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SMOOTHNESS SPACES RELATED TO MORREY SPACES - A SURVEY. I

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Abstract. We discuss different strategies to introduce smoothness spaces related to Morrey spaces.

1 Introduction

There is a rapidly increasing number of papers dealing with smoothness spaces related to Morrey spaces. It will be the aim of this survey to give an introduction to one of these approaches, namely the Nikol'skij-Besov type spaces $B^{s,\tau}_{p,q}$ and the Lizorkin-Triebel type spaces $F^{s,\tau}_{p,q}$, and to compare it to some of the other existing possibilities to introduce smoothness spaces of Nikol'skii-Besov-Lizorkin-Triebel type related to Morrey spaces. In fact, we shall consider all together eight scales of function spaces: beside of $B^{s,\tau}_{p,q}$ and $F^{s,\tau}_{p,q}$ we also discuss the scales $\mathcal{N}^s_{p,q,u}$, $\mathcal{E}^s_{p,q,u}$,$N^s_{p,q,u}$,$E^s_{p,q,u}$, $B^{s,\tau}_{p,q,\mathrm{unif}}$ and $F^{s,\tau}_{p,q,\mathrm{unif}}$ (all definitions will be given in Subsections 3.1-3.3). Whereas

$$\mathcal{E}^{s_0}_{p_0,q_0,u_0} \in \{F^{s,\tau}_{p,q} : \quad s \in \mathbb{R},\ 0 < p < \infty,\ 0 < q \le \infty,\ \tau \ge 0\}$$

and

$$\mathcal{N}^{s_0}_{p_0,\infty,u_0} \in \{B^{s,\tau}_{p,\infty} : \quad s \in \mathbb{R},\ 0 < p \le \infty,\ \tau \ge 0\}$$

for all admissible values of s_0, p_0, u_0 and q_0 hold, we have

$$\mathcal{N}^{s_0}_{p_0,q_0,u_0} \notin \{B^{s,\tau}_{p,q} : \quad s \in \mathbb{R},\ 0 < p \le \infty,\ 0 < q \le \infty,\ \tau \ge 0\}$$

for all admissible values of s_0, p_0, u_0 and $0 < q_0 < \infty$. The differences between the Nikol'skij-Besov type scale $B^{s,\tau}_{p,q}$ and the Nikol'skij-Besov-Morrey scale $\mathcal{N}^s_{p,q,u}$ $(q < \infty)$ will be discussed in certain detail.

One comment to the notation used in this paper. The situation in the literature is a little bit chaotic. At least in some cases there is no common well-accepted notation. Not only the letter for certain parameters is changing but also its position. The reader should always have a look at the used definition when comparing the results within this survey with others.

This survey is organized as follows. First of all, it consists of two parts. Part I is devoted to a discussion of the definition and some basic properties. In particular, in Section 2 of Part I we recall certain parts of the theory of spaces of functions of bounded mean oscillations. We try to explain why the space bmo (the local version of BMO) can be understood as a model case for the scales of spaces we will discuss later on. To be more precise, the specific Fourier analytic description of bmo, due to Frazier and Jawerth [26] and recalled below, is the point of departure which leads to the definition of the scales $F^{s,\tau}_{p,q}$ and $B^{s,\tau}_{p,q}$. The main properties of this four parameter scale of function spaces will be discussed in the following Section 3. Here we mainly follow the recent lecture note [102]. So we will give proofs only in exceptional cases. Many times some comments to the idea or method of proof will be given. This will be the contents of part I. In Part II we will discuss interpolation properties of these scales, in particular real interpolation of $F^{s,\tau}_{p,q}$ and $B^{s,\tau}_{p,q}$, Gagliardo-Nirenberg type inequalities, and embeddings. Part II will contain new material and is written with complete proofs. Furthermore, we will recall there some different approaches to smoothness spaces related to Morrey spaces, due to Hedberg, Netrusov and Triebel. In a final section we shall collect some open problems. Here we also add a few comments on possible generalizations.

In my opinion the theory of the spaces $F^{s,\tau}_{p,q}$ and $B^{s,\tau}_{p,q}$ is far away from being complete. In the presented survey we simply arrange what is essentially known.

Notation

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0$ the natural numbers including 0, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. $\mathbb{C}$ denotes the complex numbers and $\mathbb{R}^d$ the Euclidean d-space. All functions are assumed to be complex-valued, i.e., we consider functions $f:\ \mathbb{R}^d \to \mathbb{C}$. In general the classes of functions (distributions) are defined on $\mathbb{R}^d$. So we will drop it in notation. Let $\mathcal{S}$ denote the Schwartz space of all rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^d$. By $\mathcal{S}'$ we denote the collection of all complex-valued tempered distributions on $\mathbb{R}^d$, i.e., the topological dual of $\mathcal{S}$, equipped with the strong topology. The symbol $\mathcal{F}$ refers to the Fourier transform, $\mathcal{F}^{-1}$ to its inverse transformation, both defined on $\mathcal{S}'$. All function spaces, which we consider in this paper, are subspaces of $\mathcal{S}'$, i.e. spaces of equivalence classes w.r.t. almost everywhere equality. However, if such an equivalence class contains a continuous representative, then usually we work with this representative and call also the equivalence class a continuous function.

