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ON DIRECT VARIATIONAL FORMULATIONS FOR SECOND ORDER EVOLUTIONARY EQUATIONS

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Abstract. The existence of direct variational formulations for a wide class of second order evolutionary equations is investigated.

1 Introduction

Consider the following operator equation

$$N(u) \equiv P_{2u,t}u_{tt} + P_{1u,t}u_t + Q(t,u) = 0_V, \tag{1.1}$$

$$u \in D(N) \subseteq U \subseteq V, \quad t \in [t_0,t_1] \subset \mathbb{R}; \quad u_t \equiv D_t u \equiv \frac{d}{dt}u, \qquad u_{tt} \equiv \frac{d^2}{dt^2}u.$$

Here $\forall t \in [t_0,t_1]$, $\forall u \in U_1$ $P_{iu,t} : U_1 \to V_1$ $(i = 1,2)$ are linear operators; $Q : [t_0,t_1] \times U_1 \to V_1$ is an arbitrary operator; $D(N)$ is a domain of definition of the operator $N : D(N) \subseteq U \to V$; $U = C^2([t_0,t_1];U_1)$, $V = C([t_0,t_1];V_1)$, U_1, V_1 are linear normed spaces, $U_1 \subseteq V_1$.

Assume that for every $t \in (t_0,t_1)$ and $g(t), u(t) \in U_1$ the function $P_{1u,t}g(t)$ is continuously differentiable and $P_{2u,t}g(t)$ is twice continuously differentiable on (t_0,t_1).

Any function $u \in D(N)$ is called a solution of problem (1.1) if it satisfies equation (1.1).

In the sequel, we shall write

$$N(u) \equiv P_{2u}u_{tt} + P_{1u}u_t + Q(u) = 0_V,$$

bearing in mind that the operators P_{1u}, P_{2u} and Q may also depend on t.

First let us introduce the following concepts. Let N be an operator such that its domain of definition $D(N) \subseteq U$ and the range of values $R(N) \subseteq V$, where U and V are linear normed spaces over $\mathbb{R}$, i. e.

$$N(u) = v, \qquad u \in U, \qquad v \in V.$$

If there exists the limit

$$\delta N(u,h) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\{N(u+\varepsilon h) - N(u)\}, \qquad u \in D(N), \quad (u+\varepsilon h) \in D(N), \tag{1.2}$$

then it is called the Gâteaux variation of the operator N at the point u or the first variation of the operator N at the point u.

$\delta N(u,h)$ is homogeneous with respect to $h : \delta N(u,\lambda h) = \lambda\delta N(u,h)$, but the operator $\delta N(u,\cdot) : U \to V$ is not always additive with respect to h.

If $\delta N(u,h)$ is a linear operator with respect to h, when u is a fixed element of $D(N)$, then we say that the operator N is Gâteaux differentiable at the point u. The expression $\delta N(u,h)$ is called the Gâteaux differential and denoted by $DN(u,h)$. In this case we shall also write $DN(u,h) = N'_u h$ and say that N'_u is the Gâteaux derivative of operator N at the point u.

If N is a linear operator then $N'_u h = Nh$, i. e. the Gâteaux derivative of the linear operator coincides with it.

Further assume that for any given operator $N : D(N) \subset U \to V$ there exists its Gâteaux derivative at any point $u \in D(N)$. The domain of definition $D(N'_u)$ consisits of elements $h \in U$ such that $(u + \varepsilon h) \in D(N)$ for all ε sufficiently small. In this case $h \in D(N'_u)$ is called an admissible element.

If the Gâteaux derivative of the operator N exists, then the following equality holds

$$N(u+\varepsilon h) = N(u) + \varepsilon N'_u h + r(u,\varepsilon h), \qquad u \in D(N), \tag{1.3}$$

where for any fixed element $h \in D(N'_u)$

$$\lim_{\varepsilon\to 0} \frac{r(u,\varepsilon h)}{\varepsilon} = 0_V.$$

If the Gâteaux derivative of the operator N is known and $0_U \in D(N)$, then

$$N(u) = \int\limits_0^1 N'_{tu} u\, dt + N(0_U). \tag{1.4}$$

Note that for any linear operator $\tilde{N}_u$ which may depend on u in a nonlinear way, the Gâteaux derivative is defined by

$$\tilde{N}'_u(g;h) = \lim_{\varepsilon\to 0} \frac{\tilde{N}_{u+\varepsilon h} g - \tilde{N}_u g}{\varepsilon}. \tag{1.5}$$

The second Gâteaux derivative N''_u of the operator N is given by

$$N''_u(h_1,h_2) = \frac{\partial^2}{\partial\varepsilon^1\partial\varepsilon^2} N(u + \varepsilon^1 h_1 + \varepsilon^2 h_2)|_{\varepsilon^1=\varepsilon^2=0}. \tag{1.6}$$

In the most of applications N''_u satisfies the symmetry condition

$$N''_u(h_1,h_2) = N''_u(h_2,h_1).$$

Now we need some notation and notions for bilinear forms and potential operators.

