**EURASIAN MATHEMATICAL JOURNAL**
ISSN 2077-9879
Volume 3, Number 4 (2012), 44 – 52

# ORTHOGONALITY AND SMOOTH POINTS IN $C(K)$ AND $C_b(\Omega)$

**D.J. Kečkić**

Communicated by G. Sinnamon

**Key words:** orthogonality in the sense of James, Gateaux derivative, smooth points.

**AMS Mathematics Subject Classification:** 46G05, 46E15, 49J50.

**Abstract.** For the usual norm on spaces $C(K)$ and $C_b(\Omega)$ of all continuous functions on a compact Hausdorff space $K$ (all bounded continuous functions on a locally compact Hausdorff space $\Omega$), the following equalities are proved:

$$\lim_{t\to 0+}\frac{||f+tg||_{C(K)}-||f||_{C(K)}}{t}=\max_{x\in\{z\,|\,|f(z)|=||f||\}}\mathrm{Re}(e^{-i\arg f(x)}g(x)).$$

and

$$\lim_{t\to 0+}\frac{||f+tg||_{C_b(\Omega)}-||f||_{C_b(\Omega)}}{t}=\inf_{\delta>0}\sup_{x\in\{z\,|\,|f(z)|\geq||f||-\delta\}}\mathrm{Re}(e^{-i\arg f(x)}g(x)).$$

These equalities are used to characterize the orthogonality in the sense of James (Birkhoff) in spaces $C(K)$ and $C_b(\Omega)$ as well as to give necessary and sufficient conditions for a point on the unit sphere to be a smooth point.

## 1 Introduction

In general normed spaces, it is impossible to define the inner product that generates the initial norm, due to Jordan - von Neumann theorem, and therefore it is impossible to define classical notion of orthogonality. However, there is a simple partial replacement of the orthogonality condition, called orthogonality in the sense of James (or Birkhoff in some papers), introduced in [4], [6]

**Definition 1.** Let $X$ be a normed space, and let $x$, $y\in X$. We say that $y$ is orthogonal to $x$, if for all $\lambda$, $\mu\in\mathbf{C}$ there holds

$$||\lambda x+\mu y||\geq||\lambda x||. \tag{1.1}$$

It is obvious that one of scalars $\lambda$, $\mu$ can be omitted in (1.1). If $X$ is an inner product space, then (1.1) implies $\langle x,y\rangle=0$, i.e. orthogonality in the usual way. Note, also, that this definition is not symmetric, in general, i.e. $y$ orthogonal to $x$ might not imply $x$ orthogonal to $y$. To see this consider the vectors $(-1,0)$ and $(1,1)$ in the space $\mathbf{C}^2$ with the max-norm.

This kind of orthogonality is characterized via the Gateaux derivative of the norm in uniform convex spaces, and at smooth points of the corresponding sphere in all spaces. We briefly quote necessary definitions and statements. For further details see [1], or books [2], [3], [5].

**Definition 2.** Let $X$ be a normed space and let $x \in X$. We say that $x$ is a smooth point of the sphere centered at 0 with radius $||x||$ if there is a unique support functional, i.e. if there is a unique $\varphi \in X^*$ such that $||\varphi|| = 1$ and $\varphi(x) = ||x||$.

**Proposition 1.1.** *Let $X$ be a normed space. If the function $y \mapsto ||y||$ is Gateaux differentiable at $x$ then $x$ is a smooth point of the corresponding sphere. Moreover, its Gateaux derivative is equal to $\operatorname{Re} F_x(y)$, where $F_x$ is the unique support functional. In addition, $y$ is orthogonal to $x$ if and only if $F_x(y) = 0$.*

**Remark 1.** The norm is a function from $X$ to $\mathbb{R}$. Henceforth, its Gateaux derivative in the previous Proposition is taken considering $X$ as a linear space over $\mathbb{R}$. In other words, $x$ is a smooth point provided that

$$\lim_{\mathbb{R} \ni t \to 0} \frac{||x+ty|| - ||x||}{t}$$

exists.