If E and F are two quasi-Banach spaces, then the symbol $E \hookrightarrow F$ indicates that the embedding is continuous. By C_0^∞ we denote the set of all test functions, i.e., the set of all compactly supported and infinitely differentiable functions. If E is a quasi-Banach function space on $\mathbb{R}^d$ we denote by $E^{\ell oc}$ the collection of all functions f having the property that the products $\varphi f \in E$ for all $\varphi \in C_0^\infty$. The symbol $\mathcal{L}(E,F)$ denotes the set of all linear and bounded operators $T:\ E \to F$. In case $E = F$ we simply write $\mathcal{L}(E)$.

As usual, the symbol c denotes a positive constant which depends only on the fixed parameters d, s, τ, p, q and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we will use the symbol " $\lesssim$ " instead of "$\leq$". The meaning of $A \lesssim B$ is given by: there exists a constant $c > 0$ such that

$A \le c\,B$. The symbol $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$. Many times we shall need the following abbreviations:

$$\sigma_p := d\,\max\Big(0, \frac{1}{p} - 1\Big) \qquad \text{and} \qquad \sigma_{p,q} := d\,\max\Big(0, \frac{1}{p} - 1, \frac{1}{q} - 1\Big). \tag{1.1}$$

2 Functions of bounded mean oscillations and Lizorkin-Triebel spaces with $p = \infty$

This section has preparatory character. From our point of view, the function spaces BMO, bmo and in particular $F^s_{\infty,q}$ are the point of departure for the generalizations $F^{s,\tau}_{p,q}$ and $B^{s,\tau}_{p,q}$ which we will discuss in detail in Section 3.

2.1 Functions of bounded mean oscillations

In 1961 John and Nirenberg [38] introduced the class BMO. A locally integrable function f on $\mathbb{R}^d$ belongs to BMO (has bounded mean oscillation) if

$$\|f\|_{\mathrm{BMO}} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx < \infty\,,$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axes. Here f_Q denotes the mean-value of f on Q, i.e.,

$$f_Q := \frac{1}{|Q|} \int_Q f(x)\, dx\,.$$

Of course, $\|\cdot\|_{\mathrm{BMO}}$ is not a norm. To turn it into a norm we have to calculate modulo constants, i.e., one has to consider classes of functions

$$[f]_1 := \{f + c : \quad c \in \mathbb{C}\}$$

instead of functions. Nowadays BMO has established as a good substitute of L_∞ in harmonic analysis. Of course, $L_\infty \hookrightarrow$ BMO and the embedding is strict, since $\log|x| \in$ BMO. Polynomials, except constants, do not belong to BMO. Of certain importance for us is the Fourier-analytic description of BMO.

The Fourier-analytic description of BMO

Let $\varphi \in C_0^\infty$ be a function such that

$$\varphi(x) := 1 \qquad \text{if} \quad \frac{1}{2} \le |x| \le 2$$

and

$$\varphi(x) := 0 \qquad \text{if either} \quad |x| \le \frac{1}{4} \quad \text{or} \quad |x| \ge 4\,.$$

Then, with $\varphi_j(x) := \varphi(2^{-j+1}x)$, $j \in \mathbb{Z}$, we can construct a smooth dyadic homogeneous decomposition of unity. Indeed, an elementary calculation yields

$$\sum_{j=-\infty}^{\infty} \varphi_j(x) = 1 \qquad \text{for all} \quad x \neq 0\,. \tag{2.1}$$

Here *dyadic* refers to the fact that supp φ_j is contained in the dyadic annulus

$$2^{j-3} \le |x| \le 2^{j+1}\,, \qquad j \in \mathbb{Z}\,.$$

For $f \in \mathcal{S}'$ the product $\varphi_j \cdot \mathcal{F}f$ belongs to $\mathcal{S}'$ as well and consequently, by the famous Paley-Wiener-Schwartz theorem, $\mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)](x)$ is an analytic function (can be extended to ...). Let $\mathcal{P}$ denote the collection of all polynomials. We put

$$[f] := \{f + p : \quad p \in \mathcal{P}\}\,, \qquad f \in \mathcal{S}'\,.$$

Because of (2.1) we obtain, that for every $f \in \mathcal{S}'$ there exists a polynomial p such that

$$f + p = \sum_{j\in\mathbb{Z}} \mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)] \qquad (\text{convergence in } \mathcal{S}')\,. \tag{2.2}$$

Sometimes (2.2) is called a Littlewood-Paley decomposition of f (or at the same time of $[f]$). Those Littlewood-Paley decompositions are the basis for many function spaces, see in particular Section 3. However, for the moment we need a more general concept.

Proposition 2.1. *Let $(\varphi_j)_j$ be the smooth dyadic decomposition of unity defined above. Then we have the following equivalence. A locally integrable function $f \in \mathcal{S}'$ belongs to* BMO *if, and only if, there exists a sequence $(f_j)_j$ of L_∞-functions such that*

$$f = \sum_{j=-\infty}^{\infty} \mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f_j(\xi)](x) \tag{2.3}$$

and

$$\Big\| \Big(\sum_{j=-\infty}^{\infty} |f_j(x)|^2 \Big)^{1/2} \Big\|_{L_\infty} < \infty\,. \tag{2.4}$$

Remark 1. (i) The formulation in Proposition 2.1 requires an interpretation. Since the origin does not belong to the support of $\varphi_j\,\mathcal{F}f_j$ for all $j \in \mathbb{Z}$, the right-hand side in (2.3) does not see polynomials. Hence, the better frame here is to calculate modulo polynomials of arbitrary order. By $\mathcal{S}'/\mathcal{P}$ we denote the associated quotient space. Then the correct formulation is as follows: The class $[g]_1$, associated to a locally integrable function g, belongs to BMO if, and only if, in the class $[g]$ there exists a representative f such that (2.3) (with convergence in $\mathcal{S}'$) and (2.4) hold.
(ii) A proof of Proposition 2.1 has been given by Triebel in 1978 in his booklet [81, Theorem 3.2.2].

