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WEIGHTED ESTIMATE FOR A CLASS OF MATRICES ON THE CONE OF MONOTONE SEQUENCES

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Abstract. Weighted estimate for a class of non-negative lower triangular matrices has been established on the cone of monotone sequences.

1 Introduction

Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $u = \{u_i\}_{i=1}^{\infty}$, $v = \{v_i\}_{i=1}^{\infty}$ be positive sequences of real numbers. Let $l_{p,v}$ be the space of sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers such that

$$\|f\|_{p,v} := \left(\sum_{i=1}^{\infty} v_i |f_i|^p\right)^{\frac{1}{p}} < \infty, \qquad 1 < p < \infty.$$

Let $K^-_{p,v}$ be the cone of non-negative and non-increasing sequences $f = \{f_i\}_{i=1}^{\infty}$ from the $l_{p,v}$ space, briefly

$$K^-_{p,v} = \{0 \le f \downarrow : \quad f \in l_{p,v}\}.$$

We consider inequality of the following form

$$\left(\sum_{i=1}^{\infty} u_i \left(\sum_{j=1}^{i} a_{i,j} f_j\right)^q\right)^{\frac{1}{q}} \le C \left(\sum_{i=1}^{\infty} v_i f_i^{\,p}\right)^{\frac{1}{p}}, \qquad \forall f \in K^-_{p,v}, \tag{1.1}$$

where C is a positive constant independent of f and $(a_{i,j})$ is a non-negative triangular matrix with entries $a_{i,j} \ge 0$ for $i \ge j \ge 1$ and $a_{i,j} = 0$ for $i < j$.

For $a_{i,j} \equiv 1$, $i \ge j \ge 1$ inequality (1.1) was studied in [2] for $1 < p, q < \infty$.

In [5] necessary and sufficient conditions for the validity of (1.1) have been obtained for $1 < p \le q < \infty$ under the assumption that there exists $d \ge 1$ such that the inequalities

$$\frac{1}{d}(a_{i,k} + a_{k,j}) \le a_{i,j} \le d(a_{i,k} + a_{k,j}), \qquad i \ge k \ge j \ge 1 \tag{1.2}$$

hold.

A sequence $\{a_i\}_{i=1}^{\infty}$ is called almost non-decreasing (non-increasing), if there exists $c>0$ such that $ca_i \geq a_k$ ($a_k \leq ca_j$) for all $i \geq k \geq j \geq 1$.

In [3], [6] estimate (1.1) for all $f \in l_{p,v}$ was studied under the assumption that there exist $d \geq 1$ and a sequence of positive numbers $\{\omega_k\}_{k=1}^{\infty}$, and a non-negative matrix $(b_{i,j})$, where $b_{i,j}$ is almost non-decreasing in i and almost non-increasing in j, such that the inequalities

$$\frac{1}{d}(b_{i,k}\omega_j + a_{k,j}) \leq a_{i,j} \leq d(b_{i,k}\omega_j + a_{k,j}) \tag{1.3}$$

hold for all $i \geq k \geq j \geq 1$.

In [7], [8] inequality (1.1) for all $f \in l_{p,v}$ was considered under the assumption that there exist $d \geq 1$, a sequence of positive numbers $\{\omega_k\}_{k=1}^{\infty}$, and a non-negative matrix $(b_{i,j})$, whose entries $b_{i,j}$ are almost non-decreasing in i and almost non-increasing in j such that the inequalities

$$\frac{1}{d}(a_{i,k} + b_{k,j}\omega_i) \leq a_{i,j} \leq d(a_{i,k} + b_{k,j}\omega_i) \tag{1.4}$$

hold for all $i \geq k \geq j \geq 1$.

Conditions (1.3) and (1.4) include condition (1.2), and complement each another.

Notation: If M and K are real valued functionals of sequences, then the symbol $M \ll K$ means that there exists $c>0$ such that $M \leq cK$, where c is a constant which does not depend on the arguments of M and K. If $M \ll K \ll M$, then we write $M \approx K$.

In [2] there was established a statement which allows to reduce inequality (1.1) on the cone of monotone sequences to inequality (1.1) on the cone of non-negative sequences from $l_{p,v}$.