**Proposition 1.2.** *Let $X$ be a normed space. If its dual space $X^*$ is strictly convex then all its points are smooth.*

In spaces which dual space is not uniformly convex there are more difficulties, since they contain points that are not smooth, and consequently, Gauteaux derivative might not exist. Nevertheless, we can handle with such spaces via the $\varphi$-Gateaux derivative introduced in [7].

**Definition 3.** Let $X$ be a normed space and let $x$, $y \in X$. $\varphi$-Gateaux derivative of norm at $x$ in $y$ and $\varphi$ direction is

$$D_{\varphi,x}(y) = \lim_{t \to 0+} \frac{||x + te^{i\varphi}y|| - ||x||}{t}. \tag{1.2}$$

**Remark 2.** The limit in (1.2) always exists, due to convexity of the function $t \mapsto ||x + te^{i\varphi}y||$.

**Proposition 1.3.** *The vector $y$ is orthogonal to $x$ if and only if*

$$\inf_{0 \le \varphi < 2\pi} D_{\varphi,x}(y) \ge 0.$$

**Remark 3.** This is a refinement of [5, Theorem 50], which allows to work with points that are not smooth in *real* spaces.

The previous result is used in [7] and [8] to characterize the orthogonality in the sense of James in classical Banach spaces $L^1$, $c_0$ as well as in Banach spaces of nuclear operators $\mathfrak{S}_1$, compact operators $\mathfrak{S}_\infty$ and all operators $B(H)$ (each on a Hilbert space).

The aim of this note is to give characterization of the orthogonality in the sense of James in classical Banach spaces $C(K)$ and $C_b(\Omega)$ of all continuous functions on a compact Hausdroff space $K$ and all bounded continuous functions on a locally compact Hausdorff space $\Omega$.

It is worth saying that there is another way to prove Theorem 2.1, stated in Section 2, different from that presented in this note, and which follows from the general Theory of Birkhoff-James orthogonality.

Namely, [5, Theorem 15] (slightly reformulated) asserts that

$$\lim_{t\to 0+}\frac{||x+ty||-||x||}{t}=\sup_{\omega\in J(x)}\operatorname{Re}\omega(y), \tag{1.3}$$

where

$$J(x)=\{\omega\in X^*\mid \operatorname{Re}\omega(x)=||\omega||\,||x||,\ ||\omega||=1\}$$

Combining this and (1.2) we have

**Proposition 1.4.** *The vector $y$ is orthogonal to $x$ if and only if*

$$\inf_{0\le\varphi<2\pi}\sup_{\omega\in J(x)}\operatorname{Re}\omega(e^{i\varphi}y)\ge 0.$$

Given $f\in C(K)$, we can easily identify the set $J(f)$ with the set of all complex measures on $K$ supported on $E_0=\{x\in K\mid |f(x)|=||f||\}$, such that $\mathrm{d}\mu_1=e^{i\arg f}\,\mathrm{d}\mu$ is a probabilistic measure (positive and of total mass equals 1).

Thus, Theorem 2.1 can be proved considering atomic measures supported on a singleton $\{x\}$, $x\in E_0$, for one inequality, and by

$$\operatorname{Re}\int_K g\,\mathrm{d}\mu=\operatorname{Re}\int_{E_0}e^{-i\arg f}g\,\mathrm{d}\mu_1\le\max_{x\in E_0}\operatorname{Re}(e^{-i\arg f(x)}g(x)),$$

for the other.

Regardless of the previous consideration, we shall give, in Section 2, an elementary proof of Theorem 2.1 which gives an explicit expression of the limit in (1.3) in $C(K)$. This approach can be easily adapted to the space $C_b(\Omega)$, in Section 3, whereas Proposition 1.4 is not easy to apply to this space, due to the well-known fact that its dual space can hardly be described.

