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GENERALIZATIONS OF BORG'S UNIQUENESS THEOREM TO THE CASE OF NONSEPARATED BOUNDARY CONDITIONS

A.M. Akhtyamov, V.A. Sadovnichy, Ya.T. Sultanaev

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Abstract. Inverse Sturm–Liouville problems and generalizations of Borg's uniqueness theorem to the case of general boundary conditions are considered. Chudov, Marchenko, Krein, Karaseva and authors' generalizations are adduced. New generalizations of Borg, Marchenko and Karaseva's uniqueness theorem to the case of nonseparated boundary conditions are obtained. Appropriate examples and counterexample are given.

1 Introduction. The generalizations of Borg's uniqueness theorem (Chudov, Marchenko, Krein and Karaseva's uniqueness theorems)

Inverse Sturm–Liouville problems were first studied in Ambarzumijan's 1929 paper [2], where he considered the boundary value problem

$$-y'' + q(x)\,y = \lambda\,y, \qquad y'(0) = y'(\pi) = 0$$

and showed that, if $\int_0^\pi q(x)\,dx = 0$ and the eigenvalues are $1^2, 2^2, \ldots$, then $q(x)$ vanishes identically. This work showed that the boundary value problem can be recovered from the set of its eigenvalues.

However, in 1946, Borg [6] established

Borg's Theorem 1 (1946). *The problem*

$$-y'' + q(x)\,y = \mu\,y, \qquad y'(0) - h\,y(0) = y'(\pi) + H\,y(\pi) = 0,$$

(where the coefficients h and H are not necessarily zero) cannot generally be recovered from the set of its eigenvalues. For case of $h \neq 0$ the recovery of $q(x)$ is possible if we additionally know the set of eigenvalues of the auxiliary problem

$$-y'' + q(x)\,y = \lambda\,y, \qquad y(0) = y'(\pi) + H\,y(\pi) = 0.$$

More detailed formulation of Borg's theorem is as follows:
Borg's Theorem 2 (1946) [6]. *Let the equations*

$$-y'' + q(x)\, y = \lambda\, y \tag{1.1}$$

$$-y'' + \widetilde{q}(x)\, y = \lambda\, y \tag{1.2}$$

have the same spectrum for the boundary conditions

$$a\, y(0) + b\, y'(0) = 0, \quad c\, y(\pi) + d\, y'(\pi) = 0, \tag{1.3}$$

and for the boundary conditions

$$a\, y(0) + b\, y'(0) = 0, \quad c_1\, y(\pi) + d_1\, y'(\pi) = 0. \tag{1.4}$$

If

$$d\, d_1 = 0, \quad |d| + |d_1| > 0, \tag{1.5}$$

then $q(x) = \widetilde{q}(x)$ *almost everywhere on* $[0, \pi]$.

In the paper of L.A. Chudov [7] it is shown, that it is possible to get rid of restrictive conditions (1.5). He proved the following uniqueness theorem.
Chudov's Theorem (1949) [7]. *Let equations (1.1) and (1.2) have the same spectrum for boundary conditions (1.3) and (1.4),*

$$\begin{vmatrix} c & c_1 \\ d & d_1 \end{vmatrix} \neq 0. \tag{1.6}$$

Then $q(x) = \widetilde{q}(x)$ *almost everywhere on the* $[0, \pi]$.

Consider the special case of the problem:

$$-y'' + q(x)\, y = \lambda\, y, \quad y'(0) - h\, y(0) = 0, \quad y'(\pi) + H\, y(\pi) = 0, \tag{1.7}$$

By $S(q, h, H)$ denote the spectrum of boundary problem (1.7).

It is follows by Chudov's theorem that if

$$S(q, h, H) = S(\widetilde{q}, h, H), \quad S(q, h_1, H) = S(\widetilde{q}, h_1, H),$$

for some functions $q(x)$ and $\widetilde{q}(x)$, and numbers h, h_1 ($h \neq h_1$) and H, then $q(x) = \widetilde{q}(x)$ ($0 \leq x \leq \pi$) almost everywhere.

V.A. Marchenko (1950) [22] showed, that not only $q(x)$, but also the numbers h, h_1, H are uniquely recovered using spectra $S(q, h, H)$ and $S(q, h_1, H)$.

T.M. Karaseva (1953) [14] generalized the result of Marchenko for the case, when $q(x)$ is a complex-valued function, and the numbers h, h_1, H are complex. This T.M. Karaseva's result we will call **Borg, Marchenko and Karaseva's uniqueness theorem**.