Theorem A. [2] *Let $1<p,\ q<\infty$. Let $V_k = \sum\limits_{i=1}^{k} v_i$. Then inequality* (1.1) *is equivalent to the following inequalities*

$$\left(\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}\sum_{i=j}^{\infty} a_{i,j}g_i\right)^{p'}\left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}}\right)\right)^{\frac{1}{p'}} \leq \widetilde{C}\left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{1-q'}\right)^{\frac{1}{q'}}, \quad \forall g \geq 0, \tag{1.5}$$

if $V_\infty = \lim\limits_{k\to\infty} V_k = \infty$,

$$\left(\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}\sum_{i=j}^{\infty} a_{i,j}g_i\right)^{p'}\left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}}\right)\right)^{\frac{1}{p'}} \tag{1.6}$$

$$+\left(\sum_{j=1}^{\infty}\sum_{i=j}^{\infty} a_{i,j}g_i\right)\left(\sum_{k=1}^{\infty} v_k\right)^{-\frac{1}{p}} \leq \widehat{C}\left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{1-q'}\right)^{\frac{1}{q'}}, \quad \forall g \geq 0,$$

if $V_\infty < \infty$.

For the proof of our main theorem we will need the following results for the discrete weighted Hardy inequality.

Theorem B. ([1], [4]) *Let* $1 < p \le q < \infty$. *Let* $\{\alpha_j\}_{j=1}^\infty$ *be a non-negative sequence of real numbers. Then the inequality*

$$\left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i}\alpha_j f_j\right)^q u_i\right)^{\frac{1}{q}} \le C\left(\sum_{i=1}^{\infty} f_i^p v_i\right)^{\frac{1}{p}}, \qquad 0 \le f \in l_{p,v} \tag{1.7}$$

holds if and only if

$$H := \sup_{n\ge1}\left(\sum_{j=n}^{\infty}u_j\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n}\alpha_i^{p'}v_i^{1-p'}\right)^{\frac{1}{p'}} < \infty.$$

Moreover, $H \approx C$, *where* C *is the best constant in* (1.7).

Theorem C. [7] *Let* $1 < p \le q < \infty$ *and the entries of the matrix* $(a_{i,j})$ *satisfy assumption* (1.4). *Inequality* (1.1) *holds for* $f \in l_{p,v}$ *if and only if* $B = \max\{B_1,\ B_2\} < \infty$, *where*

$$B_1 = \sup_{n\ge1}\left(\sum_{j=1}^{n}v_j^{1-p'}\right)^{\frac{1}{p'}}\left(\sum_{i=n}^{\infty}a_{i,n}^q u_i\right)^{\frac{1}{q}}$$

and

$$B_2 = \sup_{n\ge1}\left(\sum_{j=1}^{n}b_{n,j}^{p'}v_j^{1-p'}\right)^{\frac{1}{p'}}\left(\sum_{i=n}^{\infty}\omega_i^q u_i\right)^{\frac{1}{q}}.$$

Moreover, $B \approx C$, *where* C *is the best constant in* (1.1).

Theorem D. ([1], [4]) *Let* $1 < q < p < \infty$. *Then inequality* (1.7) *holds if and only if*

$$H_1 = \left(\sum_{k=1}^{\infty}\left(\sum_{i=k}^{\infty}u_i\right)^{\frac{p}{p-q}}\left(\sum_{j=1}^{k}\alpha_j^{p'}v_j^{1-p'}\right)^{\frac{p(q-1)}{p-q}}\alpha_k^{p'}v_k^{1-p'}\right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $H_1 \approx C$, *where* C *is the best constant in* (1.7).

Theorem E. [8] *Let* $1 < q < p < \infty$. *Let the entries of the matrix* $(a_{i,j})$ *satisfy assumption* (1.4). *Then inequality* (1.1) *holds for* $f \in l_{p,v}$ *if and only if* $E = \max\{E_1,\ E_2\} < \infty$, *where*

$$E_1 = \left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i}b_{i,j}^{p'}v_j^{1-p'}\right)^{\frac{q(p-1)}{p-q}}\left(\sum_{k=i}^{\infty}\omega_k^q u_k\right)^{\frac{q}{p-q}}\omega_i^q u_i\right)^{\frac{p-q}{pq}},$$

$$E_2 = \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} v_j^{1-p'} \right)^{\frac{p(q-1)}{p-q}} \left(\sum_{k=i}^{\infty} a_{k,i}^q u_k \right)^{\frac{p}{p-q}} v_i^{1-p'} \right)^{\frac{p-q}{pq}}.$$

Moreover, $E \approx C$, where C is the best constant in (1.1).

