle \psi, f \rangle = \langle \psi, g \rangle$ for all $\psi \in \mathscr{D}$ supported in $M$.

We denote by $\mathscr{S} = \mathscr{S}(\mathbb{R}, \mathbb{C})$ the linear space of all infinitely many times differentiable functions $\psi\colon \mathbb{R} \to \mathbb{C}$ that satisfy the condition

$$\forall m, n = 0, 1, 2, \dots \quad \exists C \quad \forall x \qquad |x^m \psi^{(n)}(x)| < C$$

(see, e. g., [47, p. 153] for details). We say that a sequence $\psi_k \in \mathscr{S}$ *converges* to a function $\psi \in \mathscr{S}$ if

> for all $m, n = 0, 1, 2, \dots$ the sequence of the functions $x \mapsto x^m \psi_k^{(n)}(x)$ uniformly converges to the function $x \mapsto x^m \psi^{(n)}(x)$.

A linear (vector-valued) functional $f\colon \mathscr{S} \to \mathbb{E}$ is called *continuous* if $\langle \psi_k, f \rangle$ converges to $\langle \psi, f \rangle$ whenever $\psi_k$ converges to $\psi$ in $\mathscr{S}$. Every continuous linear functional $f\colon \mathscr{S} \to \mathbb{E}$ is called [47, p. 155] a *tempered distribution* with values in $\mathbb{E}$. We denote the set of all tempered distributions by the symbol $\mathscr{S}' = \mathscr{S}'(\mathbb{R}, \mathbb{E})$. Since $\mathscr{D} \subseteq \mathscr{S}$ and the convergence in $\mathscr{D}$ implies the convergence in $\mathscr{S}$, the space $\mathscr{S}'$ can be considered as a subspace of $\mathscr{D}'$. It is easy to show that the operation of differentiation takes $\mathscr{S}$ and $\mathscr{S}'$ into themselves.

We recall [47, p. 165] that the Fourier transform $\mathcal{F}$ is well defined on $\mathscr{S}$ and $\mathscr{S}'$.

A distribution of the *class* $\mathscr{D}'_+(\alpha) = \mathscr{D}'_+(\mathbb{E}, \alpha)$, $\alpha \in \mathbb{R}$, is [47, p. 181] a distribution $f \in \mathscr{D}'$ that is equal to zero on $(-\infty, 0)$ and after the multiplication by the function $t \mapsto e^{-\sigma t}$, for any $\sigma > \alpha$, falls into the space $\mathscr{S}'$. Obviously $\mathscr{D}'_+(\alpha) \subseteq \mathscr{D}'_+(\beta)$ for $\alpha \leq \beta$.

The *Laplace transform* of a function $f \in \mathscr{D}'_+(\alpha)$ is [40, ch. 3], [47, p. 183] the function

$$\mathcal{L}(\sigma + i\omega) = \mathcal{F}\big(f_\sigma\big)(\omega), \qquad \sigma > \alpha,$$

where $\mathcal{F}\colon \mathscr{S}' \to \mathscr{S}'$ is the Fourier transform and $f_\sigma(t) = e^{-\sigma t} f(t)$.

The (*operator*) *impulse response* of differential equation (1.1) is a distribution $H \in \mathscr{D}'\big(\mathbf{B}(Y, X), \alpha\big)$ that is equal to zero on $(-\infty, 0)$ and satisfies the equation

$$FH'(t) - GH(t) = \mathbf{1}\delta(t),$$

where $\delta$ is the Dirac function. The (*scalar*) *impulse response* of problem (1.2) is the distribution $h$ that coincides with the solution $y$ of the problem

$$\begin{aligned} Fx' &= Gx + b\delta(t), \\ y &= \langle x, d \rangle, \end{aligned}$$

and is equal to zero on $(-\infty, 0)$. We recall that the impulse response is important, because the solution of problem (1.2) can be expressed in terms of convolution with the impulse response:

$$y(t) = \int_{-\infty}^{+\infty} h(t - s) bu(s)\, ds.$$

**Proposition 4.1.** *Let $H$ be the operator impulse response of differential equation* (1.1). *Then the function $h(t) = \langle H(t)b, d\rangle$ is a scalar impulse response of problem* (1.2).

*Proof.* We set $x(t) = H(t)b$. We prove that $x$ satisfies the equation $Fx' = Gx + b\delta(t)$. We have

$$Fx'(t) - Gx(t) = FH'(t)b - GH(t)b = \big(FH'(t) - GH(t)\big)b = \mathbf{1}\delta(t)b = b\delta(t).$$

Obviously $y(t) = \langle x(t), d\rangle = \langle H(t)b, d\rangle$. $\square$

**Theorem 4.1** ([28]). *A sufficient condition for the existence of the impulse response $H \in \mathscr{D}'_+(\alpha)$ of equation* (1.1) *is that the half plane $\operatorname{Re}\lambda > \alpha$ is contained in the resolvent set $\rho(F, G)$ and the resolvent of the pencil satisfies the condition*

$$\forall\sigma > \alpha \quad \exists w \in \mathbb{Z} \quad \exists C \qquad \|R_\lambda\| \le C(1 + |\lambda|^w) \quad \textit{for } \operatorname{Re}\lambda > \sigma. \tag{4.1}$$

*In this case $R_{(\cdot)}$ is the Laplace transform of $H$.*

**Lemma 4.1** ([31, p. 509], [11]). *The inverse Laplace transform of the function $\lambda \mapsto \frac{1}{(\lambda-\mu)^{j+1}}$, $j = 0, 1, \dots,$ is the function*

$$t \mapsto \frac{t^j}{j!}\, e^{\mu t}\eta(t),$$

*where $\eta$ is the Heaviside function*

$$\eta(t) = \begin{cases} 1, & \text{for } t > 0,\\ 0, & \text{for } t \le 0. \end{cases}$$

*The inverse Laplace transform of the function $\lambda \mapsto \lambda^j$, $j = 0, 1, \dots,$ is the function*

$$t \mapsto \delta^{(j)}(t).$$

**Lemma 4.2** ([31, p. 515]). *Let a function $Z$ of a complex variable* (*taking its values in a Banach space*) *have the series expansion*

$$Z(\lambda) = \sum_{k=1}^{\infty} \frac{C_k}{\lambda^k}$$

*about infinity. Then the inverse Laplace transform of $Z$ can be represented in the form*

$$z(t) = \Big(\sum_{k=1}^{\infty} C_k \frac{t^{k-1}}{(k-1)!}\Big)\eta(t).$$

**Theorem 4.2.** *Let infinity be a pole of the pencil resolvent of order $w_0 - 1$. Then the impulse response of the class $\mathscr{D}'_+(\alpha)$, where $\alpha \in \mathbb{R}$, lies to the right of the spectrum $\sigma(F, G)$, exists and can be represented in the form*

$$H(t) = H_r(t) + H_s(t),$$

*where $H_r$ and $H_s$ are the inverse Laplace transforms of the summands $R_{r,(\cdot)}$ and $R_{s,(\cdot)}$ from Corollary 3.2. They can be represented in the form of the sums*

$$\begin{aligned} H_r(t) &= \Big(\Pi + At + \frac{t^2}{2!}A^{2\odot} + \frac{t^3}{3!}A^{3\odot} + \dots\Big)\eta(t), \\ H_s(t) &= -N\delta(t) - N^{2\odot}\delta'(t) - N^{3\odot}\delta''(t) - N^{4\odot}\delta'''(t) - \dots - N^{w_0\odot}\delta^{(w_0-1)}(t), \end{aligned} \tag{4.2}$$

*where $\Pi$, $A$, and $N$ are the same as in Corollary 3.2. Moreover,*

$$H_r(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda F - G)^{-1}\,d\lambda, \tag{4.3}$$

*where $\Gamma$ is an oriented envelope* [21, p. 166] *of the* (*ordinary*) *spectrum $\sigma(F, G)$ of the pencil with respect to $\infty$, in other words $\Gamma$ is bounded and surrounds the* (*ordinary*) *spectrum $\sigma(F, G)$ anticlockwise.*

Under assumptions of Theorem 4.2 we call the functions $H_r$ and $H_s$ *regular* and *singular* parts of the impulse response, respectively (in analogy with terms customary for the theory of distributions).