Definition 1. A mapping $\Phi(\cdot,\cdot) : V \times U \to \mathbb{R}$ is said to be a bilinear form if it is linear with respect to every argument.

Definition 2. A bilinear form $\Phi(\cdot,\cdot) : V \times V \to \mathbb{R}$ is called symmetric if

$$\Phi(v,g) = \Phi(g,v) \quad \forall g,v \in V.$$

Consider a bilinear form

$$\Phi(\cdot,\cdot) \equiv \int\limits_{t_0}^{t_1} <\cdot,\cdot> dt : V \times U \to \mathbb{R} \tag{1.7}$$

such that the bilinear mapping $\Phi_1(\cdot,\cdot) \equiv <\cdot,\cdot>$ satisfies the following conditions:

$$<v_1(t), v_2(t)> = <v_2(t), v_1(t)> \qquad \forall v_1(t), v_2(t) \in V_1, \tag{1.8}$$

$$D_t <v(t), g(t)> = <D_t v(t), g(t)> + <v(t), D_t g(t)> \qquad \forall v, g \in C^1([t_0,t_1]; U_1). \tag{1.9}$$

If $v = v(x,t), x \in \Omega \subset \mathbb{R}^n, t \in (t_0,t_1), U_1 = V_1 = C(\overline{\Omega})$, then we can take for example

$$<v,g> = \int\limits_{\Omega} v(x,t) g(x,t)\, dx. \tag{1.10}$$

Definition 3. The operator $N : D(N) \subset U \to V$ is said to be potential on the set $D(N)$ with respect to the bilinear form $\Phi(\cdot,\cdot) : V \times U \to \mathbb{R}$, if there exists a functional $F_N : D(F_N) = D(N) \to \mathbb{R}$ such that

$$\delta F_N[u,h] = \Phi(N(u),h) \qquad \forall u \in D(N), \quad \forall h \in D(N'_u).$$

The functional F_N is called the potential of the operator N, and in its turn the operator N is called the gradient of the functional F_N. In this case we shall write $N = grad_\Phi F_N$.

An element $u \in D(N)$ such that $\delta F_N[u,h] = 0 \quad \forall h \in D(N'_u)$, is said to be a critical point of the functional F_N.

The following theorem is needed in the sequel.

Theorem 1.1. [3, 5] *Consider the operator $N : D(N) \subset U \to V$ and the bilinear form $\Phi(\cdot,\cdot) : V \times U \to \mathbb{R}$ such that for any fixed elements $u \in D(N), \quad g,h \in D(N'_u)$ the function $\psi(\varepsilon) = \Phi(N(u+\varepsilon h), g)$ belongs to the class $C^1[0,1]$. For N to be potential on the convex open set $D(N)$ with respect to Φ it is necessary and sufficient that*

$$\Phi(N'_u h, g) = \Phi(N'_u g, h) \qquad \forall u \in D(N), \quad \forall g,h \in D(N'_u). \tag{1.11}$$

Under this condition the potential F_N is given by

$$F_N[u] = \int\limits_0^1 \Phi(N(u_0 + \lambda(u-u_0)), u-u_0)\, d\lambda + F_N[u_0], \tag{1.12}$$

where u_0 is a fixed element of $D(N)$.