## 2 The space $C(K)$

**Theorem 2.1.** *Let $K$ be a compact Hausdorff space, let $C(K)$ be the Banach space of all continuous complex valued functions on $K$, with the usual norm $||f||=\max_{x\in K}|f(x)|$, and let $f$, $g\in C(K)$. We have*

$$\lim_{t\to 0+}\frac{||f+tg||-||f||}{t}=\max_{x\in\{z\mid |f(z)|=||f||\}}\operatorname{Re}(e^{-i\arg f(x)}g(x)). \tag{2.1}$$

*Proof.* Consider the set $E_\delta = \{x \in K \mid |f(x)| \geq ||f|| - \delta\}$. This is, clearly, increasing family, i.e. if $\delta_1 < \delta_2$ then $E_{\delta_1} \subseteq E_{\delta_2}$. We claim that $||f + tg||$ for $t$ small enough, depends only on values of $g$ at points from $E_\delta$. Indeed, if $t < \delta/4||g||$ and $x \notin E_\delta$ then

$$|f(x) + tg(x)| \leq |f(x)| + t||g|| \leq ||f|| - \delta + \delta/4 = ||f|| - 3\delta/4,$$

whereas for the same $t$, and for any $x \in E_{\delta/2} \subseteq E_\delta$ we have

$$|f(x) + tg(x)| > |f(x)| - t||g|| > ||f|| - \delta/2 - \delta/4 = ||f|| - 3\delta/4.$$

Hence, taking into account that $E_\delta$ is a compact set and $f + tg$ is a continuous function

$$||f + tg|| = \max_{x \in E_\delta} |f(x) + tg(x)|, \tag{2.2}$$

for $t < \delta/4||g||$.

We shall, first, find an upper bound of the limit in (2.1). For $t$ small enough, we have

$$\begin{aligned}\frac{||f+tg|| - ||f||}{t} &= \frac{||f+tg||^2 - ||f||^2}{t(||f+tg|| + ||f||)} = \\ &= \frac{\max_{x \in E_\delta}(|f(x)|^2 + 2t \operatorname{Re} \overline{f(x)} g(x) + t^2|g(x)|^2) - ||f||^2}{t(||f+tg|| + ||f||)} \leq \\ &\leq \frac{\max_{x \in E_\delta}(2 \operatorname{Re} \overline{f(x)} g(x) + t|g(x)|^2)}{(||f+tg|| + ||f||)} \leq \\ &\leq \frac{\max_{x \in E_\delta}(2 \operatorname{Re} \overline{f(x)} g(x)) + t||g||^2}{(||f+tg|| + ||f||)},\end{aligned}$$

which, taking the limit as $t \to 0+$ becomes

$$\lim_{t \to 0+} \frac{||f+tg|| - ||f||}{t} \leq \frac{\max_{x \in E_\delta}(\operatorname{Re} \overline{f(x)} g(x))}{||f||}. \tag{2.3}$$

Let us find a lower bound for the limit in (2.1). Denote $E_0 = \bigcap_{\delta > 0} E_\delta = \{x \in K \mid |f(x)| = ||f||\}$. For any $x \in E_0$ we have $|f(x) + tg(x)| \leq ||f + tg||$, and hence

$$\frac{||f+tg|| - ||f||}{t} \geq \frac{|f(x) + tg(x)| - |f(x)|}{t} = |f(x)| \frac{|1 + tg(x)/f(x)| - 1}{t}.$$

Since $\lim_{t \to 0+} \frac{|1+tz|-1}{t} = \operatorname{Re} z$ for each $z \in \mathbf{C}$, as it is easy to see, we obtain for any $x \in E_0$

$$\lim_{t \to 0+} \frac{||f+tg|| - ||f||}{t} \geq |f(x)| \operatorname{Re}(g(x)/f(x)) = \operatorname{Re}(e^{-i \arg f(x)} g(x)). \tag{2.4}$$