*Proof.* Sequence (3.1) from Corollary 3.2 uniformly converges in some neighbourhood of infinity and, thus, determines in it a bounded function. This implies that $R_\lambda = R_{r,\lambda} + R_{s,\lambda}$ satisfies the estimate

$$\|R_\lambda\| \le C|\lambda|^{w_0-1}$$

in a neighbourhood of infinity. This yields condition (4.1). Consequently, by Theorem 4.1 the impulse response exists.

We apply the properties of the Laplace transform to the power series from Corollary 3.2. We apply (Lemma 4.1) directly the inverse Laplace transform to the finite sum

$$R_{s,\lambda} = -N - \lambda N^{2\odot} - \lambda^2 N^{3\odot} - \dots - \lambda^{w_0-1}N^{w_0\odot}.$$

By Lemma 4.2 the inverse Laplace transform of the series

$$R_{r,\lambda} = \frac{\Pi}{\lambda} + \frac{A^{1\odot}}{\lambda^2} + \frac{A^{2\odot}}{\lambda^3} + \frac{A^{3\odot}}{\lambda^4} + \dots$$

coincides with the function

$$H_r(t) = \Big(\Pi + At + \frac{t^2}{2!}A^{2\odot} + \frac{t^3}{3!}A^{3\odot} + \dots\Big)\eta(t).$$

The same result is obtained if one computes integral (4.3) with the help of expansion (3.1). □

**Theorem 4.3.** *In the notation of Theorem* 3.2 *the operator impulse response of equation* (1.1) *can be represented in the form*

$$H(t) = \sum_{i=1}^{q}\sum_{j=1}^{w_i} N_i^{(j-1)\odot}\frac{t^{j-1}}{(j-1)!}e^{\mu_i t}\eta(t) - \sum_{j=0}^{w_0-1} N_0^{(j+1)\odot}\delta^{(j)}(t), \tag{4.4}$$

*where $\eta$ is the Heaviside function.*

*Proof.* By formula (3.2) it is seen that the pencil resolvent satisfies the estimate from Theorem 4.1. It remains to apply Theorem 3.2 and the properties (Lemmas 4.1 and 4.2) of the Laplace transform. □

**Corollary 4.1.** *In the notation of Theorem* 3.2 *the scalar impulse response of problem* (1.2) *can be represented in the form*

$$h(t)=\sum_{i=1}^{q}\sum_{j=1}^{w_i}\langle N_i^{(j-1)\odot}b,d\rangle\,\frac{t^{j-1}}{(j-1)!}e^{\mu_i t}\eta(t)-\sum_{j=0}^{w_0-1}\langle N_0^{(j+1)\odot}b,d\rangle\,\delta^{(j)}(t). \tag{4.5}$$

*Proof.* The assertion follows from Proposition 4.1 and Theorem 4.3. □

## 5 Interpolation methods

In this Section we define a class of interpolation methods of approximate solution of problem (1.2). It is assumed that infinity is a pole of order $w_0-1$ of the pencil resolvent.

If the operator $F$ is invertible, then, by Proposition 3.3 and Theorem 4.1, the impulse responses $H$ and $h$ are determined by the explicit formulas (the inverse Laplace transform)

$$H(t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\lambda t}R_\lambda\,d\lambda,\qquad h(t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\lambda t}\langle R_\lambda b,d\rangle\,d\lambda. \tag{5.1}$$

With the aid of Jordan's lemma [31, p. 436] these integrals can be transformed into the contour integrals

$$H(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R_\lambda\,d\lambda,\qquad h(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\langle R_\lambda b,d\rangle\,d\lambda,\qquad t>0, \tag{5.2}$$

where $\Gamma$ surrounds the pencil spectrum $\sigma(F,G)$. Integrals (5.2) can be naturally interpreted [10, 21, 37] as a result of application of the functional calculus, induced by the pencil, to the function $\lambda\mapsto e^{\lambda t}$. Two representations (5.1) and (5.2) give rise to two different ideas of approximate methods. Since we interprets (5.1) as the Laplace transform, it is natural to replace the factor $\lambda\mapsto\langle R_\lambda b,d\rangle$ by a more simple function. This way results in Krylov subspace *projection* methods discussed in Section 6. Since we interprets (5.2) as the functional calculus, it is natural to approximate the factor $\lambda\mapsto e^{\lambda t}$ by a more simple function. If the latter approximation is constructed on the basis of interpolation considerations, then we call this approach *interpolational*. (The main result of this paper, Theorem 7.1, asserts that projection methods are special cases of interpolation ones.)

Unfortunately, if the operator $F$ is not invertible, then both formula (5.1) and (5.2) stop working. The literal interpretation of formula (5.1) is impossible, since, by Corollary 3.2, the improper integrals in (5.1) become divergent for sure. On the other hand, formula (5.2) can not result in the singular part (4.2) of the impulse response, because after the contour integration (separately for each $t$) $\delta$-function can not arise.

To remedy this trouble, we propose to change slightly the meaning of formula (5.2). Namely, we propose to calculate integrals (5.2) simultaneously for all $t$. More precisely, we propose to calculate the integral not of the scalar function $\lambda \mapsto e^{\lambda t}$ depending on the parameter $t$, but of the vector-function that assigns to each $\lambda$ the function $t \mapsto e^{\lambda t}$ considered as an element of an appropriate functional space. With the context of such an interpretation we may replace the function $t \mapsto e^{\lambda t}$ by a distribution. Moreover, it is possible to fulfil this replacement so that the contour integral gives the same result as Theorems 4.2 and 4.3. This Section in devoted to the description of such a construction: the impulse response is represented in the form of a special exponential function EXP of the pencil (Theorem 5.1).

We fix $\alpha \in \mathbb{R}$ that lies to the right of the real parts of all points from $\sigma(F, G)$. We denote by the symbol $\mathbf{O}\big(\bar\sigma(F, G)\big) = \mathbf{O}\big(\bar\sigma(F, G), \mathscr{D}'_+(\alpha)\big)$ the set of all analytic[7] functions $f$, where every $f$ is defined on an open neighbourhood $U$ of the set $\bar\sigma(F, G)$ and takes its values in the algebra $\mathscr{D}'_+(\alpha) = \mathscr{D}'_+\big(\mathbb{C}, \alpha\big)$, and $f(\infty) = 0$ if $\infty \in \bar\sigma(F, G)$. We say that two functions $f_1 \colon U_1 \to \mathscr{D}'_+(\alpha)$ and $f_2 \colon U_2 \to \mathscr{D}'_+(\alpha)$ are *equivalent* if there exists an open neighbourhood $U \subseteq U_1 \cap U_2$ of the set $\bar\sigma(F, G)$ such that $f_1$ and $f_2$ coincide on $U$. One can easily show that this is really an equivalence relation. Thus the elements of $\mathbf{O}\big(\bar\sigma(F, G)\big)$ are, strictly speaking, classes of equivalent functions.

For our aims, the most important example of a function in $\mathbf{O}\big(\bar\sigma(F, G), \mathscr{D}'_+(\alpha)\big)$ is the function

$$\mathrm{EXP}(\lambda)(t) = \begin{cases} e^{\lambda t}\eta(t) & \text{about } \sigma(F, G), \\ \sum_{i=1}^{w_0} \dfrac{\delta^{(i-1)}(t)}{\lambda^i} & \text{about } \infty, \end{cases}$$

where $\eta$ is the Heaviside function, $\delta$ is the Dirac function, and $w_0 - 1$ is the order of infinity as a pole of the pencil resolvent.