2 Main results

Denoting by $(...)^*$ the operator adjoint to the operator $(...)$, we shall prove

Theorem 2.1. *Suppose that $D_t^* = -D_t$ on $D(N_u')$; then for operator (1.1) to be potential on $D(N)$ with respect to bilinear form (1.7) it is necessary and sufficient that on $D(N_u')$*

$$P_{2u} - P_{2u}^* = 0, \tag{2.1}$$

$$P_{2u}^{*\prime}(\cdot; u_t) = 0, \tag{2.2}$$

$$-2\frac{\partial P_{2u}^*}{\partial t} + P_{1u}^* + P_{1u} = 0, \tag{2.3}$$

$$-\frac{\partial^2 P_{2u}^*}{\partial t^2} + \frac{\partial P_{1u}^*}{\partial t} + Q_u' - Q_u'^{\,*} = 0, \tag{2.4}$$

$$-\left(\frac{\partial P_{2u}^*}{\partial t}\right)_u' (\cdot; u_t) - \frac{\partial P_{2u}^{*\prime}}{\partial t}(\cdot; u_t) + P_{1u}^{*\prime}(\cdot; u_t) + P_{1u}'(u_t; \cdot) - [P_{1u}'(u_t; \cdot)]^* = 0, \tag{2.5}$$

$$P_{2u}'(u_{tt}; \cdot) - P_{2u}^{*\prime}(\cdot; u_{tt}) - [P_{2u}'(u_{tt}; \cdot)]^* = 0. \tag{2.6}$$

$$\forall u \in D(N), \quad \forall t \in [t_0, t_1].$$

Proof. Using (1.1) and (1.5), we get

$$N_u' h = P_{2u} h_{tt} + P_{2u}'(u_{tt}; h) + P_{1u} h_t + P_{1u}'(u_t; h) + Q_u' h.$$

In this case (1.11) can be written in the form

$$\int_{t_0}^{t_1} < P_{2u} h_{tt} + P_{2u}'(u_{tt}; h) + P_{1u} h_t + P_{1u}'(u_t; h) + Q_u' h, g > \, dt$$

$$= \int_{t_0}^{t_1} < P_{2u} g_{tt} + P_{2u}'(u_{tt}; g) + P_{1u} g_t + P_{1u}'(u_t; g) + Q_u' g, h > \, dt,$$

or

$$\int_{t_0}^{t_1} \{< P_{2u} h_{tt} + P_{2u}'(u_{tt}; h) + P_{1u} h_t + P_{1u}'(u_t; h) + Q_u' h, g >$$

$$- < D_t^2(P_{2u}^* h) + [P_{2u}'(u_{tt}; \cdot)]^* h - D_t(P_{1u}^* h) + [P_{1u}'(u_t; \cdot)]^* h + Q_u'^* h, g >\} dt = 0 \tag{2.7}$$

$$\forall u \in D(N), \quad \forall g, h \in D(N_u').$$

Taking into account the second Gâteaux derivative, we obtain

$$D_t^2(P_{2u}^* h) = D_t[D_t(P_{2u}^* h)] = D_t \left[P_{2u}^* h_t + \frac{\partial P_{2u}^*}{\partial t} h + P_{2u}^{*\prime}(h; u_t) \right]$$

$$= P^*_{2u} h_{tt} + 2\frac{\partial P^*_{2u}}{\partial t} h_t + 2P^{*\prime}_{2u}(h_t; u_t) + \frac{\partial^2 P^*_{2u}}{\partial t^2} h + \left(\frac{\partial P^*_{2u}}{\partial t}\right)'_u (h; u_t) + P^{*\prime\prime}_{2u}(h; u_t; u_t)$$

$$+P^{*\prime}_{2u}(h; u_{tt}) + \frac{\partial P^{*\prime}_{2u}}{\partial t}(h; u_t). \tag{2.8}$$

From (2.8), (2.7) we get the following:

$$\int_{t_0}^{t_1} < P_{2u} h_{tt} + P'_{2u}(u_{tt}; h) + P_{1u} h_t + P'_{1u}(u_t; h) + Q'_u h - P^*_{2u} h_{tt} - 2\frac{\partial P^*_{2u}}{\partial t} h_t$$

$$-2P^{*\prime}_{2u}(h_t; u_t) - \frac{\partial^2 P^*_{2u}}{\partial t^2} h - \left(\frac{\partial P^*_{2u}}{\partial t}\right)'_u (h; u_t) - P^{*\prime\prime}_{2u}(h; u_t; u_t) - P^{*\prime}_{2u}(h; u_{tt})$$

$$-\frac{\partial P^{*\prime}_{2u}}{\partial t}(h; u_t) - [P'_{2u}(u_{tt}; \cdot)]^* h + P^*_{1u} h_t + \frac{\partial P^*_{1u}}{\partial t} h + P^{*\prime}_{1u}(h; u_t) - [P'_{1u}(u_t; \cdot)]^* h$$