M.G. Krein (1951) [16, 17] gave another generalization of Chudov's theorem. He showed, that a nonnegative summable function $\rho(x)$ ($0 \leq x \leq \pi$) is uniquely recovered by two spectra $S(\rho, h, H)$ and $S(\rho, h_1, H)$ of the eigenvalue problems for the equation

$$y'' + \lambda\, \rho(x) y = 0, \tag{1.8}$$

with the boundary conditions $y'(0)-h\,y(0)=0$, $y'(\pi)+H\,y(\pi)=0$ and $y'(0)-h_1\,y(0)=0$, $y'(\pi)+H\,y(\pi)=0$.

Note, that equation (1.7) by change of variables can be reduced to equation (1.8). The function $\rho(x)$ obtained by this procedure is not only positive, but is also absolutely continuous with its first derivative.

Inverse Sturm-Liouville problems with separated boundary conditions were considered in the works of A. Andrew [3], P.A. Binding and B.A. Watson [5], N. Guliyev [9], A. Kammanee and C. Böckmann [12], A. Kostenko, A. Sakhnovich and G. Tesch [15], N. Levinson [18], B.M. Levitan and M.G. Gasymov [19, 20, 21], V.A. Marchenko [23, 24], L. Nizhnik [27], A.M. Savchuk and A.A. Shkalikov [40], L.A. Sakhnovich [39], I.V. Stankevich [41], A.N. Tihonov [42], V.A. Yurko [45], and other works. The main methods of inverse spectral problems were developed: Borg's method, the transformations operator method, the spectral maps method, and others.

It has taken a long time to obtain the uniqueness theorems for nonselfadjoint spectral problems with nonseparated boundary conditions on a finite interval. The first result in this direction was obtained by V.A. Sadovnichy only in 1972 [30] (20 years after the statement of the problem). After that several uniqueness theorems for inverse selfadjoint problems with nonseparated boundary conditions were proved (P.A. Binding and H. Volkmer [4], M.G. Gasymov, I.M. Guseinov, I.M. Nabiev [8, 10, 11], V.A. Marchenko [25], O.A. Plaksina [28, 29], V.A. Yurko [43, 44], and others) and for inverse nonselfadjoint problems with nonseparated boundary conditions (V.A. Sadovnichy, Ya.T. Sultanaev, A.M. Akhtyamov, B.E. Kanguzhin [13, 31, 32, 33, 34, 35]). However, the results obtained for nonselfadjoint problems were not direct generalizations of Borg's classical uniqueness theorem.

Many authors attempted to generalize Borg's classical result for nonselfadjoint problems with nonseparated boundary conditions since 1949. But only in 2009 in [36, 37] a generalization of Borg's Theorem 1 for the case of inverse nonselfadjoint problems with general boundary conditions, including nonseparated ones, was obtained. This generalization is formulated below in Section 2.

In Section 3 we prove the new generalizations. They are generalizations of Borg, Marchenko and Karaseva's uniqueness theorem.

2 Generalizations of Borg's Theorem 1 for an operator pencil with nonseparated boundary conditions

Consider the following three boundary value problems.

L:

$$ly = y'' + (s^2 + i\,s\,q_1(x) + q(x))\,y = 0, \tag{2.1}$$

$$U_1(y) = y'(0) + (a_{11} + i\,s\,a_{12})\,y(0) + (a_{13} + i\,s\,a_{14})\,y(\pi) = 0, \tag{2.2}$$

$$U_1(y) = y'(\pi) + (a_{21} + i\,s\,a_{22})\,y(0) + (a_{23} + i\,s\,a_{24})\,y(\pi) = 0, \tag{2.3}$$

L^1 :

$$\begin{aligned} & y'' + (\mu^2 + i\,\mu\, q_1(x) + q(x))\, y = 0, \\ V_1(y) &= y'(0) + (a_{11} + i\,\mu\, a_{12})\, y(0) = 0, \\ V_2(y) &= y'(\pi) + (a_{23} + i\,\mu\, a_{24})\, y(\pi) = 0, \end{aligned}$$

и L^2 :

$$\begin{aligned} & y'' + (\nu^2 + i\,\nu\, q_1(x) + q(x))\, y = 0, \\ & y(0) = 0, \\ V_2(y) &= y'(\pi) + (a_{23} + i\,\nu\, a_{24})\, y(\pi) = 0, \end{aligned}$$

where $q \in L_1[0,\pi]$ and $q_1 \in W_1[0,\pi]$ are complex-valued functions and a_{ij} $(i = 1,2; j = 1,2,3,4)$ are complex numbers with $a_{12} \neq \pm 1$, $a_{24} \neq \pm 1$.

The inverse problem is formulated as follows.