2 Main results

We define

$$V_k = \sum_{i=1}^{k} v_i, \quad A_{ik} = \sum_{j=1}^{k} a_{i,j}, \quad B_{ik} = \sum_{j=1}^{k} b_{i,j},$$

$$C_1 = \sup_{s\in\mathbb{N}} V_s^{-\frac{1}{p}} \left(\sum_{i=1}^{s} A_{ii}^q u_i \right)^{\frac{1}{q}},$$

$$C_2 = \sup_{s\in\mathbb{N}} \left(\sum_{k=1}^{s} k^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \left(\sum_{i=s}^{\infty} a_{i,s}^q u_i \right)^{\frac{1}{q}},$$

$$C_3 = \sup_{s\in\mathbb{N}} \left(\sum_{k=1}^{s} B_{sk}^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{1}{p'}} \left(\sum_{i=s}^{\infty} \omega_i^q u_i \right)^{\frac{1}{q}},$$

$$F_1 = \left(\sum_{k=1}^{\infty} V_k^{\frac{q}{q-p}} \left(\sum_{i=1}^{k} A_{ii}^q u_i \right)^{\frac{q}{p-q}} A_{kk}^q u_k \right)^{\frac{p-q}{pq}},$$

$$F_2 = \left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} w_i^q u_i \right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{k} B_{jj}^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p(q-1)}{p-q}} B_{kk}^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p-q}{pq}},$$

$$F_3 = \left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} j^{p'} b_{k,j}^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{i=k}^{\infty} w_i^q u_i \right)^{\frac{q}{p-q}} w_k^q u_k \right)^{\frac{p-q}{pq}},$$

$$F_4 = \left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} a_{i,k}^q u_i \right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{k} j^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p(q-1)}{p-q}} k^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}} \right) \right)^{\frac{p-q}{pq}}.$$

Theorem 2.1. *Let $1 < p \le q < \infty$. Let the entries of the matrix $(a_{i,j})$ satisfy assumption* (1.4)*. Then inequality* (1.1) *holds if and only if $C_0 = \max\{C_1,\ C_2,\ C_3\} < \infty$. Moreover, $C_0 \approx C$, where C is the best constant in* (1.1).

Theorem 2.2. *Let $1 < q < p < \infty$. Let the entries of the matrix $(a_{i,j})$ satisfy assumption* (1.4)*. Then inequality* (1.1) *holds if and only if $F_0 = \max\{F_1,\ F_2,\ F_3,\ F_4\} < \infty$. Moreover, $F_0 \approx C$, where C is the best constant in* (1.1)*.*

Proof of Theorem 2.1. We consider two cases separately: $V_\infty = +\infty$ and $V_\infty < +\infty$.

1. Let $V_\infty = +\infty$. Then based on Theorem A inequality (1.1) holds if and only if the following inequality holds

$$\left(\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}\sum_{i=j}^{\infty}a_{i,j}g_i\right)^{p'}\left(V_k^{-\frac{p'}{p}}-V_{k+1}^{-\frac{p'}{p}}\right)\right)^{\frac{1}{p'}} \le \widetilde{C}\left(\sum_{i=1}^{\infty}g_i^{q'}u_i^{1-q'}\right)^{\frac{1}{q'}}, \quad \forall g \ge 0. \quad (2.1)$$

Moreover, $\widetilde{C} \approx C$, where C is the best constant in (1.1).

Since $a_{i,j}$, g_i are non-negative and according to assumption (1.4) we have

$$\sum_{j=1}^{k}\sum_{i=j}^{\infty}a_{i,j}g_i = \sum_{j=1}^{k}\sum_{i=j}^{k}a_{i,j}g_i + \sum_{j=1}^{k}\sum_{i=k+1}^{\infty}a_{i,j}g_i \approx \sum_{i=1}^{k}A_{ii}g_i + \sum_{i=k}^{\infty}g_i\sum_{j=1}^{k}a_{i,j}$$

$$\approx \sum_{i=1}^{k}A_{ii}g_i + k\sum_{i=k}^{\infty}a_{i,k}g_i + B_{kk}\sum_{i=k}^{\infty}\omega_i g_i. \quad (2.2)$$