$$-Q'^*_u h, g > \; dt = 0.$$

Thus condition (2.7) is represented in the form

$$\int_{t_0}^{t_1} < \{(P_{2u} - P^*_{2u})\, D_{tt} + \left(P_{1u} - 2\frac{\partial P^*_{2u}}{\partial t} - 2P^{*\prime}_{2u}(\cdot; u_t) + P^*_{1u}\right) D_t$$

$$+P'_{2u}(u_{tt}; \cdot) + P'_{1u}(u_t; \cdot) + Q'_u - \frac{\partial^2 P^*_{2u}}{\partial t^2} - \left(\frac{\partial P^*_{2u}}{\partial t}\right)'_u (\cdot; u_t) - P^{*\prime\prime}_{2u}(\cdot; u_t; u_t)$$

$$-P^{*\prime}_{2u}(\cdot; u_{tt}) - \frac{\partial P^{*\prime}_{2u}}{\partial t}(\cdot; u_t) - [P'_{2u}(u_{tt}; \cdot)]^* + \frac{\partial P^*_{1u}}{\partial t} + P^{*\prime}_{1u}(\cdot; u_t) - [P'_{1u}(u_t; \cdot)]^*$$

$$-Q'^*_u\} h, g > dt = 0 \qquad \forall u \in D(N), \quad \forall g, h \in D(N'_u).$$

This is fulfilled identically if and only if

$$\Big[(P_{2u} - P^*_{2u})\, D_{tt} + \left(P_{1u} - 2\frac{\partial P^*_{2u}}{\partial t} - 2P^{*\prime}_{2u}(\cdot; u_t) + P^*_{1u}\right) D_t$$

$$+P'_{2u}(u_{tt}; \cdot) + P'_{1u}(u_t; \cdot) + Q'_u - \frac{\partial^2 P^*_{2u}}{\partial t^2} - \left(\frac{\partial P^*_{2u}}{\partial t}\right)'_u (\cdot; u_t) - P^{*\prime\prime}_{2u}(\cdot; u_t; u_t)$$

$$-P^{*\prime}_{2u}(\cdot; u_{tt}) - \frac{\partial P^{*\prime}_{2u}}{\partial t}(\cdot; u_t) - [P'_{2u}(u_{tt}; \cdot)]^* + \frac{\partial P^*_{1u}}{\partial t} + P^{*\prime}_{1u}(\cdot; u_t) - [P'_{1u}(u_t; \cdot)]^*$$

$$-Q'^*_u \,\Big]\, h = 0_V \qquad \forall u \in D(N), \quad \forall h \in D(N'_u).$$

The necessary and sufficient conditions for this equality to be valid are that conditions (2.1) - (2.6) hold. □

Remark 1. By (2.1), (1.4), and (2.2) it follows that

$$P_{2u} h - P_{20} h = \int_0^1 \frac{d}{d\varepsilon} P_{2\varepsilon u} h \, d\varepsilon = \int_0^1 P'_{2\varepsilon u}(h; u)\, d\varepsilon = 0,$$

i.e. the operator P_{2u} does not depend on u on $D(N'_u)$.

Therefore conditions (2.1) - (2.6) are equivalent to the following ones:

$$P_2 - P_2^* = 0, \tag{2.9}$$

$$-2\frac{\partial P_2}{\partial t} + P_{1u}^* + P_{1u} = 0, \tag{2.10}$$

$$\frac{\partial^2 P_2}{\partial t^2} - \frac{\partial P_{1u}}{\partial t} + Q'_u - {Q'_u}^* = 0, \tag{2.11}$$

$$-P'_{1u}(\cdot; u_t) + P'_{1u}(u_t; \cdot) - [P'_{1u}(u_t; \cdot)]^* = 0, \tag{2.12}$$

$$P'_{2u}(u_{tt}; \cdot) - [P'_{2u}(u_{tt}; \cdot)]^* = 0. \tag{2.13}$$

Theorem 2.2. *If $D_t^* = -D_t$ on $D(N'_u)$, then conditions (2.1) – (2.6) hold if and only if equation (1.1) can be represented in the form*

$$N(u) \equiv P_{2u} u_{tt} + P_{1u} u_t + Q(u) \tag{2.14}$$

$$\equiv (-\mathcal{R}_2 - \mathcal{R}_2^*) u_{tt} + (\mathcal{R}'^*_{1u} - \mathcal{R}'_{1u} - 2\frac{\partial \mathcal{R}_2^*}{\partial t}) u_t + grad_{\Phi_1} \mathcal{B}[u] - \frac{\partial \mathcal{R}_1}{\partial t}(u) + \frac{\partial^2 \mathcal{R}_2}{\partial t^2} u = 0_V \quad \forall u \in D(N).$$