Problem. Given the respective eigenvalues $\{s_k\}$, $\{\mu_k\}$, and $\{\nu_k\}$ of problems L, L^1, and respectively L^2 , find the coefficients of pencil L, i.e., the coefficients $q(x)$, $q_1(x)$, a_{ij} $(i = 1,2;\ j = 1,2,3,4)$.

In what follows, a problem of the type L but with different coefficients in the equation and with different parameters in the boundary forms is denoted by $\widetilde{L}$.

Throughout this section, we assume that a symbol with a tilde in problem L denotes an object similar to that in problem $\widetilde{L}$.

Theorem 2.1. [1, 36, 37] (the duality of the solution). *If* $\{s_k\} = \{\widetilde{s}_k\}$, $\{p_k\} = \{\widetilde{p}_k\}$, $\{\nu_k\} = \{\widetilde{\nu}_k\}$, *then either* $q(x) = \widetilde{q}(x)$, $q_1(x) = \widetilde{q}_1(x)$, *and* $a_{ij} = \widetilde{a}_{ij}$, $(i = 1,2;$ $j = 1,2,3,4)$, *or* $q(x) = \widetilde{q}(x)$, $q_1(x) = \widetilde{q}_1(x)$, $a_{11} = \widetilde{a}_{11}$, $a_{12} = \widetilde{a}_{12}$, $a_{13} = -\widetilde{a}_{21}$, $a_{14} = -\widetilde{a}_{22}$, $a_{21} = -\widetilde{a}_{13}$, $a_{22} = -\widetilde{a}_{14}$, $a_{23} = \widetilde{a}_{23}$, *and* $a_{24} = \widetilde{a}_{24}$.

As a special case of Theorem 2.1, we obtain Theorem 2.2.

Theorem 2.2. (uniqueness of a solution). *If* $\{s_k\} = \{\widetilde{s}_k\}$, $\{\mu_k\} = \{\widetilde{\mu}_k\}$, $\{\nu_k\} = \{\widetilde{\nu}_k\}$, $a_{13} = \widetilde{a}_{13}$, *and* $a_{14} = \widetilde{a}_{14}$, *then* $L = \widetilde{L}$.

Thus, given three spectra, the coefficients of pencil (2.1)–(2.3), are uniquely determined if a_{13} and a_{14} are known.

As a special case of Theorem 2.2, we obtain uniqueness Borg's Theorem 1. Indeed, problems L, L^1, and L^2 for $q_1(x) \equiv 0$, $a_{13} = a_{14} = a_{12} = a_{22} = a_{24} = 0$, $a_{11} = -h$, $a_{23} = H$ are formulated as follows.

$L{=}L^1$: $\quad -y'' + q(x)\, y = \lambda\, y, \qquad y'(0) - h\, y(0) = y'(\pi) + H\, y(\pi) = 0 \quad (\lambda = s^2),$

L^2: $\quad -y'' + q(x)\, y = \mu\, y, \qquad y(0) = y'(\pi) + H\, y(\pi) = 0 \quad (\mu = \nu^2).$

So, Borg's Theorem 1 is a particular case of Theorem 2.2. Indeed, problem L coincides with problem L^1 for separated conditions ($a_{12} = a_{21} = 0$). Therefore, to recover problems $L{=}L^1$ and L^2 uniquely, we can use only two spectra (the spectra of problems $L{=}L^1$ and L^2).

In [37, 36] it is shown the existence of inverse problem solution, the solution method of this inverse problems is found, examples and counterexamples are given.

These results generalize the results of many authors. Note also that the obtained theorems are true not only for operators, but also for operator pencils. These results are not only generalizations of Borg's Theorem 1 to the case of general boundary conditions, but to the operator pencil case too. This is very important, because the problems of voltage and current force oscillations in a wire lead to operator L. Identification of the function $q_1(x)$ by the eigenvalues means in this case identification of the place and volume of current leakage in a wire by the natural frequencies of current force oscillation, measured in one of accessible places, lying far from an inaccessible place.

3 Generalizations of Borg, Marchenko and Karaseva's theorem

In this section new generalizations of Borg's theorem are obtained. These results are generalizations of Borg, Marchenko and Karaseva's Theorem and were announced by the authors in [38].

Let L_0 denote the following Sturm-Liouville eigenvalue problem:

Problem L_0:

$$ly = -y'' + q(x)\,y = \lambda\,y = s^2\,y, \tag{3.1}$$

$$U_1(y) = y'(0) + a_{11}\,y(0) + a_{12}\,y(\pi) = 0, \tag{3.2}$$

$$U_2(y) = y'(\pi) + a_{21}\,y(0) + a_{22}\,y(\pi) = 0 \tag{3.3}$$

($x \in [0,\pi]$, $y = y(x) \in C^2[0,\pi]$, $q(x)$ is a summable function, and a_{ij}, i, $j = 1, 2$ are complex constants).