Therefore,

$$\left(\sum_{j=1}^{k}\sum_{i=j}^{\infty}a_{i,j}g_i\right)^{p'} \approx \left(\sum_{i=1}^{k}A_{ii}g_i\right)^{p'} + \left(k\sum_{i=k}^{\infty}a_{i,k}g_i\right)^{p'} + \left(B_{kk}\sum_{i=k}^{\infty}\omega_i g_i\right)^{p'}.$$

Substituting the last inequality in the left hand side of inequality (2.1) we have

$$\left(\sum_{k=1}^{\infty}\left[\left(\sum_{i=1}^{k}A_{ii}g_i\right)^{p'} + \left(k\sum_{i=k}^{\infty}a_{i,k}g_i\right)^{p'} + \left(B_{kk}\sum_{i=k}^{\infty}\omega_i g_i\right)^{p'}\right]\left(V_k^{-\frac{p'}{p}}-V_{k+1}^{-\frac{p'}{p}}\right)\right)^{\frac{1}{p'}}$$

$$\le \widetilde{C}_0\left(\sum_{i=1}^{\infty}g_i^{q'}u_i^{1-q'}\right)^{\frac{1}{q'}}, \quad \forall g \ge 0, \quad (2.3)$$

which is equivalent to the inequality (2.1). Moreover, $\widetilde{C} \approx \widetilde{C}_0$.

Inequality (2.3) holds if and only if the following inequalities hold simultaneously

$$\left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}A_{ii}g_i\right)^{p'}\left(V_k^{-\frac{p'}{p}}-V_{k+1}^{-\frac{p'}{p}}\right)\right)^{\frac{1}{p'}} \le \widetilde{C}_1\left(\sum_{i=1}^{\infty}g_i^{q'}u_i^{1-q'}\right)^{\frac{1}{q'}}, \quad \forall g \ge 0, \quad (2.4)$$

$$\left(\sum_{k=1}^{\infty}\left(k\sum_{i=k}^{\infty}a_{i,k}g_i\right)^{p'}\left(V_k^{-\frac{p'}{p}}-V_{k+1}^{-\frac{p'}{p}}\right)\right)^{\frac{1}{p'}} \le \widetilde{C}_2\left(\sum_{i=1}^{\infty}g_i^{q'}u_i^{1-q'}\right)^{\frac{1}{q'}}, \quad \forall g \ge 0, \quad (2.5)$$

$$\left(\sum_{k=1}^{\infty}\left(B_{kk}\sum_{i=k}^{\infty}\omega_i g_i\right)^{p'}\left(V_k^{-\frac{p'}{p}}-V_{k+1}^{-\frac{p'}{p}}\right)\right)^{\frac{1}{p'}}\leq \widetilde{C}_3\left(\sum_{i=1}^{\infty}g_i^{q'}u_i^{1-q'}\right)^{\frac{1}{q'}},\quad \forall g\geq 0. \tag{2.6}$$

Moreover,

$$\widetilde{C}\approx \max\{\widetilde{C}_1,\ \widetilde{C}_2,\ \widetilde{C}_3\}. \tag{2.7}$$

In (2.5) and (2.6) by passing to the dual inequalities we obtain

$$\left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}a_{k,i}\varphi_i\right)^{q}u_k\right)^{\frac{1}{q}}\leq \widetilde{C}_2\left(\sum_{i=1}^{\infty}\varphi_i^p\, i^{-p}\left(V_i^{-\frac{p'}{p}}-V_{i+1}^{-\frac{p'}{p}}\right)^{-\frac{p}{p'}}\right)^{\frac{1}{p}},\quad \forall \varphi\geq 0. \tag{2.8}$$

$$\left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}\varphi_i\right)^{q}\omega_k^q u_k\right)^{\frac{1}{q}}\leq \widetilde{C}_3\left(\sum_{k=1}^{\infty}\varphi_k^p B_{kk}^{-p}\left(V_k^{-\frac{p'}{p}}-V_{k+1}^{-\frac{p'}{p}}\right)^{-\frac{p}{p'}}\right)^{\frac{1}{p}},\quad \forall \varphi\geq 0. \tag{2.9}$$