We define the mapping $\Phi \colon \mathbf{O}\big(\bar\sigma(F, G), \mathscr{D}'_+(\alpha)\big) \to \mathscr{D}'_+\big(\mathbf{B}_{(F,G)}(Y, X), \alpha\big)$ by the formula

$$\Phi(f) = \frac{1}{2\pi i} \int_\Gamma f(\lambda) R_\lambda \, d\lambda, \tag{5.3}$$

where $\Gamma$ is an oriented envelope [21, p. 166] of the augmented spectrum $\bar\sigma(F, G)$ of the pencil with respect to the function $f$ domain complement. Thus, $\Gamma$ surrounds the ordinary spectrum $\sigma(F, G)$ anticlockwise and surrounds infinity clockwise.

**Theorem 5.1.** *Let infinity be a pole of order $w_0 - 1$ of the pencil resolvent. Then the operator impulse response can be represented in the form*

$$H = \Phi(\mathrm{EXP}).$$

*Proof.* In fact, the direct integration

$$\Phi(\mathrm{EXP}) = \frac{1}{2\pi i} \int_\Gamma \mathrm{EXP}(\lambda) R_\lambda \, d\lambda \tag{5.4}$$

gives, according to Corollary 3.2, the same result as Theorem 4.2. □

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[7] A function with values in a locally convex space [9] is called *analytic* if the limit of its difference quotient exists with respect to the convergence in the space. The function $f$ is *analytic at infinity* if the function $f_1(\lambda) = f\big(\frac{1}{\lambda}\big)$ is analytic at zero.

We emphasize that, if $\infty \notin \bar\sigma(F, G)$, then integral (5.4) can be understood pointwise (i. e., for every $t \in \mathbb{R}$ separately), and therefore it turns into the usual formula $H(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} R_\lambda\,d\lambda$.

**Corollary 5.1.** *Let infinity be a pole of order $w_0 - 1$ of the pencil resolvent. Then the scalar impulse response can be represented in the form*

$$h = \bigl\langle \Phi(\mathrm{EXP})\,b, d\bigr\rangle.$$

*Proof.* The assertion follows from Theorem 5.1 and Proposition 4.1. □

The *derivative* of order $k$ of the function $f$ *at infinity* is the $k$-th derivative of the function $f_1(\lambda) = f\bigl(\frac{1}{\lambda}\bigr)$ at zero.

**Proposition 5.1.** *Let $f, g \in \mathbf{O}\bigl(\bar\sigma(F, G), \mathscr{D}'_+(\alpha)\bigr)$. Let the augmented spectrum of the pencil $\lambda \mapsto \lambda F - G$ consist of the points $\mu_1, \dots, \mu_q \in \mathbb{C}$ having the multiplicities*[8] *$w_1, \dots, w_q$, and the point $\mu_0 = \infty$ having the multiplicity $w_0$. In this case $\Phi(f) = \Phi(g)$ if and only if $f^{(k)}(\mu_i) = g^{(k)}(\mu_i)$, $i = 1, \dots, q$, $k = 0, 1, \dots, w_i - 1$, and $f^{(k)}(\infty) = g^{(k)}(\infty)$, $k = 1, \dots, w_0$*[9]*.*

*Proof.* The proof is reduced to the direct calculations with the use of formula (3.2) and the rules for calculating the residues. □

Let the *interpolation points* $\mu_0 = \infty$ and $\mu_i \in \mathbb{C}$, $i = 1, 2, \dots, q$, and their *multiplicities* $w_i = 0, 1, 2, \dots, q$[10] be given. We denote by $r \in \mathbf{O}\bigl(\bar\sigma(F, G), \mathscr{D}'_+(\alpha)\bigr)$ a function that coincides with EXP at the points $\mu_i$ together with its derivatives up to order $w_i$ inclusive, $i = 0, 1, \dots, q$. The *interpolation method* of approximate solution of problem (1.2) is a calculation of the impulse response by the formula

$$h \approx \bigl\langle \Phi(r)\,b, d\bigr\rangle,$$

cf. Corollary 5.1.

The simplest example of the function $r$ is a function that coincides with $t \mapsto e^{\lambda t}\eta(t)$ about several points of the spectrum and equals zero about the other points; such a choice of $r$ leads to a version of the Fourier method. If one takes a polynomial as $r$, then [19, 20] the interpolation method becomes an explicit Runge-Kutta method, and if one takes a rational function as $r$, then [19, 20] the interpolation method becomes an implicit Runge-Kutta method. The discussion of using polynomials and rational functions of the best approximation as $r$ can be found in [27]; see also [12], where the coefficients of the best approximation are calculated.

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[8]See the definition of multiplicity between Theorem 2.2 and its proof.

[9]The equality $f(\infty) = g(\infty) = 0$ takes place according to the definition of $\mathbf{O}\bigl(\bar\sigma(F, G), \mathscr{D}'_+(\alpha)\bigr)$.

[10]It is convenient to allow the possibility that $w_i = 0$. For example, the condition $w_0 = 0$ means that infinity is not really a point of interpolation.

## 6 Krylov subspace projection methods

In this Section we give a description of considered projection methods of approximate solution of problem (1.2) and prove some related identities.

Let us return to problems (1.2) and (1.3). Let $F, G \in \mathbf{B}(X, Y)$ and $\widehat{F}, \widehat{G} \in \mathbf{B}(\widehat{X}, \widehat{Y})$. If the dimensions of the spaces $\widehat{X}$ and $\widehat{Y}$ are smaller than the dimensions of the spaces $X$ and $Y$, and in the same time the solution $\hat{y}$ of problem (1.3) is in some sense close to the solution $y$ of problem (1.2), then problem (1.3) is called a *reduced-order* problem relative to problem (1.2). In standard notation for problem (1.3), we shall use the symbol $\widehat{\ }$; for example, $\widehat{R}_\lambda = (\lambda\widehat{F} - \widehat{G})^{-1}$, $\widehat{H}$, $\widehat{\Phi}$, $\widehat{\mathbf{1}}$, $\widehat{\odot}$.

We discuss the methods (see, for example [1, 2, 7, 8, 14, 16, 18, 33, 39, 45]) for construction of a reduced-order problem (1.3), which are called *projection* ones. Let $\widehat{X}$ and $\widehat{Y}$ be Banach spaces of a finite dimension $n$, and $V\colon \widehat{X} \to X$ and $\Lambda\colon Y \to \widehat{Y}$ be some bounded linear operators. We assume that the parameters in formula (1.3) are defined according to the rule[11]

$$\widehat{F} = \Lambda F V, \qquad \widehat{G} = \Lambda G V, \qquad \hat{b} = \Lambda b, \qquad \hat{d} = V^* d. \tag{6.1}$$

The traditional motivation for the choice of the operators $V$ and $\Lambda$ is based on Corollary 6.1, see below. It is discussed at the end of the Section.

The following Proposition 6.1, in view of Proposition 6.2, shows that the solution of problem (1.3) is determined not by the operators $V$ and $\Lambda$ themselves, but only by the images of the operators $V$ and $\Lambda^*$.

**Proposition 6.1.** *Let $S\colon \widehat{X} \to \widehat{X}$ and $Q\colon \widehat{Y} \to \widehat{Y}$ be arbitrary invertible operators. Let us set $V_1 = VS$, $\Lambda_1 = Q\Lambda$,*

$$\widehat{F}_1 = \Lambda_1 F V_1, \qquad \widehat{G}_1 = \Lambda_1 G V_1, \qquad \hat{b}_1 = \Lambda_1 b, \qquad \hat{d}_1 = V_1^* d.$$

*Then the solution of the problem*

$$\begin{aligned} \widehat{F}_1 \hat{x}_1' &= \widehat{G}_1 \hat{x}_1 + \hat{b}_1 u(t), \\ \hat{y}_1(t) &= \langle \hat{x}_1, \hat{d}_1 \rangle \end{aligned}$$

*coincides with the solution of problem* (1.3).