Along with problem L_0, we consider two problems with separated boundary conditions:

Problem L_1 :

$$\begin{aligned} ly &= -y'' + q(x)\,y = \lambda\,y, \\ U_{1,1}(y) &= y'(0) + a_{11}\,y(0) = 0, \\ U_{2,1}(y) &= y'(\pi) + (a_{21} + a_{22})\,y(\pi) = 0. \end{aligned}$$

Problem L_2 :

$$\begin{aligned} ly &= -y'' + q(x)\,y = \lambda\,y, \\ U_{1,2}(y) &= y'(0) + a\,y(0) = 0, \\ U_{2,2}(y) &= y'(\pi) + (a_{21} + a_{22})\,y(\pi) = 0, \end{aligned}$$

where a is a constant deferent from a_{11}.

Let λ_k, μ_k, and ν_k be the eigenvalues of problems L, L_1, L_2 respectively, indexed in increasing order of their absolute values.

An inverse problem for L_0 is formulated as follows.

Inverse problem. Let the potential function $q(x)$ and the coefficients of the boundary conditions in problems L_j $(j = 0, 1, 2)$ be unknown, while the eigenvalues of problems L_j $(j = 0, 1, 2)$ be given. The goal is to find $q(x)$ and the boundary conditions in problems L_j $(j = 0, 1, 2)$ from their eigenvalues.

In what follows, a problem of type L_j with different coefficients in the equation and with different parameters in the boundary forms is denoted by $\widetilde{\mathrm{L}}_j$. Moreover, if a certain symbol denotes an object in problem L_j, the same symbol with a tilde denotes its counterpart in problem $\widetilde{\mathrm{L}}_j$.

Theorem 3.1. *Let $a_{11} \neq a$, $\widetilde{a}_{11} \neq \widetilde{a}$. If the eigenvalues of problems* L_j *are equal to those of problems* $\widetilde{\mathrm{L}}_j$ *counting their algebraic multiplicities, then the coefficients of the equations and the constants in the boundary conditions of* L_j *and* $\widetilde{\mathrm{L}}_j$ *($j = 0, 1, 2$) are also equal to each other; i.e., $q(x) = \widetilde{q}(x)$, $a = \widetilde{a}$, and $a_{ij} = \widetilde{a}_{ij}$ for $i, j = 1, 2$.*

Proof of Theorem 3.1 Applying Borg, Marchenko and Karaseva's uniqueness theorem ([6, 14, 19, p. 9]) to problems L_1 and L_2, we obtain

$$q(x) = \widetilde{q}(x), \qquad a_{11} = \widetilde{a}_{11}, \qquad a = \widetilde{a}, \qquad a_{21} + a_{22} = \widetilde{a}_{21} + \widetilde{a}_{22}. \tag{3.4}$$

To complete the proof, we have to show that $a_{12} = \widetilde{a}_{12}$, $a_{21} = \widetilde{a}_{21}$, and $a_{22} = \widetilde{a}_{22}$.

Let $y_1(x, \lambda)$ and $y_2(x, \lambda)$ be linearly independent solutions of equation (3.1) that satisfy the conditions

$$y_1(0, \lambda) = 1, \quad y_1'(0, \lambda) = 0, \quad y_2(0, \lambda) = 0, \quad y_2'(0, \lambda) = 1. \tag{3.5}$$

Then we have the asymptotic formulas

$$\begin{aligned} y_1(x, \lambda) &= \cos sx + \tfrac{1}{s}\, u(x) \sin sx + \mathcal{O}\left(\tfrac{1}{s^2}\right), \\ y_2(x, \lambda) &= \tfrac{1}{s} \sin sx - \tfrac{1}{s^2}\, u(x) \cos sx + \mathcal{O}\left(\tfrac{1}{s^3}\right), \\ y_1'(x, \lambda) &= -s \sin sx + u(x) \cos sx + \mathcal{O}\left(\tfrac{1}{s}\right), \\ y_2'(x, \lambda) &= \cos sx + \tfrac{1}{s}\, u(x) \sin sx + \mathcal{O}\left(\tfrac{1}{s^2}\right) \end{aligned} \tag{3.6}$$

where $u(x) = \frac{1}{2} \int_0^x q(t)\, dt$, for sufficiently large $\lambda = s^2 \in \mathbb{R}$ ([26, pp. 62–65]).