The operators $\mathcal{R}_1, \mathcal{R}_2, \mathcal{B}$ are defined by

$$\Phi(\mathcal{R}_1(u), u_t) = \int_{t_0}^{t_1} \int_0^1 < -P_{1\tilde{u}(\lambda)}(u - u_0), \frac{\partial \tilde{u}(\lambda)}{\partial t} > d\lambda dt, \tag{2.15}$$

$$\Phi(\mathcal{R}_2 u_t, u_t) = \int_{t_0}^{t_1} \int_0^1 < -P_2(u_t - u_{0_t}), \frac{\partial \tilde{u}(\lambda)}{\partial t} > d\lambda dt, \tag{2.16}$$

$$\mathcal{B}[u] = \int_0^1 < Q(\tilde{u}(\lambda)) + \lambda \frac{\partial P_{1\tilde{u}(\lambda)}}{\partial t}(u - u_0) - \lambda \frac{\partial^2 P_2}{\partial t^2}(u - u_0), u - u_0 > d\lambda, \tag{2.17}$$

where $\tilde{u}(\lambda) = u_0 + \lambda(u - u_0)$; u_0 *is a fixed element of* $D(N)$.

Proof. If $D_t^* = -D_t$ on $D(N'_u)$ and conditions (2.1) - (2.6) hold, then by Theorem 2.1 it follows that operator (1.1) is potential with respect to (1.7). This implies that we can construct the corresponding functional.

$$F_N[u] = \int_{t_0}^{t_1} \int_0^1 \Big[< P_{2\tilde{u}(\lambda)} \frac{\partial^2 \tilde{u}(\lambda)}{\partial t^2}, u - u_0 >$$

$$+ < P_{1\tilde{u}(\lambda)} \frac{\partial \tilde{u}(\lambda)}{\partial t}, u - u_0 > + < Q(\tilde{u}(\lambda)), u - u_0 > \Big] d\lambda dt + F_N[u_0].$$

Let us consider the integral

$$J_2[u] = \int_{t_0}^{t_1}\int_0^1 < P_{2\tilde{u}(\lambda)}\frac{\partial^2 \tilde{u}(\lambda)}{\partial t^2}, u - u_0 > d\lambda dt.$$

Since D_t is skew-symmetric, we obtain

$$J_2[u] = \int_{t_0}^{t_1}\int_0^1 < P_{2\tilde{u}(\lambda)}u_{0_{tt}} + \lambda P_{2\tilde{u}(\lambda)}(u_{tt} - u_{0_{tt}}), u - u_0 > d\lambda dt$$

$$= \int_{t_0}^{t_1}\int_0^1 \big[< P_{2\tilde{u}(\lambda)}u_{0_{tt}}, u - u_0 >$$

$$- < \lambda(u_t - u_{0_t}), \frac{\partial P_{2\tilde{u}(\lambda)}}{\partial t}(u - u_0) + P_{2\tilde{u}(\lambda)}(u_t - u_{0_t}) >\Big] d\lambda dt$$

$$= \int_{t_0}^{t_1}\int_0^1 \Big[P_{2\tilde{u}(\lambda)}u_{0_{tt}}, u - u_0 > + < \lambda(u - u_0), \frac{\partial^2 P_{2\tilde{u}(\lambda)}}{\partial t^2}(u - u_0) >$$

$$+ < 2\lambda(u - u_0), \frac{\partial P_{2\tilde{u}(\lambda)}}{\partial t}(u_t - u_{0_t}) > + < \lambda(u - u_0), P_{2\tilde{u}(\lambda)}(u_{tt} - u_{0_{tt}}) >\Big] d\lambda dt$$

$$= \int_{t_0}^{t_1}\int_0^1 \Big[< P_{2\tilde{u}(\lambda)}\frac{\partial^2 \tilde{u}(\lambda)}{\partial t^2}, u - u_0 >$$

$$+ \lambda < \frac{\partial^2 P_{2\tilde{u}(\lambda)}}{\partial t^2}(u - u_0) + 2\frac{\partial P_{2\tilde{u}(\lambda)}}{\partial t}(u_t - u_{0_t}), u - u_0 >\Big] d\lambda dt$$

$$= \int_{t_0}^{t_1}\int_0^1 \Big[- < \frac{\partial P_2}{\partial t}(u - u_0), \frac{\partial \tilde{u}(\lambda)}{\partial t} > - < P_2(u_t - u_{0_t}), \frac{\partial \tilde{u}(\lambda)}{\partial t} >$$

$$+ \lambda < \frac{\partial^2 P_2}{\partial t^2}(u - u_0) + 2\frac{\partial P_2}{\partial t}(u_t - u_{0_t}), u - u_0 >\Big] d\lambda dt.$$