*Proof.* The proof is reduced to the change $\hat{x} = S\hat{x}_1$ and the multiplication of the equation in problem (1.3) by $Q$. □

**Lemma 6.1.** *Let operators $U\colon \widehat{X} \to X$ and $V\colon X \to \widehat{X}$ satisfy the normalization condition*

$$UV = \widehat{\mathbf{1}}, \tag{6.2}$$

*where $\widehat{\mathbf{1}}\colon \mathbb{C}^n \to \mathbb{C}^n$ is the identity mapping. Then $VU$ is a projector on the image of the operator $V$ parallel to the kernel of the operator $U$.*

[11] We stress that, even if $X = Y$ and $F = \mathbf{1}$, then $\widehat{F} = \Lambda V$, which is not obligatorily an identity operator.

*Proof.* The fact that the operator $VU$ is a projector follows from the identity

$$(VU)^2 = V(UV)U = V\widehat{\mathbf{1}}U = VU.$$

Obviously, $\operatorname{Im} VU \subseteq \operatorname{Im} V$. The fact that the whole image of the operator $V$ is contained in the image of this projector follows from the identity

$$UV = \widehat{\mathbf{1}} = \widehat{\mathbf{1}}^2 = U(VU)V$$

and dimensional considerations.

The embedding $\operatorname{Ker} VU \supseteq \operatorname{Ker} U$ is obvious. The fact that the kernel of $VU$ is not wider than the kernel of $U$ follows from the proved identity $U(VU)V = \widehat{\mathbf{1}}$ and dimensional considerations. □

**Proposition 6.2.** *Every bounded linear operator $W\colon \widehat{Y}^* \to Y^*$ has a unique pre-conjugate operator, i. e., an operator $\Lambda\colon Y \to \widehat{Y}$ such that $W = \Lambda^*$.*

*Proof.* We consider $W^*\colon Y^{**} \to \widehat{Y}^{**}$. The space $\widehat{Y}$ is finite dimensional. Hence it is reflexive. Therefore one may think that $\widehat{Y}^{**} = \widehat{Y}$ and, thus, $W^*\colon Y^{**} \to \widehat{Y}$. We denote by $\Lambda\colon Y \to \widehat{Y}$ the restriction of $W^*\colon Y^{**} \to \widehat{Y}$ to $Y \subseteq Y^{**}$. By the definition of a conjugate operator, for any $\hat{y}^* \in \widehat{Y}^*$ and $y^{**} \in Y^{**}$ we have

$$\langle y^{**}, W\hat{y}^*\rangle = \langle W^*y^{**}, \hat{y}^*\rangle.$$

Especially, this equality is valid for all $\hat{y}^* \in \widehat{Y}^*$ and $y \in Y \subseteq Y^{**}$, i. e.,

$$\langle y, W\hat{y}^*\rangle = \langle \Lambda y, \hat{y}^*\rangle.$$

The last identity means that $W = \Lambda^*$.

The uniqueness of the pre-conjugate operator follows from the equality of the norms of an operator and its conjugate. □

Proposition 6.2 shows that one may specify $\Lambda$ by imposing restrictions on $\Lambda^*$, as it is suggested in Proposition 6.3.

**Proposition 6.3.**

(a) [18, Lemma 3.2] *Let $\lambda_j \in \mathbb{C}$ be not both in the spectrum of the pencil $\lambda \mapsto \lambda F - G$ and in the spectrum of the pencil $\lambda \mapsto \lambda\widehat{F} - \widehat{G}$. Let the image of the operator $V$ contain the vectors $R^{k\odot}_{\lambda_j} b$, $k = 1, \dots, \varkappa$. Then,*

$$R^{k\odot}_{\lambda_j} b = V\widehat{R}^{k\widehat{\odot}}_{\lambda_j}\hat{b}, \qquad k = 1, \dots, \varkappa. \tag{6.3}$$

(b) *Let the operators $F$ and $\widehat{F}$ be invertible. Let the image of the operator $V$ contain the vectors*[12] $(F^{-1}GF^{-1})^{k\odot} b$, *$k = 0, 1, \dots, \varkappa - 1$*[13]. *Then,*

$$(F^{-1}GF^{-1})^{k\odot} b = V(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{k\widehat{\odot}}\hat{b}, \qquad k = 0, 1, \dots, \varkappa - 1. \tag{6.4}$$

[12] By $(F^{-1}GF^{-1})^{0\odot}$ we mean $\mathbf{1}_\odot = F^{-1}$.

[13] In order to make the number of the conditions be equal to $\varkappa$, we assume here and below that the numbering finishes at $\varkappa - 1$.

(c) [18, Lemma 3.3] *Let $\lambda_j \in \mathbb{C}$ be not both in the spectrum of the pencil $\lambda \mapsto \lambda F - G$ and in the spectrum of the pencil $\lambda \mapsto \lambda\widehat{F} - \widehat{G}$. Let the image of the operator $\Lambda^*$ contain the vectors $(R^{m\odot}_{\lambda_j})^* d$, $m = 1, \dots, \chi$. Then,*

$$(R^{m\odot}_{\lambda_j})^* d = \Lambda^* (\widehat{R}^{m\widehat{\odot}}_{\lambda_j})^* \hat{d}, \qquad m = 1, \dots, \chi.$$

(d) *Let the operators $F$ and $\widehat{F}$ be invertible. Let the image of the operator $\Lambda^*$ contain the vectors $\big((F^{-1}GF^{-1})^{m\odot}\big)^* d$, $m = 0, 1, \dots, \chi - 1$. Then,*

$$\big[(F^{-1}GF^{-1})^{m\odot}\big]^* d = \Lambda^* \big[(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{m\widehat{\odot}}\big]^* \hat{d}, \qquad m = 0, 1, \dots, \chi - 1.$$

*Proof.* (a) We consider the auxiliary operator

$$U = \widehat{R}_{\lambda_j} \Lambda(\lambda_j F - G) = \big[\Lambda(\lambda_j F - G)V\big]^{-1} \Lambda(\lambda_j F - G).$$

Obviously, $UV = \widehat{\mathbf{1}}$. Therefore by Lemma 6.1 the operator $VU$ is a projector on the image of the operator $V$.

We prove the equality (6.3) for $k = 1, 2, \dots, \varkappa$, by induction on $k$. For $k = 1$ we have

$$V\widehat{R}_{\lambda_j}\hat{b} = V\widehat{R}_{\lambda_j}\Lambda b = V\widehat{R}_{\lambda_j}\Lambda(\lambda_j F - G)R_{\lambda_j} b = VUR_{\lambda_j} b = R_{\lambda_j} b.$$

Further we suppose that equality (6.3) is true for some $k$. Let us prove the analogous equality for $k+1$. We have

$$\begin{aligned} V\widehat{R}^{(k+1)\widehat{\odot}}_{\lambda_j}\hat{b} &= V\widehat{R}_{\lambda_j} \,\widehat{\odot}\, \widehat{R}^{k\widehat{\odot}}_{\lambda_j}\hat{b} = V\widehat{R}_{\lambda_j}\widehat{F}\widehat{R}^{k\widehat{\odot}}_{\lambda_j}\hat{b} = V\widehat{R}_{\lambda_j}\Lambda F V\widehat{R}^{k\widehat{\odot}}_{\lambda_j}\hat{b} \\ &= V\widehat{R}_{\lambda_j}\Lambda(\lambda_j F - G)R_{\lambda_j} F R^{k\odot}_{\lambda_j} b = V[\widehat{R}_{\lambda_j}\Lambda(\lambda_j F - G)]R_{\lambda_j} F R^{k\odot}_{\lambda_j} b \\ &= VUR_{\lambda_j} F R^{k\odot}_{\lambda_j} b = VUR_{\lambda_j} \odot R^{k\odot}_{\lambda_j} b = VUR^{(k+1)\odot}_{\lambda_j} b = R^{(k+1)\odot}_{\lambda_j} b. \end{aligned}$$

(b) The proof is analogous to that of (a). We consider the auxiliary operator

$$U = \widehat{F}^{-1}\Lambda F = \big(\Lambda F V\big)^{-1}\Lambda F.$$

It is straightforward to verify that $UV = \widehat{\mathbf{1}}$. Therefore by Lemma 6.1 the operator $VU$ is a projector on the image of the operator $V$.