The eigenvalues of problem L are the roots of the entire function

$$\Delta(\lambda) = \begin{vmatrix} U_1(y_1(x, \lambda)) & U_1(y_2(x, \lambda)) \\ U_2(y_1(x, \lambda)) & U_2(y_2(x, \lambda)) \end{vmatrix}, \tag{3.7}$$

and the algebraic multiplicity of an eigenvalue is equal to the multiplicity of the same root of $\Delta(\lambda)$ ([26, p. 29]). It follows that

$$\begin{aligned} \Delta(\lambda) = {} & (a_{11} + a_{12}\, y_1(\pi, \lambda)) \cdot (y_2'(\pi, \lambda) + a_{22}\, y_2(\pi, \lambda)) - \\ & -(1 + a_{12}\, y_2(\pi, \lambda)) \cdot (y_1'(\pi, \lambda) + a_{21} + a_{22}\, y_1(\pi, \lambda)). \end{aligned} \tag{3.8}$$

Substituting the asymptotic formulas for $y_1(x, \lambda)$ and $y_2(x, \lambda))$ in (3.8) yields

$$\begin{aligned} \Delta(\lambda) \;=\; & a_{11} \cos \sqrt{\lambda}\pi + a_{12} + \sqrt{\lambda} \sin \sqrt{\lambda}\pi - u(\pi) \cos \sqrt{\lambda}\pi - \\ & -a_{21} - a_{22} \cos \sqrt{\lambda}\pi + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \end{aligned}$$

It can be seen that $\Delta(\lambda)$ is an entire function of order $\frac{1}{2}$. Moreover, according to the assumptions of the theorem, the eigenvalues of L_0 and $\widetilde{\mathrm{L}}_0$ counted taking into account their algebraic multiplicities are equal to each other. Therefore, the Hadamard factorization theorem implies that $\Delta(\lambda) \equiv C\, \widetilde{\Delta}(\lambda)$, where C is a nonzero constant. It follows that

$$\begin{aligned} \Delta(\lambda) - C\widetilde{\Delta}(\lambda) \;\equiv\; & \\ \equiv\; & (a_{11} - \widetilde{a}_{11}\, C) \cos \sqrt{\lambda}\pi + (a_{12} - \widetilde{a}_{12}\, C) + \\ & +(1 - C) \sqrt{\lambda} \sin \sqrt{\lambda}\pi - (1 - C)\, u(\pi) \cos \sqrt{\lambda}\pi + \\ & -(a_{21} - \widetilde{a}_{21}\, C) - (a_{22} - \widetilde{a}_{22}\, C) \cos \sqrt{\lambda}\pi + \\ & +(1 - C)\, \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) \;\equiv\; 0. \end{aligned} \tag{3.9}$$

Here, 1, $\sin\sqrt{\lambda}\pi$, $\cos\sqrt{\lambda}\pi$, $\sqrt{\lambda}\cdot\sin\sqrt{\lambda}\pi$, and $\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$ are linearly independent functions of λ. (This can easily be verified using the definition of linearly independent functions.) Therefore, $C=1$ and

$$\begin{gathered}(a_{12}-\widetilde{a}_{12})-(a_{21}-\widetilde{a}_{21})+\\ +(a_{11}-\widetilde{a}_{11}-a_{22}+\widetilde{a}_{22})\cos\sqrt{\lambda}\pi+\mathcal{O}\left(\tfrac{1}{\lambda}\right)\equiv 0.\end{gathered}\tag{3.10}$$

Combining this with (3.4) gives $a_{12}=\widetilde{a}_{12}$, $a_{21}=\widetilde{a}_{21}$, $a_{22}=\widetilde{a}_{22}$. □

Note that Borg, Marchenko and Karaseva's Theorem [14] is a special case of Theorem 3.1. Indeed, in the case of separated conditions ($a_{12}=a_{21}=0$), problem L_0 coincides with L_1. Therefore, problems L_0=L_1 and L_2 can be uniquely recovered using only two spectra (namely, those of L_0=L_1 and L_2). Moreover, a stronger result than Theorem 3.1 holds true. Given two sufficiently large numbers N_1 and N_2 of different parity, let the corresponding eigenvalues of problem L_0 be denoted by λ_{N_1} and λ_{N_2}.

Theorem 3.2. *Let $a_{11}\neq a$ and $\widetilde{a}_{11}\neq\widetilde{a}$. If the eigenvalues of problems L_1 and $\widetilde{\mathrm{L}}_1$ counted taking into account their algebraic multiplicities coincide, so do the eigenvalues of problems L_2 and $\widetilde{\mathrm{L}}_2$ and, additionally, $\lambda_{N_1}=\widetilde{\lambda}_{N_1}$ and $\lambda_{N_2}=\widetilde{\lambda}_{N_2}$, then the coefficients of the equations and the constants in the boundary conditions in problems L_j and $\widetilde{\mathrm{L}}_j$ ($j=0,1,2$) coincide as well; i.e., $q(x)=\widetilde{q}(x)$, $a=\widetilde{a}$, and $a_{ij}=\widetilde{a}_{ij}$ for $i,j=1,2$.*

Proof of Theorem 3.2. Applying Borg, Marchenko and Karaseva's uniqueness theorem ([6, 14, 19, p. 9]) to problems L_1 and L_2, yields (3.4).