Since the left hand side in (2.3) does not depend on $\delta > 0$, and the left hand side in (2.4) does not depend on $x \in E_0$, we have

$$\max_{x \in E_0} \operatorname{Re}(e^{-i \arg f(x)} g(x)) \leq \lim_{t \to 0+} \frac{||f+tg|| - ||f||}{t} \leq \inf_{\delta > 0} \frac{\max_{x \in E_\delta}(\operatorname{Re} \overline{f(x)} g(x))}{||f||}. \tag{2.5}$$

To finish the proof, it is enough to prove that the right hand side in (2.5) is equal to its left hand side. Indeed, there is a sequence $\delta_n \to 0+$ and $x_n \in E_{\delta_n}$ such that $(\mathrm{Re}\,\overline{f(x_n)}g(x_n))/||f||$ tends to the right hand side of (2.5). Since $K$ is a compact set, there is its subsequence that tends to some $x_0 \in K$, which we shall also denote by $x_n$ in order to simplify notations. Moreover, $x_0 \in E_0$ because $\overline{E_\delta}$ is an increasing sequence of sets. Using the continuity of $f$ and $g$ we obtain that $(\mathrm{Re}\,\overline{f(x_n)}g(x_n))/||f|| \to (\mathrm{Re}\,\overline{f(x_0)}g(x_0))/||f|| = \mathrm{Re}(e^{-i\arg f(x_0)}g(x_0))$. So we get that for some $x_0 \in E_0$ we have

$$\max_{x\in E_0}\mathrm{Re}(e^{-i\arg f(x)}g(x)) \le \lim_{t\to 0+}\frac{||f+tg||-||f||}{t} \le \\ \le \mathrm{Re}(e^{-i\arg f(x_0)}g(x_0)) \le \max_{x\in E_0}\mathrm{Re}(e^{-i\arg f(x)}g(x)).$$

The proof is complete. □

**Corollary 2.1.** *The following three conditions are mutually equivalent:*

*$(i)$ The function $g$ is orthogonal to $f$ in the space $C(K)$;*

*$(ii)$ The values of the function $\overline{f(x)}g(x)$ on the set $E_0 = \{x \in K \mid |f(x)| = ||f||\}$ are not contained in an open half plain (in* **C***) with boundary that contains the origin.*

*$(iii)$ There exists a probability measure (i.e. positive and of full measure equals 1) $\mu$ with support contained in $E_0$ such that*

$$\int_K \overline{f(x)}g(x)\,\mathrm{d}\mu(x) = 0. \tag{2.6}$$

*If $|f|$ attains its norm at the single point, say $x_0$, then $g$ is orthogonal to $f$ if and only if $g(x_0) = 0$.*

*Proof.* By Theorem 2.1 and Proposition 1.3, $g$ is orthogonal to $f$ if and only if

$$\inf_{0\le\varphi<2\pi}\max_{x\in E_0}\mathrm{Re}(e^{i\varphi}e^{-i\arg f(x)}g(x)) \ge 0.$$

This is equivalent to the condition that the set $\{e^{-i\arg f(x)}g(x) \mid x \in E_0\}$ contains at least one value with nonnegative real part under all rotations around the origin. This is equivalent to the condition $(ii)$ (note that for $x \in E_0$ $|f(x)| = ||f||$ is constant).

Further, condition $(ii)$ is equivalent to the property that the closed convex hull of the set $F = \{\overline{f(x)}g(x) \mid x \in E_0\}$ contains the origin. The convex hull of the set $F$ consists of points of the form $\int_K \overline{f(x)}g(x)\,\mathrm{d}\lambda(x)$, where $\lambda$ is a probability measure supported on a finite subset of $E_0$. Therefore, it have to be

$$0 = \lim_{n\to+\infty}\int_K \overline{f(x)}g(x)\,\mathrm{d}\lambda_n(x),$$

for some sequence $\lambda_n$. Since, according to Alaoglu Theorem, the unit sphere in $C(K)^* \cong \mathcal{M}(K)$ is weakly-$*$ compact, there exists $\mu = \lim_{k\to+\infty}\lambda_{n_k}$, such that (2.6) is valid. The support of $\mu$ is contained in $E_0$ obviously. Thus $(ii) \Longrightarrow (iii)$.