We prove the equality (6.4) for $k = 0, 1, 2, \dots, \varkappa - 1$, by induction on $k$. For $k = 0$ we have

$$V\widehat{F}^{-1}\hat{b} = V\widehat{F}^{-1}\Lambda b = V\widehat{F}^{-1}\Lambda F F^{-1} b = V[\widehat{F}^{-1}\Lambda F]F^{-1} b = VUF^{-1} b = F^{-1} b.$$

Next we suppose that equality (6.4) is true for some $k$. We prove the analogous equality for $k+1$. We have

$$\begin{aligned}
V(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{(k+1)\widehat{\odot}}\hat{b} &= V(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})\widehat{\odot}(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{k\widehat{\odot}}\hat{b}\\
&= V(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})\widehat{F}(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{k\widehat{\odot}}\hat{b}\\
&= V(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})\Lambda FV(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{k\widehat{\odot}}\hat{b}\\
&= V\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1}\Lambda F(F^{-1}GF^{-1})^{k\odot}b\\
&= V\widehat{F}^{-1}[\Lambda GV]\widehat{F}^{-1}\Lambda F(F^{-1}GF^{-1})^{k\odot}b\\
&= V[\widehat{F}^{-1}\Lambda F]F^{-1}GV(\widehat{F}^{-1}\Lambda F)(F^{-1}GF^{-1})^{k\odot}b\\
&= VUF^{-1}GVU(F^{-1}GF^{-1})^{k\odot}b\\
&= VUF^{-1}G(F^{-1}GF^{-1})^{k\odot}b\\
&= VUF^{-1}GF^{-1}F(F^{-1}GF^{-1})^{k\odot}b\\
&= VU(F^{-1}GF^{-1})\odot(F^{-1}GF^{-1})^{k\odot}b\\
&= VU(F^{-1}GF^{-1})^{(k+1)\odot}b = (F^{-1}GF^{-1})^{(k+1)\odot}b.
\end{aligned}$$

(c) and (d) are deduced form (a) and (b) via a change of notation. □

**Proposition 6.4.**

(a) [18, Theorem 3.1] *Let $\lambda_j\in\mathbb{C}$ be not both in the spectrum of the pencil $\lambda\mapsto\lambda F-G$ and in the spectrum of the pencil $\lambda\mapsto\lambda\widehat{F}-\widehat{G}$. Let the image of the operator $V$ contain the vectors $R^{k\odot}_{\lambda_j}b$, $k=1,\dots,\varkappa$, and the image of the operator $\Lambda^*$ contain the vectors $\bigl(R^{m\odot}_{\lambda_j}\bigr)^*d$, $m=1,\dots,\chi$. Then,*

$$\langle R^{l\odot}_{\lambda_j}b,d\rangle=\langle\widehat{R}^{l\widehat{\odot}}_{\lambda_j}\hat{b},\hat{d}\rangle,\qquad l=1,\dots,\varkappa+\chi.$$

(b) *Let both the operator $F$ and the operator $\widehat{F}$ be invertible. Let the image of the operator $V$ contain the vectors $(F^{-1}GF^{-1})^{k\odot}b$, $k=0,1,\dots,\varkappa-1$, and the image of the operator $\Lambda^*$ contain the vectors $\bigl((F^{-1}GF^{-1})^{m\odot}\bigr)^*d$, $m=0,1,\dots,\chi-1$. Then,*

$$\bigl\langle(F^{-1}GF^{-1})^{l\odot}b,d\bigr\rangle=\bigl\langle(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{l\widehat{\odot}}\hat{b},\hat{d}\bigr\rangle,\qquad l=0,1,\dots,\varkappa+\chi-1.$$

*Proof.* (a) By virtue of Proposition 6.3(a), for $l=1,\dots,\varkappa$ we have

$$\langle R^{l\odot}_{\lambda_j}b,d\rangle=\langle V\widehat{R}^{l\widehat{\odot}}_{\lambda_j}\hat{b},d\rangle=\langle\widehat{R}^{l\widehat{\odot}}_{\lambda_j}\hat{b},V^*d\rangle=\langle\widehat{R}^{l\widehat{\odot}}_{\lambda_j}\hat{b},\hat{d}\rangle.$$

In a similar manner by Proposition 6.3(c), for $l=1,\dots,\chi$ we have

$$\langle R^{l\odot}_{\lambda_j}b,d\rangle=\langle b,\bigl(R^{l\odot}_{\lambda_j}\bigr)^*d\rangle=\langle b,\Lambda^*\bigl(\widehat{R}^{l\widehat{\odot}}_{\lambda_j}\bigr)^*\hat{d}\rangle=\langle\widehat{R}^{l\widehat{\odot}}_{\lambda_j}\Lambda b,\hat{d}\rangle=\langle\widehat{R}^{l\widehat{\odot}}_{\lambda_j}\hat{b},\hat{d}\rangle.$$

Let $l$ be represented in the form $l=k+m$, where $k=1,\dots,\varkappa$ and $m=1,\dots,\chi$. Then by Proposition 6.3(a,c) we have

$$\begin{aligned}
\langle R^{(m+k)\odot}_{\lambda_j}b,d\rangle&=\langle R^{m\odot}_{\lambda_j}\odot R^{k\odot}_{\lambda_j}b,d\rangle=\langle R^{m\odot}_{\lambda_j}FR^{k\odot}_{\lambda_j}b,d\rangle=\langle FR^{k\odot}_{\lambda_j}b,\bigl(R^{m\odot}_{\lambda_j}\bigr)^*d\rangle\\
&=\langle FV\widehat{R}^{k\widehat{\odot}}_{\lambda_j}\hat{b},\Lambda^*(\widehat{R}^{m\widehat{\odot}}_{\lambda_j})^*\hat{d}\rangle=\langle\widehat{R}^{m\widehat{\odot}}_{\lambda_j}\Lambda FV\widehat{R}^{k\widehat{\odot}}_{\lambda_j}\hat{b},\hat{d}\rangle\\
&=\langle\widehat{R}^{m\widehat{\odot}}_{\lambda_j}\widehat{F}\widehat{R}^{k\widehat{\odot}}_{\lambda_j}\hat{b},\hat{d}\rangle=\langle\widehat{R}^{m\widehat{\odot}}_{\lambda_j}\widehat{\odot}\widehat{R}^{k\widehat{\odot}}_{\lambda_j}\hat{b},\hat{d}\rangle=\langle\widehat{R}^{(m+k)\widehat{\odot}}_{\lambda_j}\hat{b},\hat{d}\rangle.
\end{aligned}$$

(b) By Proposition 6.3(b) for $l=0,1,\dots,\varkappa-1$ we have

$$\begin{aligned}\big\langle (F^{-1}GF^{-1})^{l\odot}b,d\big\rangle &= \big\langle V(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{l\widehat{\odot}}\hat{b},d\big\rangle = \big\langle (\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{l\widehat{\odot}}\hat{b},V^*d\big\rangle\\ &= \big\langle (\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{l\widehat{\odot}}\hat{b},\hat{d}\big\rangle.\end{aligned}$$

In a similar fashion by Proposition 6.3(d) for $l=0,1,\dots,\chi-1$ we have

$$\begin{aligned}\big\langle (F^{-1}GF^{-1})^{l\odot}b,d\big\rangle &= \big\langle b,\big[(F^{-1}GF^{-1})^{l\odot}\big]^*d\big\rangle = \big\langle b,\Lambda^*\big[(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{l\widehat{\odot}}\big]^*\hat{d}\big\rangle\\ &= \big\langle \Lambda b,\big[(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{l\widehat{\odot}}\big]^*\hat{d}\big\rangle = \big\langle (\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{l\widehat{\odot}}\hat{b},\hat{d}\big\rangle.\end{aligned}$$