(2.4) and (2.9) are Hardy type inequalities. Hence, by Theorem B inequalities (2.4) and (2.9) hold if and only if the following conditions hold respectively

$$\sup_{s\in\mathbb{N}}\left(\sum_{k=s}^{\infty}\left(V_k^{-\frac{p'}{p}}-V_{k+1}^{-\frac{p'}{p}}\right)\right)^{\frac{1}{p'}}\left(\sum_{i=1}^{s}A_{ii}^q u_i\right)^{\frac{1}{q}} \tag{2.10}$$

$$=\sup_{s\in\mathbb{N}}V_s^{-\frac{1}{p}}\left(\sum_{i=1}^{s}A_{ii}^q u_i\right)^{\frac{1}{q}}=C_1<\infty,$$

$$\sup_{s\in\mathbb{N}}\left(\sum_{i=s}^{\infty}\omega_i^q u_i\right)^{\frac{1}{q}}\left(\sum_{k=1}^{s}B_{kk}^{p'}\left(V_k^{-\frac{p'}{p}}-V_{k+1}^{-\frac{p'}{p}}\right)\right)^{\frac{1}{p'}}=C_4<\infty. \tag{2.11}$$

Moreover,

$$C_1\approx \widetilde{C}_1,\quad C_4\approx \widetilde{C}_3. \tag{2.12}$$

By using Theorem C inequality (2.8) holds if and only if the following conditions hold

$$\sup_{s\in\mathbb{N}}\left(\sum_{k=1}^{s}k^{p'}\left(V_k^{-\frac{p'}{p}}-V_{k+1}^{-\frac{p'}{p}}\right)\right)^{\frac{1}{p'}}\left(\sum_{i=s}^{\infty}a_{i,s}^q u_i\right)^{\frac{1}{q}}=C_2<\infty, \tag{2.13}$$

$$\sup_{s\in\mathbb{N}}\left(\sum_{k=1}^{s}b_{s,k}^{p'}k^{p'}\left(V_k^{-\frac{p'}{p}}-V_{k+1}^{-\frac{p'}{p}}\right)\right)^{\frac{1}{p'}}\left(\sum_{i=s}^{\infty}\omega_i^q u_i\right)^{\frac{1}{q}}=C_5<\infty \tag{2.14}$$

and

$$\widetilde{C}_2 \approx \max\{C_2,\ C_5\}. \tag{2.15}$$

Since $b_{i,j}$ is almost non-decreasing in i and almost non-increasing in j, for $s \geq k$ we have

$$C_3 \approx C_4 + C_5. \tag{2.16}$$

By (2.10),(2.11), (2.13), (2.14) and (2.16) we obtain that inequalities (2.4)-(2.6) hold if and only if $C_0 = \max\{C_1,\ C_2,\ C_3\} < \infty$. Moreover, $C_0 \approx \max\{\widetilde{C}_1,\ \widetilde{C}_2,\ \widetilde{C}_3\}$, which implies that $C_0 \approx \widetilde{C}$. Since $\widetilde{C} \approx C$ we get $C_0 \approx C$. The last equivalence gives the statement of Theorem 2.1 in the case $V_\infty = \infty$.

2. Let $V_\infty < +\infty$. By Theorem A inequality (1.1) holds if and only if along with inequality (2.1) the following inequality holds

$$\left(\sum_{k=1}^{\infty}\sum_{i=k}^{\infty} a_{i,k} g_i\right)\left(\sum_{i=1}^{\infty} v_i\right)^{-\frac{1}{p}} \leq \widehat{C}\left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{1-q'}\right)^{\frac{1}{q'}}, \quad \forall g \geq 0. \tag{2.17}$$

Moreover, $C \approx \max\{\widetilde{C}, \widehat{C}\}$.

Since $a_{i,j}$, g_i are non-negative, changing the order of summation in the left hand side of (2.17) we obtain

$$\left(\sum_{i=1}^{\infty} g_i A_{ii}\right) \leq \widehat{C} V_\infty^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} g_i^{q'} u_i^{1-q'}\right)^{\frac{1}{q'}}, \quad \forall g \geq 0.$$

By the reverse Hölder's inequality we have

$$\left(\sum_{i=1}^{\infty} A_{ii}^q u_i\right)^{\frac{1}{q}} = \widehat{C} V_\infty^{\frac{1}{p}},$$

consequently

$$V_\infty^{-\frac{1}{p}} \left(\sum_{i=1}^{\infty} A_{ii}^q u_i\right)^{\frac{1}{q}} = \widehat{C}. \tag{2.18}$$