The next step has been done by Frazier and Jawerth in their famous paper [26] in 1990. They have found a bit easier Fourier-analytic description of BMO. To describe this we need dyadic cubes. A cube Q such that

$$Q = Q_{j,k} := \{x \in \mathbb{R}^d :\, 2^{-j} k_\ell \le x_\ell < 2^{-j}(k_\ell + 1)\,,\ \ell = 1, \ldots, d\}\,,$$

for some $j \in \mathbb{Z}$ and some $k \in \mathbb{Z}^d$ is called *dyadic*. The collection of all dyadic cubes will be denoted by $\mathcal{Q}$. For a given cube Q the number $\ell(Q)$ is its side-length. To each dyadic cube we associate one more number, namely

$$j_Q := -\log_2 \ell(Q)\,, \qquad Q \in \mathcal{Q}\,.$$

Proposition 2.2. *Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity. A class $[f] \in \mathcal{S}'/\mathcal{P}$ belongs to* BMO *if, and only if,*

$$\|f\|^*_{\mathrm{BMO}} := \sup_{Q \in \mathcal{Q}} \left\{ \frac{1}{|Q|} \int_Q \sum_{j=j_Q}^{\infty} |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|^2\, dx \right\}^{1/2} < \infty.$$

Remark 2. As in Proposition 2.1 BMO has to be interpreted as a subset of $\mathcal{S}'/\mathcal{P}$, or, with other words, the class $[g]_1$, associated to a locally integrable function g, belongs to BMO if, and only if, in the class $[g]$ there exists a representative f such that $\|f\|^*_{\mathrm{BMO}} < \infty$.

2.2 Functions of local bounded mean oscillations

Less known than BMO is the following local variant bmo.

Definition 1. *A locally integrable function f on $\mathbb{R}^d$ belongs to* bmo *(has local bounded mean oscillations) if $f \in$ BMO and*

$$\|f\|_{\mathrm{bmo}} := \|f\|_{\mathrm{BMO}} + \sup_Q \frac{1}{|Q|} \int_Q |f(x)|\, dx < \infty\,,$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axes and side-length $\ell(Q) \le 1$.

Obviously $\|\cdot|\mathrm{bmo}\|$ is a norm, we do not calculate modulo constants this time. Furthermore, $L_\infty \hookrightarrow \mathrm{bmo} \hookrightarrow \mathrm{BMO}$ and all embeddings are proper. Of course, the second embedding requires an interpretation. But here it is enough to associate to each $f \in \mathrm{bmo}$ the class $[f]_1$. The function $\log|x|$ does not belong to bmo, but $\psi(x)\, \log|x| \in$ bmo, where $\psi \in C_0^\infty$.

The Fourier-analytic description of bmo

Let $\psi \in C_0^\infty$ be a function such that

$$\psi(x) := 1 \quad \text{if} \quad |x| \le 1 \qquad \text{and} \qquad \psi(x) := 0 \quad \text{if} \quad |x| \ge \frac{3}{2}\,. \tag{2.5}$$

Then, with $\varphi_0 := \psi$,

$$\varphi(x) := \varphi_0(x/2) - \varphi_0(x) \qquad \text{and} \qquad \varphi_j(x) := \varphi(2^{-j+1}x)\,, \quad j \in \mathbb{N}\,, \tag{2.6}$$

we have

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}^d\,.$$

This time we have constructed an inhomogeneous smooth dyadic decomposition of unity. The main difference to a homogeneous smooth dyadic decomposition of unity consists in the fact that this time a certain neighborhood of the origin belongs to exactly one support of the functions φ_j, $j \in \mathbb{N}_0$. Again Triebel [81] has found the Fourier-analytic description of *bmo*.

Proposition 2.3. *Let $(\varphi_j)_j$ be the smooth dyadic decomposition of unity defined in (2.5), (2.6). Then we have the following equivalence. A locally integrable function $f \in \mathcal{S}'$ belongs to* bmo *if, and only if, there exists a sequence $(f_j)_j$ of L_∞-functions such that*

$$f = \sum_{j=0}^{\infty} \mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F} f_j(\xi)](x) \tag{2.7}$$

and

$$\Big\| \Big(\sum_{j=0}^{\infty} |f_j(x)|^2 \Big)^{1/2} \Big\|_{L_\infty} < \infty\,. \tag{2.8}$$

Remark 3. This time there is no need for an interpretation. Since $0 \in \operatorname{supp} \varphi_0$ the right-hand side in (2.7) is sensitive with respect to polynomials.

Also Frazier and Jawerth have considered the nonhomogeneous situation and proved the following characterization of bmo, see [26].