Let $l$ be represented in the form $l=k+m$, where $k=0,1,\dots,\varkappa-1$ and $m=0,1,\dots,\chi-1$. Then by Proposition 6.3(b,d) we have

$$\begin{aligned}\big\langle (F^{-1}GF^{-1})^{(m+k+1)\odot}b,d\big\rangle &= \big\langle (F^{-1}GF^{-1})^{m\odot}\odot(F^{-1}GF^{-1})\odot(F^{-1}GF^{-1})^{k\odot}b,d\big\rangle\\ &= \big\langle (F^{-1}GF^{-1})^{m\odot}F(F^{-1}GF^{-1})F(F^{-1}GF^{-1})^{k\odot}b,d\big\rangle\\ &= \big\langle (F^{-1}GF^{-1})^{m\odot}G(F^{-1}GF^{-1})^{k\odot}b,d\big\rangle\\ &= \big\langle G(F^{-1}GF^{-1})^{k\odot}b,\big[(F^{-1}GF^{-1})^{m\odot}\big]^*d\big\rangle\\ &= \big\langle GV(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{k\widehat{\odot}}\hat{b},\Lambda^*\big[(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{m\widehat{\odot}}\big]^*\hat{d}\big\rangle\\ &= \big\langle (\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{m\widehat{\odot}}\Lambda GV(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{k\widehat{\odot}}\hat{b},\hat{d}\big\rangle\\ &= \big\langle (\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{m\widehat{\odot}}\widehat{G}(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{k\widehat{\odot}}\hat{b},\hat{d}\big\rangle\\ &= \big\langle (\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{m\widehat{\odot}}\widehat{F}\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1}\widehat{F}(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{k\widehat{\odot}}\hat{b},\hat{d}\big\rangle\\ &= \big\langle (\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{m\widehat{\odot}}\widehat{\odot}(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})\widehat{\odot}(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{k\widehat{\odot}}\hat{b},\hat{d}\big\rangle\\ &= \big\langle (\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{(m+k+1)\widehat{\odot}}\hat{b},\hat{d}\big\rangle. \quad\square\end{aligned}$$

We call the function

$$\tilde{h}(\lambda)=\langle R_\lambda b,d\rangle=\big\langle(\lambda F-G)^{-1}b,d\big\rangle \tag{6.5}$$

the (*scalar*) *frequency response* of problem (1.2). By Theorem 4.1 the frequency response is the Laplace transform of the impulse response. We denote the (scalar) frequency response of reduced-order problem (1.3) by the symbol $\tilde{\hat{h}}$.

**Corollary 6.1.**

(a) *Let $\lambda_j\in\mathbb{C}$ be not both in the spectrum of the pencil $\lambda\mapsto\lambda F-G$ and in the spectrum of the pencil $\lambda\mapsto\lambda\widehat{F}-\widehat{G}$. Let the image of the operator $V$ contain the vectors $R_{\lambda_j}^{k\odot}b$, $k=1,\dots,\varkappa$, and the image of the operator $\Lambda^*$ contain the vectors $\big(R_{\lambda_j}^{m\odot}\big)^*d$, $m=1,\dots,\chi$. Then the first terms of the Taylor expansion about the point $\lambda_j$ of the frequency response $\tilde{\hat{h}}$ of reduced-order problem* (1.3) *coincides up to the term $(\lambda-\lambda_j)^{\varkappa+\chi-1}$ inclusive with the corresponding terms of the expansion of the frequency response $\tilde{h}$ of initial problem* (1.2).

(b) *Let both the operator $F$ and the operator $\widehat{F}$ be invertible. Let the image of the operator $V$ contain the vectors $(F^{-1}GF^{-1})^{k\odot}b$, $k=0,1,\dots,\varkappa-1$, and the image of the operator $\Lambda^*$ contain the vectors $\big[(F^{-1}GF^{-1})^{m\odot}\big]^*d$, $m=0,1,\dots,\chi-1$. Then the first terms of the Laurent expansion about infinity of the frequency response $\tilde{\hat{h}}$ of reduced-order problem* (1.3) *coincides up to the term $\lambda^{-(\varkappa+\chi)+1}$ inclusive with the corresponding terms of the expansion of the frequency response $\tilde{h}$ of initial problem* (1.2).

*Proof.* (a) From Corollary 3.1 and formula (6.5) it is seen that the series expansions about the point $\lambda_j$ of the frequency response $\tilde{h}$ of initial problem (1.2) and the frequency response $\tilde{\hat{h}}$ of reduced-order problem (1.3) have the following forms

$$\tilde{h}(\lambda)=\sum_{l=0}^{\infty}(\lambda_j-\lambda)^l\langle R_{\lambda_j}^{(l+1)\odot}b,d\rangle, \tag{6.6}$$
$$\tilde{\hat{h}}(\lambda)=\sum_{l=0}^{\infty}(\lambda_j-\lambda)^l\langle \widehat{R}_{\lambda_j}^{(l+1)\widehat{\odot}}\hat{b},\hat{d}\rangle.$$

The end of the proof follows from Proposition 6.4.

(b) From Propositions 3.3 and formula (6.5) it follows that the Laurent expansion about infinity of the frequency response $\tilde{h}(\lambda)$ of initial problem (1.2) and the frequency response $\tilde{\hat{h}}(\lambda)$ of reduced-order problem (1.3) have the following forms

$$\tilde{h}(\lambda)=\sum_{l=1}^{\infty}\frac{1}{\lambda^l}\big\langle(F^{-1}GF^{-1})^{(l-1)\odot}b,d\big\rangle, \tag{6.7}$$
$$\tilde{\hat{h}}(\lambda)=\sum_{l=1}^{\infty}\frac{1}{\lambda^l}\big\langle(\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{(l-1)\widehat{\odot}}\hat{b},\hat{d}\big\rangle.$$

The end of the proof follows from Proposition 6.4. □

We return to the discussion of construction methods for the operators $\Lambda$ and $V$.

The calculation of the coefficients in expansions (6.6) and (6.7) can often be easily fulfilled. That is why one can desire, starting with the knowledge of these coefficients, to construct an interpolation approximation to the frequency response $\tilde{h}$ by a rational function.

Corollary 6.1 shows that an effect of a frequency response interpolation arises if one uses projection methods, where the operators $\Lambda^*$ and $V$ are constructed according to Proposition 6.3. A practical realization (see, for example, [1, 2, 8, 13, 14, 15, 18, 24, 33, 44]) of this idea is usually based on the employment of the Krylov subspaces and the usage of different modifications of the Lanczos and Arnoldi methods [23, 34, 48].

## 7 Reduction of projection methods to interpolation ones

In this Section we show (Theorem 7.1) that the discussed projection methods of the impulse response approximation are equivalent to the approximation of the function

$\Phi(\mathrm{EXP})$ (see Theorem 5.1) by the function $\Phi(r)$, where $r$ is a rational function that approximates EXP in interpolational sense on the extended complex plane. Moreover, from this point of view it turns out that reduced-order problem (1.3) only implicitly specifies interpolation points and their multiplicities.

In this Section we assume that infinity is a pole of the pencil resolvent $\lambda \mapsto (\lambda F - G)^{-1}$.