Since λ_{N_1} and λ_{N_2} are eigenvalues of L, we have

$$\Delta(\lambda_{N_i})=0,\qquad i=1,2,\tag{3.11}$$

where $\Delta(\lambda)$ is the characteristic determinant (3.8) of problem L.

It is well known that $\sqrt{\lambda_{N_i}}=N_i+\mathcal{O}\left(\frac{1}{N_i}\right)$, $i=1,2$. Substituting this expression in (3.11) and taking into account (3.6) and (3.8), we obtain

$$a_{22}\cdot e_i+a_{21}-a_{12}=a_{11}\cdot e_i-y_1'(\pi,\lambda_{N_i})+\mathcal{O}\left(\frac{1}{N_i}\right),\quad i=1,2,\tag{3.12}$$

where

$$e_i=\begin{cases}+1 \text{ if } N_i \text{ is even},\\ -1 \text{ if } N_i \text{ is odd}.\end{cases}$$

Similarly, for problem $\widetilde{\mathrm{L}}$,

$$\widetilde{a}_{22}\cdot e_i+\widetilde{a}_{21}-\widetilde{a}_{12}=a_{11}\cdot e_i-y_1'(\pi,\lambda_{N_i})+\mathcal{O}\left(\frac{1}{N_i}\right),\quad i=1,2.\tag{3.13}$$

(Here, $\widetilde{y}_1'(\pi,\widetilde{\lambda}_{N_i})=y_1'(\pi,\lambda_{N_i})$ since $\widetilde{\lambda}_{N_i}=\lambda_{N_i}$ and $\widetilde{q}(x)=q(x)$). Subtracting equation (3.13) term-by-term from equation (3.12) with $e_i=+1$ and using (3.4), we see that $\widetilde{a}_{12}-a_{12}=\mathcal{O}\left(\frac{1}{N_i}\right)$. Passing to the limit as $N_i\to\infty$, in the last equality yields

$$\widetilde{a}_{12}=a_{12}.\tag{3.14}$$

Subtracting equation (3.12) term-by-term from (3.13) with $e_i = -1$ and passing to the limit as $N_i \to \infty$ yields $-\widetilde{a}_{22} + \widetilde{a}_{21} = -a_{22} + a_{21}$. Combining this with (3.4), we finally have

$$a_{21} = \widetilde{a}_{21}, \quad a_{22} = \widetilde{a}_{22}. \qquad \square$$

Since the proof of Theorem 3.2 is based on an asymptotic relation between the eigenvalues, the basic questions are whether or not the following assumptions of Theorem 3.2 can be omitted:

(i) the indices N_1 and N_2 (of the eigenvalues λ_{N_2} and λ_{N_2}) are sufficiently large,

(ii) the eigenvalue indices have different parities.

Below are two counterexamples. The first implies that Theorem 3.2 does not hold if N_1 or N_2 fails to be sufficiently large. The second counterexample shows that, if N_1 and N_2 in Theorem 3.2 are assumed to be sufficiently large numbers of the same parity, then problem L_0 is not uniquely recovered.

Counterexample 1. (Numbers N_1 and N_2 of different parity, but N_1 is not sufficiently large). Problems L_0 and $\widetilde{\mathrm{L}}_0$ with the coefficients $q(x) = \widetilde{q}(x) = 0$, $a = \widetilde{a} \neq 0$, $a_{11} = \widetilde{a}_{11} = a_{12} = 0$, $\widetilde{a}_{12} = -1/2$, $a_{21} = a_{22} = 1/2$, $\widetilde{a}_{21} = 1/4$, and $\widetilde{a}_{22} = 3/4$ do not coincide. However, $\mathrm{L}_1 = \widetilde{\mathrm{L}}_1$, $\mathrm{L}_2 = \widetilde{\mathrm{L}}_2$, $\lambda_{N_1} = \widetilde{\lambda}_{N_1}$, and $\lambda_{N_2} = \widetilde{\lambda}_{N_2}$, where numbers N_1 and N_2 of different parity. Theorem 3.2 does not hold, because N_1 or N_2 fails to be sufficiently large.