Proposition 2.4. *Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity defined in (2.5), (2.6). A locally integrable function $f \in \mathcal{S}'$ belongs to* bmo *if, and only if,*

$$\|f\|^*_{\mathrm{bmo}} := \sup_{\substack{Q \in \mathcal{Q} \\ \ell(Q) \le 1}} \left\{ \frac{1}{|Q|} \int_Q \sum_{j=j_Q}^{\infty} |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F} f(\xi)](x)|^2\, dx \right\}^{1/2} < \infty.$$

2.3 The inhomogeneous Lizorkin-Triebel spaces with $p = \infty$

Originally the definition of the scale of Lizorkin-Triebel spaces $F^s_{p,q}$ was restricted to values of $p < \infty$. At an early stage of the theory Triebel [82, 2.1.4] had shown that the naive extension of the Fourier-analytic definition is not meaningful. In his book [83] from 1983 he defined for the first time the spaces $F^s_{\infty,q}$, $1 < q < \infty$ (with some forerunners in [81], there denoted by $L^s_{\infty,q}$). The point of view, he had chosen, has been a completion of the duality relation

$$\Big(F^s_{p,q}\Big)' = F^{-s}_{p',q'}\,, \qquad 1 < p < \infty\,, \qquad 1 < q < \infty\,, \tag{2.9}$$

see [83, 2.11.2]. His definition was oriented on two facts:

- the duality relation $\Big(h_1\Big)' = \text{bmo}$, where h_1 denotes the local Hardy space, proved by Goldberg [28] in 1979;
- the identification of h_1 as $F^0_{1,2}$, for which we refer to Bui Huy Qui [9].

Probably one should mention here as well the famous and earlier known homogeneous counterparts of these assertions:

- the duality relation $\Big(H_1\Big)' = \text{BMO}$, where H_1 denotes the real Hardy space, proved by Fefferman [22] in 1971;
- the identification of H_1 as $\dot{F}^0_{1,2}$, observed by Peetre [62, 64] around 1974.

With these facts at hand Triebel introduced $F^s_{\infty,q}$, $1 < q < \infty$ in the spirit of Proposition 2.3, extending the validity of (2.9) to $p = 1$. We skip this and concentrate on the Frazier-Jawerth approach to these classes in [26]. In the spirit of Proposition 2.4 they used the following definition.

Definition 2. *Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity as defined in (2.5), (2.6). Let $0 < q < \infty$ and $s \in \mathbb{R}$. Then $F^s_{\infty,q}$ is the collection of all distributions $f \in \mathcal{S}'$ such that*

$$\| f \|_{F^s_{\infty,q}} := \sup_{\substack{Q \in \mathcal{Q}\\ \ell(Q) \le 1}} \left\{ \frac{1}{|Q|} \int_Q \sum_{j=j_Q}^{\infty} 2^{jsq}\, |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|^q \, dx \right\}^{1/q} < \infty. \tag{2.10}$$

Remark 4. (i) The classes $F^s_{\infty,q}$ do not depend on the chosen decomposition of unity in the sense of equivalent quasi-norms. $F^s_{\infty,q}$ is a quasi-Banach space (Banach space if $q \ge 1$). In case of $1 < q < \infty$ we have coincidence of the two approaches. For all these statements we refer to [26].
(ii) Proposition 2.4 yields $F^0_{\infty,2} = \text{bmo}$ in the sense of equivalent norms.
(iii) Replacing

$$\sup_{\substack{Q \in \mathcal{Q}\\ \ell(Q) \le 1}} \qquad \text{simply by} \qquad \sup_{Q \in \mathcal{Q}} \tag{2.11}$$

we get an equivalent quasi-norm in $F^s_{\infty,q}$. This is a consequence of an easy calculation.

2.4 Lizorkin-Triebel and Nikol'skij-Besov spaces on $\mathbb{R}^d$

For convenience of the reader we also recall the Fourier-analytic definition of Nikol'skij-Besov and Lizorkin-Triebel spaces on $\mathbb{R}^d$.

Definition 3. *Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity as defined in (2.5), (2.6). Let $0 < q \le \infty$ and $s \in \mathbb{R}$.*
(i) *Let $0 < p \le \infty$. Then the Nikol'skij-Besov space $B^s_{p,q}$ is the collection of all distributions $f \in \mathcal{S}'$ such that*

$$\| f \|_{B^s_{p,q}} := \left\{ \sum_{j=0}^{\infty} 2^{jsq}\, \| \, \mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)] \, \|^q_{L_p} \right\}^{1/q} < \infty. \tag{2.12}$$

(ii) *Let* $0 < p < \infty$. *Then the Lizorkin-Triebel space* $F^s_{p,q}$ *is the collection of all distributions* $f \in \mathcal{S}'$ *such that*

$$\|f\|_{F^s_{p,q}} := \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq}\, |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|^q \Big)^{1/q} \Big\|_{L_p} < \infty. \tag{2.13}$$

Remark 5. (i) For the nowadays well-developed theory of Nikol'skij-Besov and Lizorkin-Triebel spaces on $\mathbb{R}^d$ we refer to the monographs [3, 4, 5, 60, 65, 83, 84, 86].
(ii) We shall use the convention

$$F^s_{\infty,\infty} := B^s_{\infty,\infty}\,, \qquad s \in \mathbb{R}\,. \tag{2.14}$$

Definition 2 combined with the Fourier-analytic definition of the Lizorkin-Triebel and Nikol'skij-Besov spaces are the sources for the following far-reaching generalization.