For the integral $J_1[u] = \int_{t_0}^{t_1}\int_0^1 < P_{1\tilde{u}(\lambda)}\frac{\partial \tilde{u}(\lambda)}{\partial t}, u - u_0 > d\lambda dt$ we have

$$J_1[u] = \int_{t_0}^{t_1}\int_0^1 \big[< P_{1\tilde{u}(\lambda)}u_{0_t}, u - u_0 > + < \lambda P_{1\tilde{u}(\lambda)}(u_t - u_{0_t}), u - u_0 >\big] d\lambda dt$$

$$= \int_{t_0}^{t_1}\int_0^1 \big[< P_{1\tilde{u}(\lambda)}u_{0_t}, u - u_0 > - < \lambda(u - u_0), D_t(P^*_{1\tilde{u}(\lambda)}(u - u_0)) >\big] d\lambda dt.$$

Further note that

$$D_t[P^*_{1\tilde{u}(\lambda)}(u-u_0)] = D_t\left[\left(2\frac{\partial P_2}{\partial t} - P_{1\tilde{u}(\lambda)}\right)(u-u_0)\right] = 2\frac{\partial^2 P_2}{\partial t^2}(u-u_0)$$

$$+2\frac{\partial P_2}{\partial t}(u_t - u_{0_t}) - \frac{\partial P_{1\tilde{u}(\lambda)}}{\partial t}(u-u_0) - P_{1\tilde{u}(\lambda)}(u_t - u_{0_t}) - P'_{1\tilde{u}(\lambda)}\left(u-u_0; \frac{\partial \tilde{u}(\lambda)}{\partial t}\right).$$

Consequently,

$$J_1[u] = \int_{t_0}^{t_1}\int_0^1 \Big[< P_{1\tilde{u}(\lambda)}u_{0_t}, u-u_0 > - < \lambda(u-u_0), 2\frac{\partial^2 P_2}{\partial t^2}(u-u_0) >$$

$$- < \lambda(u-u_0), 2\frac{\partial P_2}{\partial t}(u_t - u_{0_t}) > + \lambda < \frac{\partial P_{1\tilde{u}(\lambda)}}{\partial t}(u-u_0), u-u_0 >$$

$$+ < \lambda P_{1\tilde{u}(\lambda)}(u_t - u_{0_t}) + \lambda P'_{1\tilde{u}(\lambda)}\left(u-u_0; \frac{\partial \tilde{u}(\lambda)}{\partial t}\right), u-u_0 >\Big] d\lambda dt.$$

Taking into consideration (2.12), we get

$$\int_{t_0}^{t_1}\int_0^1 \lambda < P'_{1\tilde{u}(\lambda)}\left(u-u_0; \frac{\partial \tilde{u}(\lambda)}{\partial t}\right), u-u_0 > d\lambda dt$$

$$= \int_{t_0}^{t_1}\int_0^1 \lambda < P'_{1\tilde{u}(\lambda)}\left(\frac{\partial \tilde{u}(\lambda)}{\partial t}; u-u_0\right) - [P'_{1\tilde{u}(\lambda)}\left(\frac{\partial \tilde{u}(\lambda)}{\partial t}; .\right)]^*(u-u_0), u-u_0 > d\lambda dt$$

$$= \int_{t_0}^{t_1}\int_0^1 \lambda < P'_{1\tilde{u}(\lambda)}\left(\frac{\partial \tilde{u}(\lambda)}{\partial t}; u-u_0\right) - P'_{1\tilde{u}(\lambda)}\left(\frac{\partial \tilde{u}(\lambda)}{\partial t}; u-u_0\right), u-u_0 > d\lambda dt = 0,$$

i.e.