Hence,

$$\widehat{C} \leq C_1.$$

Now we see that $\max\{\widetilde{C}, \widehat{C}\} \approx C_0 = \max\{C_1,\ C_2,\ C_3\}$ regardless of whether V_∞ is finite or infinite. Since $\max\{\widetilde{C},\ \widehat{C}\} \approx C$, we get $C \approx C_0 = \max\{C_1,\ C_2,\ C_3\}$. □

Proof of Theorem 2.2. We consider two cases separately: $V_\infty = +\infty$ and $V_\infty < +\infty$.

1. Let $V_\infty = +\infty$. Then in the same way using Theorem A as in the proof of Theorem 2.1 we obtain inequalities (2.4), (2.8) and (2.9).

By Theorem D inequalities (2.4), (2.9) hold if and only if the following conditions hold respectively

$$\left(\sum_{k=1}^{\infty} V_k^{\frac{q}{q-p}} \left(\sum_{i=1}^{k} A_{ii}^q u_i\right)^{\frac{q}{p-q}} A_{kk}^q u_k\right)^{\frac{p-q}{pq}} = F_1 < \infty, \tag{2.19}$$

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} w_i^q u_i\right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{k} B_{jj}^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}}\right)\right)^{\frac{p(q-1)}{p-q}} \times \right. \tag{2.20}$$

$$\left. B_{kk}^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}}\right)\right)^{\frac{p-q}{pq}} = F_2 < \infty.$$

Moreover,

$$F_1 \approx \widetilde{C}_1, \quad F_2 \approx \widetilde{C}_3. \tag{2.21}$$

The entries of the matrix $(a_{k,i})$ satisfy assumption (1.4). Therefore, by Theorem E inequality (2.8) holds if and only if the following conditions hold

$$\left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} j^{p'} b_{k,j}^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}}\right)\right)^{\frac{q(p-1)}{p-q}} \left(\sum_{i=k}^{\infty} w_i^q u_i\right)^{\frac{q}{p-q}} w_k^q u_k\right)^{\frac{p-q}{pq}} \tag{2.22}$$

$$= F_3 < \infty,$$

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} a_{i,k}^q u_i\right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{k} j^{p'} \left(V_j^{-\frac{p'}{p}} - V_{j+1}^{-\frac{p'}{p}}\right)\right)^{\frac{p(q-1)}{p-q}} \times \right. \tag{2.23}$$

$$\left. k^{p'} \left(V_k^{-\frac{p'}{p}} - V_{k+1}^{-\frac{p'}{p}}\right)\right)^{\frac{p-q}{pq}} = F_4 < \infty$$

and

$$\widetilde{C}_2 \approx \max\{F_3, \ F_4\}. \tag{2.24}$$

By (2.19), (2.20) and (2.22), (2.23) we obtain that inequalities (2.4), (2.8) and (2.9) hold if and only if $F_0 = \max\{F_1, \ F_2, \ F_3, \ F_4\} < \infty$. Moreover, $F_0 \approx \max\{\widetilde{C}_1, \ \widetilde{C}_2, \ \widetilde{C}_3\}$, which implies that $F_0 \approx \widetilde{C}$. Since $\widetilde{C} \approx C$ we get $F_0 \approx C$. The last equivalence gives the statement of Theorem 2.2 in the case $V_\infty = \infty$.

2. Let $V_\infty < +\infty$. By Theorem A inequality (1.1) holds if and only if along with inequality (2.1) inequality (2.17) holds. Moreover, $C \approx \max\{\widetilde{C}, \widehat{C}\}$.

As in the proof of Theorem 2.1 from inequality (2.17) we obtain inequality (2.18).

It is easy to prove that

$$F_1 \geq V_\infty^{-\frac{1}{p}} \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} A_{ii}^q u_i \right)^{\frac{q}{p-q}} A_{kk}^q u_k \right)^{\frac{p-q}{pq}} \gg \widehat{C}.$$

Therefore, $C \approx \max\{\widetilde{C}, \widehat{C}\} \approx F_0 = \max\{F_1,\ F_2,\ F_3,\ F_4\}$ regardless of whether V_∞ is finite or infinite. □

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