3 Inhomogeneous spaces of Nikol'skij-Besov-Lizorkin-Triebel type

This is the main section of this survey. Here we discuss one approach to smoothness spaces related to Morrey spaces.

3.1 The definition of $F^{s,\tau}_{p,q}$ and $B^{s,\tau}_{p,q}$ and some elementary properties

In comparison with $F^s_{p,q}$ and $B^s_{p,q}$ we introduce a fourth parameter τ by replacing $|Q|$ in (2.10) by $|Q|^\tau$.

Definition 4. *Let* $(\varphi_j)_j$ *be a smooth dyadic decomposition of unity as defined in (2.5), (2.6). Let* τ, $s \in \mathbb{R}$ *and* $0 < q \le \infty$.
(i) *Let* $0 < p < \infty$. *Then the inhomogeneous Lizorkin-Triebel type space* $F^{s,\tau}_{p,q}$ *is defined to be the set of all* $f \in \mathcal{S}'$ *such that*

$$\|f\|_{F^{s,\tau}_{p,q}} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \left\{ \int_Q \left[\sum_{j=\max(j_Q,0)}^{\infty} 2^{jsq} |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|^q \right]^{p/q} dx \right\}^{1/p} < \infty\,.$$

(ii) Let $0 < p \le \infty$. *Then the inhomogeneous Nikol'skij-Besov type space* $B^{s,\tau}_{p,q}$ *is defined to be the set of all* $f \in \mathcal{S}'$ *such that*

$$\|f\|_{B^{s,\tau}_{p,q}} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \left\{ \sum_{j=\max(j_Q,0)}^{\infty} \left[\int_Q (2^{js} |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|)^p \, dx \right]^{q/p} \right\}^{1/q} < \infty\,. \tag{3.1}$$

Remark 6. (i) El Baraka [16, 17, 18] introduced and investigated the scale of Nikol'skij-Besov type spaces $B^{s,\tau}_{p,q}$ in the Banach case.
(ii) Quite recently, namely 2008 and 2010, see [97] (Banach case), [98] (quasi-Banach case), Dachun Yang and Wen Yuan have introduced and investigated the homogeneous counterparts of these Lizorkin-Triebel type spaces (which means one has to use the smooth dyadic decomposition of unity in (2.1) and to calculate in $\mathcal{S}'/\mathcal{P}$). The inhomogeneous spaces $F^{s,\tau}_{p,q}$ are considered for the first time in [102].
(iii) Many times the scale $F^{s,\tau}_{p,q}$ behave as the scale $B^{s,\tau}_{p,q}$. In those situations, to avoid unattractive repetitions, we shall use the notation $A^{s,\tau}_{p,q}$ with $A \in \{F, B\}$.

There is a number of immediate consequences of this definition:

- In case $\tau < 0$, by considering $|Q| \to \infty$, we obviously obtain $A^{s,\tau}_{p,q} = \{0\}$, $A \in \{F, B\}$.
- For $\tau = 0$ we have $A^{s,0}_{p,q} = A^s_{p,q}$, $A \in \{F, B\}$.
- We always have $F^{s,\tau}_{p,p} = B^{s,\tau}_{p,p}$.
- In the definition of the scale $F^{s,\tau}_{p,q}$ the case $p = \infty$ is excluded. However, with $\tau = 1/q$ we have the identity
$$F^s_{\infty,q} = F^{s,1/q}_{q,q}\,, \qquad s \in \mathbb{R}, \quad 0 < q < \infty\,, \tag{3.2}$$
in the sense of equivalent quasi-norms, see Definition 2 and (2.11).

Some basic properties of $A^{s,\tau}_{p,q}$ are collected in the following lemma, see [102, Lemma 2.1, Proposition 2.3].

Lemma 3.1. *(i) The classes $A^{s,\tau}_{p,q}$ are quasi-Banach spaces, i. e., complete quasi-normed spaces. With $\varepsilon := \min\{1, p, q\}$ it holds*
$$\|f+g\|^{\varepsilon}_{A^{s,\tau}_{p,q}} \le \|f\|^{\varepsilon}_{A^{s,\tau}_{p,q}} + \|g\|^{\varepsilon}_{A^{s,\tau}_{p,q}}$$
for all $f, g \in A^{s,\tau}_{p,q}$.
(ii) We always have
$$\mathcal{S} \hookrightarrow A^{s,\tau}_{p,q} \hookrightarrow \mathcal{S}'\,.$$
(iii) One can replace the set $\mathcal{Q}$ by the set of all cubes with sides parallel to the axes in Definition 4 obtaining an equivalent quasi-norm on that way. With the same argument on can replace the set of all such cubes by the set of all balls.

In the next lemma we collect elementary embeddings, see [102, Proposition 2.1].