**Theorem 7.1.** *Let the points $\lambda_1, \dots, \lambda_p \in \mathbb{C}$ be not both in the spectrum of the pencil $\lambda \mapsto \lambda F - G$ and in the spectrum of the pencil $\lambda \mapsto \lambda\widehat{F} - \widehat{G}$. Let the image of the operator $V$ contain the vectors $R^{k\odot}_{\lambda_j} b$, $k = 1, \dots, \varkappa_j$, $j = 1, \dots, p$, and the image of the operator $\Lambda^*$ contain the vectors $(R^{m\odot}_{\lambda_j})^* d$, $m = 1, \dots, \chi_j$, $j = 1, \dots, p$.*

*If the operators $F$ and $\widehat{F}$ are invertible, then we additionally assume that the image of the operator $V$ contains the vectors[14] $(F^{-1}GF^{-1})^{k\odot} b$, $k = 0, 1, \dots, \varkappa_0 - 1$, and the image of the operator $\Lambda^*$ contains the vectors $\big[(F^{-1}GF^{-1})^{m\odot}\big]^* d$, $m = 0, 1, \dots, \chi_0 - 1$. If at least one of the operators $F$ or $\widehat{F}$ is not invertible, then we set $\varkappa_0 = \chi_0 = 0$.*

*Then,*

$$\hat{h} = \big\langle \Phi(r) b, d \big\rangle,$$

*where $r \in \mathbf{O}\big(\bar{\sigma}(F, G), \mathscr{D}'_+(\alpha)\big)$ is a rational function of the form*

$$r(\lambda) = \sum_{j=1}^{p} \sum_{l=1}^{\varkappa_j + \chi_j} \frac{c_{lj}}{(\lambda_j - \lambda)^l} + \sum_{l=0}^{\varkappa_0 + \chi_0 - 1} c_{l0} \lambda^l \tag{7.1}$$

*with the coeffitients[15] $c_{lj} \in \mathscr{D}'_+\big(\mathbb{C}, \alpha\big)$. The function $r$ coincides with the function* EXP *at the points $\hat{\mu}_1, \dots, \hat{\mu}_{\hat{q}} \in \mathbb{C}$ of the reduced-order pencil spectrum $\sigma(\widehat{F}, \widehat{G})$ with the derivatives up to the orders $\hat{w}_1 - 1, \dots, \hat{w}_{\hat{q}} - 1$ inclusive, and coincides with the function* EXP *at the point $\hat{\mu}_0 = \infty$ with the derivatives up to the order $\hat{w}_0$ inclusive[16]. Here $\hat{w}_0, \hat{w}_1, \dots, \hat{w}_{\hat{q}}$ are the multiplicities of the corresponding points of the augmented spectrum of the pencil $\lambda \mapsto \lambda\widehat{F} - \widehat{G}$, see Theorem* 3.2.

For the existence of a function $r$ satisfying the interpolation conditions from Theorem 7.1, see [3] and [49, § 8.3, Theorem 1].

*Proof.* By definition (5.3), Corollary 3.1, Proposition 3.3 as well as by the residues calculation rules we have

$$\Phi(r) = \sum_{j=1}^{p} \sum_{l=1}^{\varkappa_j + \chi_j} c_{lj} R^{l\odot}_{\lambda_j} + \sum_{l=0}^{\varkappa_0 + \chi_0 - 1} c_{l0} (F^{-1} G F^{-1})^{l\odot},$$

$$\widehat{\Phi}(r) = \sum_{j=1}^{p} \sum_{l=1}^{\varkappa_j + \chi_j} c_{lj} \widehat{R}^{l\widehat{\odot}}_{\lambda_j} + \sum_{l=0}^{\varkappa_0 + \chi_0 - 1} c_{l0} (\widehat{F}^{-1} \widehat{G} \widehat{F}^{-1})^{l\widehat{\odot}},$$

[14] As usual $(F^{-1}GF^{-1})^{0\odot}$ means $\mathbf{1}_\odot = F^{-1}$.

[15] It can be easily shown that $c_{lj}$ are the linear combinations of the functions $t \mapsto t^{q-1} e^{\hat{\mu}_i t} \eta(t)$ and $\delta^{(q)}$.

[16] It follows that $\mathrm{EXP}(\lambda)(t) = e^{\lambda t}\eta(t)$ in a neighbourhood of $\sigma(F, G)$ containing all points $\hat{\mu}_1, \dots, \hat{\mu}_{\hat{q}} \in \mathbb{C}$, and $\mathrm{EXP}(\lambda)(t) = \sum_{i=1}^{w_0} \dfrac{\delta^{(i-1)}(t)}{\lambda^i}$ in a neighbourhood of $\hat{\mu}_0 = \infty$.

which implies

$$\begin{aligned}\langle\Phi(r)b,d\rangle &= \sum_{j=1}^{p}\sum_{l=1}^{\varkappa_j+\chi_j} c_{lj}\big\langle R_{\lambda_j}^{l\odot}b,d\big\rangle + \sum_{l=0}^{\varkappa_0+\chi_0-1} c_{l0}\big\langle (F^{-1}GF^{-1})^{l\odot}b,d\big\rangle,\\ \langle\widehat{\Phi}(r)\hat{b},\hat{d}\rangle &= \sum_{j=1}^{p}\sum_{l=1}^{\varkappa_j+\chi_j} c_{lj}\big\langle \widehat{R}_{\lambda_j}^{l\widehat{\odot}}\hat{b},\hat{d}\big\rangle + \sum_{l=0}^{\varkappa_0+\chi_0-1} c_{l0}\big\langle (\widehat{F}^{-1}\widehat{G}\widehat{F}^{-1})^{l\widehat{\odot}}\hat{b},\hat{d}\big\rangle.\end{aligned} \tag{7.2}$$

From Proposition 6.4 it follows that the coefficients in formulas (7.2) coincide with each other. Thus we obtain the equality $\langle\widehat{\Phi}(r)\hat{b},\hat{d}\rangle = \langle\Phi(r)b,d\rangle$. From Proposition 5.1 it follows the equality $\widehat{\Phi}(\mathrm{EXP}) = \widehat{\Phi}(r)$, and from Corollary 5.1 it follows the equality $\hat{h} = \big\langle\widehat{\Phi}(\mathrm{EXP})\hat{b},\hat{d}\big\rangle$. Combining all together we obtain $\hat{h} = \big\langle\Phi(r)b,d\big\rangle$. □

**Remark 1.** In literature on simulation of linear circuits, they usually discuss the interpolation in the frequency domain based on the coincidence of the *scalar* frequency responses of problems (1.2) and (1.3) at given points $\lambda_0, \lambda_1, \dots, \lambda_p$ together with the corresponding derivatives, see Corollary 6.1. We emphasize that in contrast to such interpolation, in Theorem 7.1 we deal with the interpolation of the (other) function $\Phi$ at the (other) points $\hat{\mu}_0, \hat{\mu}_1, \dots, \hat{\mu}_q$.

**Remark 2.** Theorem 7.1 remains valid for the problem

$$\begin{aligned}Fx' &= Gx + b_1u_1 + \dots + b_\alpha u_\alpha,\\ y &= \big(\langle x,d_1\rangle,\dots,\langle x,d_\beta\rangle\big)\end{aligned}$$

with $\alpha$ inputs and $\beta$ outputs provided that the interpolation points $\lambda_j$ and their orders $\varkappa_j$ and $\chi_j$ are the same for all $b_1,\dots,b_\alpha$ and $d_1,\dots,d_\beta$. Let us formulate this statement more accuratly. We denote by $b = (b_1,\dots,b_\alpha)$ the row consisting of the vectors $b_1,\dots,b_\alpha \in Y$, and we denote by $d = (d_1,\dots,d_\beta)$ the row consisting of the vectors $d_1,\dots,d_\beta \in X^*$. We shall mean by $\langle b,d\rangle$ the $\beta\times\alpha$-matrix that consists of the entries $\langle b_j,d_i\rangle$. We define a "scalar" impulse response by the former formula $h(t) = \langle H(t)b,d\rangle$ (Proposition 4.1). The function $r$ is, as well as $\Phi(r)$, a matrix-valued function taking its values in $\mathbb{C}^{\beta\times\alpha}$. By the assumptions of Theorem 7.1 of the kind "the image of the operator $V$ contains the vectors $R_{\lambda_j}^{k\odot}b$, $k = 1,\dots,\varkappa_j$, $j = 1,\dots,p$," one should mean "the image of the operator $V$ contains the vectors $R_{\lambda_j}^{k\odot}b_i$, $k = 1,\dots,\varkappa_j$, $j = 1,\dots,p$, for all $i = 1,\dots,\alpha$". The proof still goes for this case without essential changes.

## References


[1] A.C. Antoulas, *Approximation of Large-Scale Dynamical Systems.* Philadelphia: SIAM, 2005.

[2] Z. Bai, *Krylov subspace technique for reduced-order modelling of large-scale dynamical systems.* Applied Numerical Mathematics. 43, no. 9 (2002), 9–44.

[3] G.A. Baker, P. Graves-Morris, *Pade Approximations.* Part I, II. London: Addison-Wesley Publishing Company, 1981.