Conversely, let (2.6) holds, and let $F(g) = \int_{E_0} g(x)\,\mathrm{d}\mu(x)$ Then, for any $\lambda \in \mathbf{C}$ we have

$$\begin{aligned} ||f+\lambda g||^2 =& || \,|f|^2 + 2\operatorname{Re}(\overline{f}\lambda g) + |\lambda|^2|g|^2|| \geq \\ \geq& |F(|f|^2 + 2\operatorname{Re}(\overline{f}\lambda g) + |\lambda|^2|g|^2)| = \\ =& ||f||^2 + |\lambda|^2 \int_{E_0} |g(x)|^2\,\mathrm{d}\mu(x) \geq ||f||^2, \end{aligned}$$

which leads to $(iii) \Longrightarrow (i)$.

If $f$ attains its norm at the single point $x_0$, then our set $F$ is a singleton, and its unique element must be equal to zero. □

**Corollary 2.2.** *The function $f \in C(K)$ is a smooth point of the corresponding sphere if and only if it attains its norm at the single point.*

*Proof.* Let $f$ attain its norm at a single point $x_0 \in K$. Then $E_0 = \{x_0\}$ and, by Theorem 2.1

$$\lim_{t\to 0+} \frac{||f+tg|| - ||f||}{t} = \operatorname{Re}(e^{-i\arg f(x_0)} g(x_0)),$$

implying

$$\begin{aligned} \lim_{t\to 0-} \frac{||f+tg|| - ||f||}{t} = - \lim_{t\to 0+} \frac{||f-tg|| - ||f||}{t} = \\ = -\operatorname{Re}(-e^{-i\arg f(x_0)} g(x_0)) = \operatorname{Re}(e^{-i\arg f(x_0)} g(x_0)). \end{aligned}$$

Therefore the norm is Gateaux differentiable at $f$, and the unique support functional $F_f$ is

$$F_f(g) = e^{-i\arg f(x_0)} g(x_0).$$

Let $f$ attain its norm at, at least, two different points $x_1$ and $x_2$. Then the functionals $F_j$, $j = 1, 2$ given by

$$F_j(g) = e^{-i\arg f(x_j)} g(x_j)$$

both satisfy $||F_j|| = 1$ and $F_j(f) = ||f||$. Hence, by Proposition 1.1, $f$ is not a smooth point. □

## 3 The space $C_b(\Omega)$

Let us, now, pass to the space $C_b(\Omega)$.

**Theorem 3.1.** *Let $\Omega$ be a locally compact Hausdorff space, let $C_b(\Omega)$ be the Banach space of all bounded continuous complex valued functions on $\Omega$, with the usual norm $||f|| = \sup_{x\in\Omega} |f(x)|$, and let $f$, $g \in C_b(\Omega)$. We have*

$$\lim_{t\to 0+} \frac{||f+tg|| - ||f||}{t} = \inf_{\delta>0} \sup_{x\in E_\delta} \operatorname{Re}(e^{-i\arg f(x)} g(x)), \tag{3.1}$$

*where $E_\delta = \{x \in \Omega \mid |f(x)| \geq ||f|| - \delta\}$.*

*Proof.* The proof is essentially the same as that of Theorem 2.1. The inequality (2.3) remains valid if we replace max by sup, since $E_\delta$ might not be compact.

So, it is sufficient to prove the opposite inequality. At first, note that $\inf\limits_{\delta>0}$ in (3.1) can be replaced by $\lim\limits_{\delta\to 0+}$, since the function $\delta\mapsto\sup_{\delta>0}\operatorname{Re}(e^{-i\arg f(x)}g(x))$ is nonincreasing. Further, given $\delta>0$, choose $x_\delta\in E_\delta$ such that $\sup_{x\in E_\delta}\operatorname{Re}(e^{-i\arg f(x)}g(x))<\operatorname{Re}(e^{-i\arg f(x_\delta)}g(x_\delta))+\delta$. Then $|f(x_\delta)|\to||f||$ as $\delta\to 0+$ and