Lemma 3.2. *With $q_0 \le q_1$ we have*
$$A^{s,\tau}_{p,q_0} \hookrightarrow A^{s,\tau}_{p,q_1}\,. \tag{3.3}$$
Furthermore, we have
$$B^{s,\tau}_{p,\min(p,q)} \hookrightarrow F^{s,\tau}_{p,q} \hookrightarrow B^{s,\tau}_{p,\max(p,q)} \tag{3.4}$$
and
$$A^{s,\tau}_{p,q} \hookrightarrow B^{s,\tau}_{p,\infty} \qquad A \in \{B, F\}\,. \tag{3.5}$$

Proof. The embedding (3.3) is a consequence of $\ell_{q_0} \hookrightarrow \ell_{q_1}$. Next, (3.4) follows from

$$\Big(\sum_{j=0}^\infty \| f_j \|_{L_p}^v\Big)^{1/v} \le \Big\|\Big(\sum_{j=0}^\infty |f_j|^q\Big)^{1/q}\Big\|_{L_p} \le \Big(\sum_{j=0}^\infty \| f_j \|_{L_p}^u\Big)^{1/u} \tag{3.6}$$

with $u := \min(p,q)$ and $v := \max(p,q)$ and valid for all sequences $(f_j)_j$ of measurable functions. Finally, (3.5) is implied by (3.3) and (3.4). □

3.2 A first discussion of the definition

Comparing Definition 4 with the definitions of $F^s_{\infty,q}$, $F^s_{p,q}$ and $B^s_{p,q}$ there arise a number of other possibilities to define smoothness spaces in the above spirit. Here are some of them.

(a) Replace $\sup_{Q\in\mathcal{Q}}$ by $\sup_{\substack{Q\in\mathcal{Q}\\ \ell(Q)\le 1}}$ in Definition 4, see (2.11).

(b) Replace

$$\sum_{j=\max(j_Q,0)} \qquad \text{by} \qquad \sum_{j=0} .$$

in Definition 4, see Definition 3.

(c) We concentrate on the B-case. Replace

$$\sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau}\Big\{\sum_{j=\max(j_Q,0)}^\infty \ldots \qquad \text{by} \qquad \Big\{\sum_{j=0}^\infty \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\tau q}} \ldots$$

For later use we introduce the following notation.

Definition 5. *Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity as defined in (2.5), (2.6). Let τ, $s \in \mathbb{R}$ and $0 < q, p \le \infty$. Then the space $\mathcal{B}^{s,\tau}_{p,q}$ is defined to be the set of all $f \in \mathcal{S}'$ such that*

$$\|f\|_{\mathcal{B}^{s,\tau}_{p,q}} := \Big\{\sum_{j=0}^\infty \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau}\Big[\int_Q (2^{js}|\mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)](x)|)^p\,dx\Big]^{q/p}\Big\}^{1/q} < \infty . \tag{3.7}$$

From the Definitions 4, 5 it follows immediately

$$\mathcal{B}^{s,\tau}_{p,q} \hookrightarrow B^{s,\tau}_{p,q} . \tag{3.8}$$

(d) Start with one of the known characterizations of $F^s_{p,q}$ and $B^s_{p,q}$, e.g., by differences, atoms, wavelets, approximation, etc. and replace the L_p-norm at appropriate places by

$$\sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau}\Big(\int_Q |\ldots|^p dx\Big)^{1/p} \qquad \text{or} \qquad \sup_{\substack{Q\in\mathcal{Q}\\ \ell(Q)\le 1}} \frac{1}{|Q|^\tau}\Big(\int_Q |\ldots|^p dx\Big)^{1/p} .$$

A few comments are in order. Concerning (a) we have the following.

Lemma 3.3. *Let $s \in \mathbb{R}$ and $0 < q \le \infty$.*
(i) *Let $0 < p < \infty$ and $\tau \ge 1/p$. A tempered distribution f belongs to $F^{s,\tau}_{p,q}$ if, and only if,*

$$\|f\|^{\#}_{F^{s,\tau}_{p,q}} := \sup_{\{P \in \mathcal{Q},\, |P| \le 1\}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=\max(j_P,0)}^{\infty} (2^{js} |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|)^q \right]^{p/q} dx \right\}^{1/p} < \infty .$$

Furthermore, the quasi-norms $\|f\|_{F^{s,\tau}_{p,q}}$ and $\|f\|^{\#}_{F^{s,\tau}_{p,q}}$ are equivalent.
(ii) *Let $0 < p \le \infty$ and $\tau \ge 1/p$. A tempered distribution f belongs to $B^{s,\tau}_{p,q}$ if, and only if,*

$$\|f\|^{\#}_{B^{s,\tau}_{p,q}} := \sup_{\{P \in \mathcal{Q},\, |P| \le 1\}} \frac{1}{|P|^\tau} \left\{ \sum_{j=\max(j_P,0)}^{\infty} \left[\int_P (2^{js} |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|)^p \, dx \right]^{q/p} \right\}^{1/q} < \infty .$$

Furthermore, the quasi-norms $\|f\|_{B^{s,\tau}_{p,q}}$ and $\|f\|^{\#}_{B^{s,\tau}_{p,q}}$ are equivalent.

Remark 7. An elementary proof of this lemma can be found in [102, Lemma 2.2]. Lemma 3.3 does not extend to values $\tau < 1/p$, see Remark 2.2 in [102, p. 23].

Next we would like to comment on (b) and (c). We will restrict ourselves to values of $0 \le \tau \le 1/p$, since otherwise we know the following.

Lemma 3.4. *Let $\tau > 1/p$. Assume*

$$\sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^\tau} \sup_{j=0,1,\ldots} \left[\int_Q (2^{js} |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|)^p \, dx \right]^{1/p} < \infty . \tag{3.9}$$

Then $f = 0$ a.e. follows.