$$J_1[u] = \int_{t_0}^{t_1}\int_0^1 \Big\{< P_{1\tilde{u}(\lambda)}\frac{\partial \tilde{u}(\lambda)}{\partial t}, u-u_0 > + \lambda < \frac{\partial P_{1\tilde{u}(\lambda)}}{\partial t}(u-u_0), u-u_0 >$$

$$- \lambda < 2\frac{\partial^2 P_2}{\partial t^2}(u-u_0), u-u_0 > - \lambda < 2\frac{\partial P_2}{\partial t}(u_t - u_{0_t}), u-u_0 >\Big\} d\lambda dt.$$

Hence,

$$F_N[u] = \int_{t_0}^{t_1}\int_0^1 \Big[< -P_2(u_t - u_{0_t}), \frac{\partial \tilde{u}(\lambda)}{\partial t} > - < P_{1\tilde{u}(\lambda)}(u-u_0), \frac{\partial \tilde{u}(\lambda)}{\partial t} >$$

$$+ < \frac{\partial P_2}{\partial t}(u-u_0), \frac{\partial \tilde{u}(\lambda)}{\partial t} > - \lambda < \frac{\partial^2 P_2}{\partial t^2}(u-u_0), u-u_0 >$$

$$+ < Q(\tilde{u}(\lambda)) + \lambda \frac{\partial P_{1\tilde{u}(\lambda)}}{\partial t}(u-u_0), u-u_0 > \Big] \, d\lambda dt + F_N[u_0].$$

Using (2.15) - (2.17), we get

$$F_N[u] = \int_{t_0}^{t_1} \Big[< \mathcal{R}_2 u_t, u_t > + < \mathcal{R}_1(u) - \frac{\partial \mathcal{R}_2}{\partial t} u, u_t > + \mathcal{B}[u] \Big] \, dt + F_N[u_0]. \quad (2.18)$$

It is easy to show that

$$\delta F_N[u,h] = \int_{t_0}^{t_1} [< \mathcal{R}_2 h_t + \mathcal{R}'_{1u} h, u_t > + < \mathcal{R}_2 u_t + \mathcal{R}_1(u), h_t >$$

$$- < \frac{\partial \mathcal{R}_2}{\partial t} h, u_t > - < \frac{\partial \mathcal{R}_2}{\partial t} u, h_t > + < grad_{\Phi_1} \mathcal{B}[u], h > \Big] \, dt$$

$$= \int_{t_0}^{t_1} \Big[< -\frac{\partial \mathcal{R}_2^*}{\partial t} u_t, h > - < \mathcal{R}_2^* u_{tt}, h > - < \frac{\partial \mathcal{R}_2}{\partial t} u_t, h > - < \mathcal{R}_2 u_{tt}, h >$$

$$+ < \mathcal{R}'^*_{1u} u_t, h > - < \frac{\partial \mathcal{R}_1}{\partial t}(u), h > - < \mathcal{R}'_{1u} u_t, h > - \left(\frac{\partial \mathcal{R}_2}{\partial t} \right)^* u_t, h >$$

$$+ < \frac{\partial^2 \mathcal{R}_2}{\partial t^2} u, h > + < \frac{\partial \mathcal{R}_2}{\partial t} u_t, h > + < grad_{\Phi_1} \mathcal{B}[u], h > \Big] \, dt$$

$$= \int_{t_0}^{t_1} < (-\mathcal{R}_2 - \mathcal{R}_2^*) u_{tt} + (\mathcal{R}'^*_{1u} - \mathcal{R}'_{1u} - 2 \frac{\partial \mathcal{R}_2^*}{\partial t}) u_t$$

$$+ \left(grad_{\Phi_1} \mathcal{B}[u] - \frac{\partial \mathcal{R}_1}{\partial t}(u) + \frac{\partial^2 \mathcal{R}_2}{\partial t^2} u \right), h > \, dt = \int_{t_0}^{t_1} < N(u), h > \, dt$$

$$\forall u \in D(N), \forall h \in D(N'_u).$$

Thus, the sufficiency of representation (2.14) is proved. On the other hand, if $D_t^* = -D_t$ on $D(N'_u)$ and equation (1.1) can be represented in the form (2.14), then

$$P_2 = -\mathcal{R}_2 - \mathcal{R}_2^*, \quad (2.19)$$

$$P_{1u} = \mathcal{R}'^*_{1u} - \mathcal{R}'_{1u} - 2 \frac{\partial \mathcal{R}_2^*}{\partial t}, \quad (2.20)$$

$$Q(u) = grad_{\Phi_1} \mathcal{B}[u] - \frac{\partial \mathcal{R}_1}{\partial t}(u) + \frac{\partial^2 \mathcal{R}_2}{\partial t^2} u. \quad (2.21)$$