$$\begin{aligned}\frac{||f+tg||-||f||}{t}&=\frac{||f+tg||^2-||f||^2}{t(||f+tg||+||f||)}\geq\\&\geq\frac{|f(x_\delta)|^2+2t\operatorname{Re}\overline{f(x_\delta)}g(x_\delta)+t^2|g(x_\delta)|^2-||f||^2}{t(||f+tg||+||f||)}\geq\\&\geq\frac{|f(x_\delta)|^2+2t|f(x_\delta)|(\sup_{x\in E_\delta}\operatorname{Re}(e^{-i\arg f(x)}g(x))-\delta)-||f||^2}{t(||f+tg||+||f||)}.\end{aligned}$$

Taking $\lim\limits_{\delta\to 0+}$ we obtain

$$\frac{||f+tg||-||f||}{t}\geq\frac{||f||^2+2t||f||(\inf_{\delta>0}\sup_{x\in E_\delta}\operatorname{Re}(e^{-i\arg f(x)}g(x)))-||f||^2}{t(||f+tg||+||f||)},$$

and finally, letting $t\to 0+$

$$\lim_{t\to 0+}\frac{||f+tg||-||f||}{t}\geq\inf_{\delta>0}\sup_{x\in E_\delta}\operatorname{Re}(e^{-i\arg f(x)}g(x))),$$

which finishes the proof. □

**Corollary 3.1.** *The function $g\in C_b(\Omega)$ is orthogonal to $f\in C_b(\Omega)$ if and only if there is a sequence of probability measures $\mu_n$ concentrated at $E_\delta=\{z\in\Omega\mid|f(z)|\geq||f||-\delta\}$ such that*

$$\lim_{n\to\infty}\int_\Omega\overline{f(x)}g(x)\,\mathrm{d}\mu_n(x)=0. \tag{3.2}$$

*Proof.* By Theorem 3.1 and Proposition 1.1 $g$ is orthogonal to $f$ if and only if

$$\inf_{\delta>0}\inf_{0\leq\varphi<2\pi}\sup_{x\in E_\delta}\operatorname{Re}(e^{-i\arg f(x)}g(x))\geq 0.$$

As in the proof of Corollary 2.1 we conclude that for all $\delta>0$ the closed convex hull of the set $F_\delta=\{\operatorname{Re}(e^{-i\arg f(x)}g(x))\mid x\in E_\delta\}$ contains the origin. Hence, there is a probability measure $\mu_\delta$ concentrated at the finite subset of $E_\delta$ such that

$$\left|\int_\Omega e^{-i\arg f(x)}g(x)\,\mathrm{d}\mu_\delta(x)\right|<\delta. \tag{3.3}$$

Choose a sequence $\delta_n\to 0+$, and denote $\mu_n=\mu_{\delta_n}$. Next, we estimate the difference of integral in (3.2) multiplied by $||f||$ and the integral in (3.3). Indeed, for $x\in E_\delta$ it holds $||f||-\delta\leq|f(x)|\leq||f||$ and hence

$$\left|\int_\Omega(||f||e^{-i\arg f(x)}g(x)-\overline{f(x)}g(x))\,\mathrm{d}\mu_n(x)\right|\leq\int_\Omega\delta_n|g(x)|\,\mathrm{d}\mu_n(x)<\delta_n||g||,$$

which implies (3.2).