Proof. Let $f \in \mathcal{S}'$. Suppose $|\mathcal{F}^{-1}[\varphi_{j_0}(\xi)\, \mathcal{F}f(\xi)](x_0)| > 0$ and $x_0 \in Q_{j,k_j}$ for all $j \in \mathbb{N}_0$. The function $|\mathcal{F}^{-1}[\varphi_{j_0}(\xi)\, \mathcal{F}f(\xi)](x)|$ is continuous and hence

$$\sup_{j \in \mathbb{N}_0} \left[\frac{1}{|Q_{j,k_j}|^{\tau p}} \int_{Q_{j,k_j}} (2^{j_0 s} |\mathcal{F}^{-1}[\varphi_{j_0}(\xi)\, \mathcal{F}f(\xi)](x)|)^p \, dx \right]^{1/p} = \infty .$$

If $|\mathcal{F}^{-1}[\varphi_{j_0}(\xi)\, \mathcal{F}f(\xi)](x)| = 0$ for all j and all x, then f must be the regular distribution which is vanishing a.e.. □

Remark 8. In view of (3.5) the relation in (3.9) holds for all elements in $A^{s,\tau}_{p,q}$, $A \in \{B, F\}$.

Of some importance for all what follows are the following properties with respect to (b).

Proposition 3.1. *Let $s \in \mathbb{R}$.*
(i) Let $0 < p, q < \infty$ and $0 \le \tau < 1/p$. Then the inhomogeneous Nikol'skij-Besov type space $B^{s,\tau}_{p,q}$ is the set of all $f \in \mathcal{S}'$ such that

$$\|f\|^*_{B^{s,\tau}_{p,q}} := \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^\tau} \left\{ \sum_{j=0}^\infty \left[\int_Q (2^{js} |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F} f(\xi)](x)|)^p \, dx \right]^{q/p} \right\}^{1/q} < \infty\,. \quad (3.10)$$

Furthermore, $\|f\|^*_{B^{s,\tau}_{p,q}}$ is an equivalent quasi-norm on $B^{s,\tau}_{p,q}$.
(ii) Let $0 < p \le \infty$ and $0 \le \tau \le 1/p$. Then the inhomogeneous Nikol'skij-Besov type space $B^{s,\tau}_{p,\infty}$ is the set of all $f \in \mathcal{S}'$ such that

$$\|f\|^*_{B^{s,\tau}_{p,\infty}} := \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^\tau} \sup_{j=0,1,\ldots} \left[\int_Q (2^{js} |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F} f(\xi)](x)|)^p \, dx \right]^{1/p} < \infty\,. \quad (3.11)$$

Furthermore, $\|f\|^*_{B^{s,\tau}_{p,\infty}}$ is an equivalent quasi-norm on $B^{s,\tau}_{p,\infty}$.

Proof. The proof is an exercise in working with maximal functions. As the first step we need the Peetre maximal function defined as

$$f^*_j(x) := \sup_{z \in \mathbb{R}^d} \frac{\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F} f(\xi)](x-z)}{(1+2^j|z|)^a}\,, \qquad x \in \mathbb{R}^d\,. \quad (3.12)$$

Here $f \in \mathcal{S}'$, $j \in \mathbb{N}_0$ and $a > 0$ will be chosen later on. Obviously, if $|x-y| < \sqrt{d}\, 2^{-j}$, we find

$$f^*_j(x) \le f^*_j(y) \sup_{z \in \mathbb{R}^d} \frac{(1+2^j|z-y-x|)^a}{(1+2^j|z|)^a} \le (1+\sqrt{d})^a\, f^*_j(y)\,. \quad (3.13)$$

Let $Q := Q_{j,k}$ with $j \in \mathbb{N}$ and $k \in \mathbb{Z}^d$. Then $j_Q > 0$. Let $0 \le \ell < j_Q$. There exists an unique dyadic cube $Q_{\ell,m}$ such that $Q \subset Q_{\ell,m}$. We obtain

$$\begin{aligned} \int_Q |\mathcal{F}^{-1}[\varphi_\ell(\xi)\, \mathcal{F} f(\xi)](x)|^p \, dx &\le \max_{x \in Q} |\mathcal{F}^{-1}[\varphi_\ell(\xi)\, \mathcal{F} f(\xi)](x)|^p\, |Q| \\ &\le (\inf_{y \in Q_{\ell,m}} f^*_\ell(y))^p\, (1+\sqrt{d})^{ap}\, |Q| \\ &\le \frac{1}{|Q_{\ell,m}|} \int_{Q_{\ell,m}} f^*_\ell(y)^p\, dy\, (1+\sqrt{d})^{ap}\, |Q|\,. \quad (3.14) \end{aligned}$$

This simple inequality is the basis for the proof.