Indeed, we have

problem L_0 :

$$-y'' = \lambda\, y, \qquad y'(0) - \frac{1}{2}\, y(\pi) = 0, \quad y'(\pi) + \frac{1}{2}\, y(0) + \frac{1}{2}\, y(\pi) = 0,$$

problem $\widetilde{\mathrm{L}}_0$:

$$-y'' = \lambda\, y, \qquad y'(0) - \frac{1}{2}\, y(\pi) = 0, \quad y'(\pi) + \frac{1}{4}\, y(0) + \frac{3}{4}\, y(\pi) = 0,$$

problem $\mathrm{L}_1 = \widetilde{\mathrm{L}}_1$:

$$-y'' = \lambda\, y, \qquad y'(0) = 0, \quad y'(\pi) + y(\pi) = 0,$$

problem $\mathrm{L}_2 = \widetilde{\mathrm{L}}_2$:

$$-y'' = \lambda\, y, \qquad y'(0) + a\, y(0) = 0, \quad y'(\pi) + y(\pi) = 0,$$

$$\Delta(\lambda) = \sqrt{\lambda}\, \sin\sqrt{\lambda}\,\pi - \frac{1 + \cos\sqrt{\lambda}\,\pi}{2} = \cos\frac{\sqrt{\lambda}\,\pi}{2}\left(2\sqrt{\lambda}\, \sin\frac{\sqrt{\lambda}\,\pi}{2} - \cos\frac{\sqrt{\lambda}\,\pi}{2}\right),$$

$$\widetilde{\Delta}(\lambda) = \cos\frac{\sqrt{\lambda}\,\pi}{2}\left(-\frac{3}{2}\cos\frac{\sqrt{\lambda}\,\pi}{2} + 2\left(\sqrt{\lambda} + \frac{1}{8\sqrt{\lambda}}\right)\sin\frac{\sqrt{\lambda}\,\pi}{2}\right).$$

Therefore, the eigenvalues of problem L with even indices are the roots of the function $\cos\frac{\sqrt{\lambda}\pi}{2}$, the eigenvalues of problem L with odd indices are the roots of the function $\tan\frac{\sqrt{\lambda}\pi}{2} - \frac{1}{2\sqrt{\lambda}}$, the eigenvalues of problem $\widetilde{\mathrm{L}}$ with even indices are the roots of the function $\cos\frac{\sqrt{\lambda}\pi}{2}$, the eigenvalues of problem $\widetilde{\mathrm{L}}$ with odd indices are the roots of the function $\tan\frac{\sqrt{\lambda}\pi}{2} - \frac{6\sqrt{\lambda}}{1+8\lambda}$.

The first eigenvalues of problems L and $\widetilde{\mathrm{L}}$ coincide: $\lambda_1 = \widetilde{\lambda}_1 = 1/4 < 1$. All eigenvalues with even indices of problem L and $\widetilde{\mathrm{L}}$ coincide too. The index of the first eigenvalue is odd. So, Theorem 3.2 does not hold for $N_1 = 1$ and sufficiently large even N_2 (N_1 fails to be sufficiently large).

Counterexample 2. (N_1 and N_2 are of the same parity, and can be arbitrary large). Problems L_0 and $\widetilde{\mathrm{L}}_0$ with the coefficients $q(x) = \widetilde{q}(x) = 0$, $a = \widetilde{a} \neq 0$, $a_{11} = \widetilde{a}_{11} = a_{12} = \widetilde{a}_{12} = 0$, $a_{21} = \widetilde{a}_{22} = -1$, and $a_{22} = \widetilde{a}_{21} = +1$ do not coincide, but $\sqrt{\lambda_{N_1}} = \sqrt{\widetilde{\lambda}_{N_1}}$, $\sqrt{\lambda_{N_2}} = \sqrt{\widetilde{\lambda}_{N_2}}$, where N_1 and N_2 are assumed to be sufficiently large even numbers; $\mathrm{L}_1 = \widetilde{\mathrm{L}}_1$, and $\mathrm{L}_2 = \widetilde{\mathrm{L}}_2$. Theorem 3.2 does not hold, because N_1 and N_2 are of the same parity.

Indeed, we have

problem L_0 :

$$-y'' = \lambda\, y, \qquad y'(0) = 0, \quad y'(\pi) - y(0) + y(\pi) = 0,$$

problem $\widetilde{\mathrm{L}}_0$:

$$-y'' = \lambda\, y, \qquad y'(0) = 0, \quad y'(\pi) + y(0) - y(\pi) = 0,$$

problem $\mathrm{L}_1 = \widetilde{\mathrm{L}}_1$:

$$-y'' = \lambda\, y, \qquad y'(0) = 0, \quad y'(\pi) = 0,$$

problem $\mathrm{L}_2 = \widetilde{\mathrm{L}}_2$:

$$-y'' = \lambda\, y, \qquad y'(0) + a\, y(0) = 0, \quad y'(\pi) = 0,$$

$$\Delta(\lambda) = \sqrt{\lambda}\,\sin\sqrt{\lambda}\,\pi + 1 - \cos\sqrt{\lambda}\,\pi = 2\,\sin\frac{\sqrt{\lambda}\,\pi}{2}\left(\sqrt{\lambda}\,\cos\frac{\sqrt{\lambda}\,\pi}{2} + \sin\frac{\sqrt{\lambda}\,\pi}{2}\right),$$

$$\widetilde{\Delta}(\lambda) = 2\,\sin\frac{\sqrt{\lambda}\,\pi}{2}\left(\sqrt{\lambda}\,\cos\frac{\sqrt{\lambda}\,\pi}{2} - \sin\frac{\sqrt{\lambda}\,\pi}{2}\right).$$

Therefore, the positive eigenvalues of problem L with even indices are the roots of the function $\sin\frac{\sqrt{\lambda}\pi}{2}$, the positive eigenvalues of problem L with odd indices are the roots of the function $\tan\frac{\sqrt{\lambda}\pi}{2} + \frac{1}{\sqrt{\lambda}}$, the positive eigenvalues of problem $\widetilde{\mathrm{L}}$ with even indices are the roots of the function $\sin\frac{\sqrt{\lambda}\pi}{2}$, the positive eigenvalues of problem $\widetilde{\mathrm{L}}$ with odd indices are the roots of the function $\tan\frac{\sqrt{\lambda}\pi}{2} - \frac{1}{\sqrt{\lambda}}$. All positive eigenvalues with even indices of problem L and $\widetilde{\mathrm{L}}$ coincide. Thus, if N_1 and N_2 are of the same parity and even are arbitrarily large numbers, then problem L_0 is not uniquely recovered.

Remark 1. The proof of Theorem 3.2 implies that the theorem assumptions $\lambda_{N_1} = \widetilde{\lambda}_{N_1}$ and $\lambda_{N_2} = \widetilde{\lambda}_{N_2}$ are only needed to derive $a_{12} = \widetilde{a}_{12}$ and $a_{21} = \widetilde{a}_{21}$. It follows from (3.4), (3.12), and (3.13) that, if the coefficients a_{21} and $\widetilde{a}_{21}$ in L_0 and $\widetilde{\mathrm{L}}_0$ vanish, then the theorem assumptions $\lambda_{N_1} = \widetilde{\lambda}_{N_1}$ and $\lambda_{N_2} = \widetilde{\lambda}_{N_2}$ can be replaced by one of them. If the coefficients a_{12} and $\widetilde{a}_{12}$ in L_0 and $\widetilde{\mathrm{L}}_0$ vanish, then two assumptions $\lambda_{N_1} = \widetilde{\lambda}_{N_1}$ and $\lambda_{N_2} = \widetilde{\lambda}_{N_2}$ in Theorem 3.2 can be replaced by the single one $\lambda_{N_1} = \widetilde{\lambda}_{N_1}$, where N_1 is a sufficiently large even number.

Remark 2. Borg, Marchenko and Karaseva's Theorem is a special case of Theorem 3.2. Indeed, in the case of separated conditions ($a_{12} = a_{21} = 0$, $\widetilde{a}_{12} = \widetilde{a}_{21} = 0$), problem L_0 coincides with L_1, while problem $\widetilde{\mathrm{L}}_0$ coincides with $\widetilde{\mathrm{L}}_1$. Therefore, problems L_0=L_1 and L_2 can be uniquely recovered using only two spectra (namely, those of L_0=L_1 and L_2). In this case, the assumptions $\lambda_{N_1} = \widetilde{\lambda}_{N_1}$ and $\lambda_{N_2} = \widetilde{\lambda}_{N_2}$ are not required in Theorem 3.2, since they follow from the fact that the eigenvalues of L_1 and $\widetilde{\mathrm{L}}_1$ are equal to each other.

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Azamat Akhtyamov
Bashkir State University,
ul. Frunze 32,
450074 Ufa, Russia;
and
Mavlutov Institute of Mechanics,
Russian Academy of Sciences,
71 Oktyabrya Pr,
450054 Ufa, Russia
E-mail: AkhtyamovAM@mail.ru

Victor Antonovich Sadovnichy
M.V. Lomonosov Moscow State University,
Leninskie gory,
119992 Moscow, Russia,
E-mail: rector@rector.msu.su

Yaudat Sultanaev
Mavlutov Institute of Mechanics,
Russian Academy of Sciences,
71 Oktyabrya Pr,
450054 Ufa, Russia
E-mail: sultanaevyt@bsunet.ru

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