□

Remark 2. If $0_U \in D(N)$, then the operators $\mathcal{R}_1$, $\mathcal{R}_2$ can be found by formulas

$$< \mathcal{R}_1(u), u_t >= \int\limits_0^1 < -\lambda P_{1\lambda u} u, u_t > \, d\lambda,$$

$$< \mathcal{R}_2 u_t, u_t >= \int\limits_0^1 < -\lambda P_2 u_t, u_t > \, d\lambda.$$

In the case of bilinear mapping (1.10) we have

$$\mathcal{R}_1(u) = \int\limits_0^1 -\lambda P_{1\lambda u} u \, d\lambda,$$

$$\mathcal{R}_2 = -\frac{1}{2} P_2,$$

Example. Consider the following partial differential equation:

$$N(u) \equiv u_{tt} + 2\beta v(t) u_{tx} + u_{xxxx} + v^2(t) u_{xx} + \beta v'(t) u_x = 0, \tag{2.22}$$

$$(x, t) \in Q_T = (a, b) \times (0, T),$$

where β is a constant, $v(t)$ is a fixed function and $u(x, t)$ is the unknown function.

We denote by $D(N)$ the domain of definition of the operator N in (2.22):

$$D(N) = \{u \in U = C^{2,4}_{t,x}(\overline{Q_T}) : u|_{t=0} = \phi_1(x), \; u_t|_{t=0} = \phi_2(x) \; (x \in (a, b)), \tag{2.23}$$

$$u|_{x=a} = \psi_1(t), \; u|_{x=b} = \psi_2(t) \; (t \in (0, T)\},$$

where ϕ_i, ψ_i (i=1,2) are given functions.

We denote $V = C(\overline{Q_T})$ and determine the bilinear form $\Phi(\cdot, \cdot) : V \times U \to \mathbb{R}$ by setting

$$\Phi(v, g) = \int\limits_0^T \int\limits_a^b v(x, t) g(x, t) \, dx dt. \tag{2.24}$$

Let us note that equation (2.22) has the structure of equation (1.1). Indeed, in this case

$$P_2 = I, \qquad P_1 = 2\beta v(t) D_x, \qquad Q(u) = u_{xxxx} + v^2(t) u_{xx} + \beta v'(t) u_x.$$

Let us show that conditions (2.1) – (2.6) are satisfied. We have

(2.1) $\Longrightarrow I - I = 0$,
(2.2) $\Longrightarrow 0 = 0$,
(2.3) $\Longrightarrow 2\beta v(t) D_x - 2\beta v(t) D_x = 0$,
(2.4) $\Longrightarrow -2\beta v'(t) D_x + D_x^4 + v^2(t) D_x^2 + \beta v'(t) D_x - D_x^4 - v^2(t) D_x^2 + \beta v'(t) D_x = 0$,
(2.5) $\Longrightarrow 0 = 0$,
(2.6) $\Longrightarrow 0 = 0$.

Hence, equation (2.22) can be represented in the form of the Euler-Lagrange equation. Further, using (2.15) – (2.17) one obtains

$$\mathcal{R}_2 = -\frac{1}{2}I, \qquad \mathcal{R}_1 = -\beta v(t) D_x, \qquad \mathcal{B}[u] = \frac{1}{2}\int\limits_a^b \{u_{xx}^2 - v^2(t)u_x^2\}\, dx.$$

Thus, the potential of the operator N in (2.22) can be written in the form

$$F_N[u] = \frac{1}{2}\int\limits_0^T \int\limits_a^b \{-u_t^2 - 2\beta v(t) u_x u_t + u_{xx}^2 - v^2(t)u_x^2\}\, dxdt + F_N[u_0]. \tag{2.25}$$

Let us note that potential (2.25) can be used to obtain an infinite number of first integrals of the given equation (2.22).

Suppose that $u \in C_{t,x}^{2,\infty}(\overline{Q_T})$, then the first integrals of (2.22) are

$$I_k[u] = \int\limits_a^b (-u_t - \beta v(t) u_x)\, \frac{\partial^{2k+1} u}{\partial x^{2k+1}} dx, \quad (k = 0, 1, 2, ...)$$

(see [7]).

This paper can be considered as a continuation of [4].

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