[4] N. Balabanian, *Network Synthesis.* N.-J.: Prentice-Hall, 1958.

[5] A.G. Baskakov, K.I. Chernyshov, *Spectral analysis of linear relations and degenerate operator semigroups.* Mat. Sb. (N.S.) 193, no. 11 (2002), 3–42. (in Russian; English transl. Sib. Math., 193, no. 11 (2002), 1573–1610.)

[6] S.I. Baskakov, *Lectures on Circuit Theory.* Moscow: KomKniga, 2005. (in Russian)

[7] P. Benner, V. Mehrmann, D.C. Sorensen, *Dimension Reduction of Large-Scale Systems.* Berlin–Heidelberg: Springer-Verlag, 2005.

[8] D.L. Boley, *Krylov space methods on state-space control models.* Circuit, Systems and Signal Process. 13 (1994), 733–758.

[9] N. Bourbaki, *Éléments de Mathématique. Espaces Vectoriels Topologiques.* Paris: Masson, 1981.

[10] N. Bourbaki, *Éléments de Mathématique. Théories Spectrales.* Paris: Hermann, 1967.

[11] Yu.A. Brychkov, A.P. Prudnikov, *Integral Transformations of Generalized Functions.* New York: Gordon & Breach Sci. Publ., 1989.

[12] W.J. Cody, G. Meinardus, and R.S. Varga, *Chebyshev rational approximations to* $e^{-x}$ *in* $[0,+1)$ *and applications to heat-conduction problems*, J. Approx. Theory. 2, no. 1 (1969), 50–65.

[13] P. Feldman, R.W. Freund, *Efficient linear circuit analysis by Pade approximation via the Lanczos process.* IEEE Trans. Computer-Aided Design, 14 (1995), May, 639–649.

[14] R.W. Freund, *Model reduction methods based on Krylov subspaces.* Acta Numerica. 12 (2003), 267–319.

[15] K. Gallivan, E.J. Grimme, P. Van Dooren, *A rational Lanczos algorithm for model reduction.* Numerical Algorithms, 2, no. 1-2 (1996), 33–63.

[16] K. Gallivan, A. Vandendorpe, P. Van Dooren, *Model reduction of MIMO systems via tangential interpolation.* SIAM Journal on Matrix Analysis and Applications. 26, no. 2 (2004), 328–349.

[17] F.R. Gantmacher, *The Theory of Matrices.* Moscow: Nauka, 1988. (in Russian; English transl. of the previous edition, N.-Y.: Chelsea, 1959.)

[18] E.J. Grimme, *Krylov Projection Methods for Model Order Reduction.* Ph. D. thesis. University of Illinois at Urbana–Champaign, 1997.

[19] E. Hairer, S.P. Nørsett, G. Wanner, *Solving Ordinary Differential Equations I. Nonstiff Problems.* Berlin: Springer-Verlag, 1987.

[20] E. Hairer, G. Wanner, *Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems.* Berlin: Springer-Verlag, 1996.

[21] E. Hille, R.S. Phillips, *Functional Analysis and Semi-Groups.* Providence, Rhode Island: AMS, 1957.

[22] Kh.D. Ikramov, *Matrix Pencils — Theory, Applications, and Numerical Methods.* Progress in Science and Technology, Series on Mathematical Analysis. 29. All-Union Institute for Scientific and Technical Information (VINITI). Moscow: Akad. Nauk SSSR, 1991, 3–106. (in Russian)

[23] Kh.D. Ikramov, *The Nonsymmetric Eigenvalue Problem.* Moscow: Nauka, 1991. (in Russian)

[24] Knizhnerman L., Simoncini V. *A new investigation of the extended Krylov subspace method for matrix function evaluations.* Numerical Linear Algebra With Applications. 17, no. 4 (2010), 615–638.

[25] N.D. Kopachevskii, S.G. Krein, Ngo Zui Kan, *Operator Methods in Linear Hydrodynamics: Evolution and Spectral Problems.* Moscow: Nauka, 1989. (in Russian)

[26] L.D. Kudryavtsev, *Concise Course of Mathematical Calculus.* Moscow: Nauka, 1989. (in Russian)

[27] V.G. Kurbatov, M.N. Oreshina, *Interconnect macromodelling and approximation of matrix exponent.* Analog Integrated Circuits and Signal Processing. 40, no. 1 (2004), 5–19.

[28] I.V. Kurbatova, *On generalized impulse response.* Voronezh: Vestnik VGU. Physics and Mathematics, 2007, no. 1, 148–152. (in Russian)

[29] I.V. Kurbatova, *Banach algebras associated with linear operator pencils.* Mat. Zametki, 86, no. 3 (2009), 394–401. (in Russian; English transl. Mathematical Notes. 86, no. 3 (2009), 361–367.)

[30] I.V. Kurbatova, *Pseudoresolvents, Functional Calculus, and Operator Pencils.* Voronezh: Voronezh State Univ. Research Math. Institute, 2010. (in Russian)

[31] M.A. Lavrent'ev, B.V. Shabat, *Methods of the Theory of Functions of a Complex Variable.* Moscow: Nauka, 1965. (in Russian)

[32] A.S. Markus, *Introduction to the Spectral Theory of Polynomial Operator Pencils.* Providence R.I.: AMS 1988.

[33] A. Odabasioglu, M. Celik, L. T. Pillegi, *PRIMA: Passive reduced-order interconnect macromodeling algorithm.* IEEE Trans. Computer-Aided Design. 17, no. 8 (1998), Aug, 645–654.

[34] B.N. Parlett, *The Symmetric Eigenvalue Problem.* N.-J.: Prentice-Hall, 1980.

[35] V.P. Popov, *Foundations of Circuit Theory.* Moscow: Vysshaya Shkola, 1998. (in Russian)

[36] A. Preumont, *Vibration Control of Active Structures.* Dordrecht: Kluwer Academic Publishers, 2002.

[37] W. Rudin, *Functional Analysis.* N.-Y.: McGraw-Hill Book Company, 1973.

[38] Y. Saad, *Analysis of some Krylov subspace approximations to the matrix exponential operator.* SIAM J. Numer. Anal. 29 (1992), 209–228.

[39] Y. Saad, *Iterative Methods for Sparse Linear Systems.* Boston: PWS, 1996.

[40] L. Schwartz, *Transformation de Laplace des distributions.* Séminaire Mathematique de l'Université de Lund. Tome supplémentaire dedié á M. Riesz. l'Université de Lund. 1952, 196–206.

[41] L. Schwartz, *Distributions á valeurs vectorielles.* Ann. Inst. Fourier. I. 7 (1957), 1–141; II. 8 (1957), 1–209.

[42] S. Seshu, M.B. Reed, *Linear Graphs and Electric Networks.* Mass.: Addison–Wesley Publishing Company, 1961.

[43] A.A. Shkalikov, A.V. Shkred, *The problem of steady-state oscillation of a transversally isotropic half-cylinder.* Mat. Sb. 182, no. 8 (1991), 96–118. (in Russian; English transl. Math. USSR-Sb., 62, no. 1 (1989), 97–119.)

[44] V. Simoncini, D.B. Szyld, *Recent computational developments in Krylov subspace methods for linear systems.* Numerical Linear Algebra with Applications. 14, no. 1 (2007), 1–59.

[45] C. de Villemagne, R.E. Skelton, *Model reductions using a projection formulation.* Internat. J. Control. 46, no. 6 (1987), 2141–2169.

[46] J. Vlach, K. Singhal, *Computer Methods for Circuit Analysis and Design.* Dordrecht: Kluwer Academic Publishers, 1994.

[47] V.S. Vladimirov, *Equations of Mathematical Physics.* Moscow: Nauka, 1976. (in Russian; English transl. of the previuos edition, N.-Y.: M. Dekker, 1971.)

[48] H.A. Van der Vorst, *Iterative Krylov Methods for Large Linear Systems.* Cambridge: Cambridge Univ. Press, 2003.

[49] J.L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain.* Providence, Rhode Island: AMS, 1960.

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