Conversely, let (3.2) holds. Then, as in the proof of Corollary 2.1 we have, denoting $F_n(g) = \int_\Omega g(x)\,\mathrm{d}\mu_n(x)$,

$$\begin{aligned}||f+\lambda g||^2 =&|| \, |f|^2 + 2\,\mathrm{Re}(\overline{f}\lambda g) + |\lambda|^2|g|^2|| \geq \\ \geq&|F_n(|f|^2 + 2\,\mathrm{Re}(\overline{f}\lambda g) + |\lambda|^2|g|^2)| = \\ =&||f||^2 + |\lambda|^2 \int_{E_0} |g(x)|^2\,\mathrm{d}\mu(x) + 2\,\mathrm{Re}\int_\Omega \overline{f(x)}g(x)\,\mathrm{d}\mu_n(x).\end{aligned}$$

Take the limit as $n \to +\infty$ and we get $||f + \lambda g|| \geq ||f||$, which finishes the proof. □

**Corollary 3.2.** *Let $\Omega$ be a normal space. The function $f$ is a smooth point of the corresponding sphere in $C_b(\Omega)$ if and only if:*

*($i$) $f$ attains its norm at the unique point, and*

*($ii$) There is $\delta > 0$ such that $E_\delta$ is compact set.*

*Proof.* Let ($i$) and ($ii$) holds. Then we can reduce the proof to the case $f \in C(E_\delta)$ and apply Corollary 2.2. Also, if ($i$) does not hold, then we can apply the argument from Corollary 2.2.

If ($ii$) does not hold, then there is a sequence $x_n \in \Omega$ with no accumulation points in $\Omega$ such that $|f(x_n)| \to ||f||$. Let glim denote the generalized Banach limit on the space $c$ of all convergent complex sequences. We define functionals $F_1$ and $F_2$ by

$$F_1(g) = \operatorname*{glim}_{n\to+\infty} e^{-i\arg f(x_{2n})}g(x_{2n}), \qquad F_2(g) = \operatorname*{glim}_{n\to+\infty} e^{-i\arg f(x_{2n+1})}g(x_{2n+1}).$$

Both of them satisfy $||F_j|| = 1$ and $F_j(f) = ||f||$. Since $x_n$ has no accumulation point in $\Omega$, the set $\{x_n \mid n \in \mathrm{N}\}$ is closed, and the function $h$ defined by $h(x_{2n}) = 0$ and $h(x_{2n+1}) = e^{i\arg f(x_{2n+1})}$ is continuous and bounded on it. By the Tietze Theorem it can be extended to some bounded continuous function on the whole $\Omega$. We have, then, $F_1(h) = 0$ and $F_2(h) = 1$, that is $F_1 \neq F_2$. □

**Remark 4.** The normality condition is used only in proving that $\neg(ii)$ implies that $f$ is not smooth point. All other implications hold provided only that $\Omega$ is Hausdorff locally compact.

**Acknowledgement.** The author was partially supported by MPN Republic of Serbia, grant no. 174034. The author is grateful to the reviewer who suggested to use Proposition 1.4 in order to prove Theorem 2.1 by using general theory of Birkhoff-James orthogonality.

## References


[1] T.J. Abatzoglu, *Norm derivatives on spaces of operators.* Math. Ann. 239 (1979), 129–135.

[2] C. Alsina, J. Sikorska, M. Santos Tomas, *Norm derivatives and characterizations of inner product spaces.* World scientific publishing, 2010.

[3] D. Amir, *Characterizations of inner product spaces.* Birkhauser Verlag, 1986.

[4] G. Birkhoff, *Orthogonality in linear metric spaces.* Duke Math. J. 1 (1935), 169–172.

[5] S. Dragomir, *Semi-inner products and applications.* Nova Science Pub. Inc, 2004.

[6] R.C. James, *Orthogonality and linear functionals in normed linear spaces.* Trans. Amer. Math. Soc. 61 (1947), 265–292.

[7] D.J. Kečkić, *Orthogonality in* $\mathfrak{S}_1$ *and* $\mathfrak{S}_\infty$ *spaces and normal derivations.* J Oper. Theory 51, no. 1 (2004), 89–104.

[8] D.J. Kečkić, *Gateaux derivative of* $B(H)$ *norm.* Proc. Amer. Math. Soc. 133 (2005), 2061–2067.



Dragoljub J. Kečkić
Faculty of Mathematics
University of Belgrade
Studentski trg 16-18
11000 Beograd
E-mail: dragoljub@keckic.com


Received: 22.11.2011