Step 1. Let $0 < q < \infty$. Then (3.14) yields

$$\begin{aligned}
\sum_{\ell=0}^{j_Q-1} & 2^{\ell sq}\Big(\frac{1}{|Q|^{\tau p}}\int_Q |\mathcal{F}^{-1}[\varphi_\ell(\xi)\,\mathcal{F}f(\xi)](x)|^p\,dx\Big)^{q/p} \\
\le & \sum_{\ell=0}^{j_Q-1} 2^{\ell sq}\Big(\frac{1}{|Q|^{\tau p}}\frac{|Q|}{|Q_{\ell,m}|}\int_{Q_{\ell,m}} f_\ell^*(y)^p\,dy\,(1+\sqrt{d})^{ap}\Big)^{q/p} \\
\le & (1+\sqrt{d})^{aq}\sum_{\ell=0}^{j_Q-1} 2^{\ell sq}2^{d(\ell-j)q/p}2^{-d(\ell-j)q\tau}\Big(\frac{1}{|Q_{\ell,m}|^{\tau}p}\int_{Q_{\ell,m}} f_\ell^*(y)^p\,dy\Big)^{q/p} \\
\le & (1+\sqrt{d})^{aq}\,\Big(\sum_{\ell=0}^{j_Q-1} 2^{d(\ell-j)q/p}2^{d(j-\ell)q\tau}\Big)\sup_{0\le\ell<j_Q} 2^{\ell sq}\,\Big(\frac{1}{|Q_{\ell,m}|^{\tau p}}\int_{Q_{\ell,m}} f_\ell^*(y)^p\,dy\Big)^{q/p} \\
\le & c_1\sup_{0\le\ell<j_Q} 2^{\ell sq}\,\Big(\frac{1}{|Q_{\ell,m}|^{\tau p}}\int_{Q_{\ell,m}} f_\ell^*(y)^p\,dy\Big)^{q/p},
\end{aligned}$$

since $\tau < 1/p$. By means of the maximal inequality

$$\sup_{Q\in\mathcal{Q}}\frac{1}{|Q|^\tau}\sup_{\max(j_Q,0)\le j<\infty}\Big[\int_Q (2^{js}|f_j^*(x)|)^p\,dx\Big]^{1/p} \le c_2\,\|f\|_{B^{s,\tau}_{p,\infty}}\,,$$

if $a > d/p$, see [99, Theorem 1.1, comment on p. 3809], we conclude

$$\|f\|^*_{B^{s,\tau}_{p,q}} \le c_3(\|f\|_{B^{s,\tau}_{p,q}} + \|f\|_{B^{s,\tau}_{p,\infty}}) \le c_4\,\|f\|_{B^{s,\tau}_{p,q}}\,,$$

where c_4 does not depend on f. The reverse inequality is obvious.
Step 2. Let $q = \infty$. Then by the same type of arguments the claim follows, this time valid also for $\tau = 1/p$. □

Remark 9. Part (ii) of Proposition 3.1 implies

$$\mathcal{B}^{s,\tau}_{p,\infty} = B^{s,\tau}_{p,\infty}, \qquad 0 < p \le \infty, \quad 0 \le \tau \le 1/p\,, \tag{3.15}$$

in the sense of equivalent quasi-norms.

Proposition 3.2. *Let $s \in \mathbb{R}$ and $0 < p < \infty$.*
(i) Let $0 < q < \infty$ and $0 \le \tau < 1/p$. Then the inhomogeneous Lizorkin-Triebel type space $F^{s,\tau}_{p,q}$ is the set of all $f \in \mathcal{S}'$ such that

$$\|f\|^*_{F^{s,\tau}_{p,q}} := \sup_{Q\in\mathcal{Q}}\frac{1}{|Q|^\tau}\left\{\int_Q\left[\sum_{j=0}^\infty 2^{jsq}\,|\mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)](x)|^q\right]^{p/q} dx\right\}^{1/p} < \infty\,. \tag{3.16}$$

Furthermore, $\|f\|^*_{F^{s,\tau}_{p,q}}$ is an equivalent quasi-norm on $F^{s,\tau}_{p,q}$.
(ii) Let $0 \le \tau \le 1/p$. Then the inhomogeneous Lizorkin-Triebel type space $F^{s,\tau}_{p,\infty}$ is the set of all $f \in \mathcal{S}'$ such that

$$\|f\|^*_{F^{s,\tau}_{p,\infty}} := \sup_{Q\in\mathcal{Q}}\frac{1}{|Q|^\tau}\left\{\int_Q\sup_{j=0,1,2\ldots} 2^{js}\,|\mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)](x)|^p\,dx\right\}^{1/p} < \infty\,. \tag{3.17}$$

Furthermore, $\|f\|^*_{F^{s,\tau}_{p,\infty}}$ is an equivalent quasi-norm on $F^{s,\tau}_{p,\infty}$.

Proof. We discuss the needed modifications in comparison with the B-case using the same notations as there. Essentially we have to estimate

$$C_Q := \frac{1}{|Q|^{\tau p}} \int_Q \Big[\sum_{\ell=0}^{j_Q-1} 2^{\ell s q} \, |\mathcal{F}^{-1}[\varphi_\ell(\xi) \, \mathcal{F}f(\xi)](x)|^q \Big]^{p/q} \, dx$$

for any dyadic cube $Q = Q_{j,k}$ such that $j_Q \ge 1$. Using (3.14) we find

$$\begin{aligned} C_Q & \le (1+\sqrt{d})^{ap} \, \frac{1}{|Q|^{\tau p}} \int_Q \Big[\sum_{\ell=0}^{j_Q-1} 2^{\ell s q} \inf_{y \in Q_{\ell,m}} |f_\ell^*(y)|^q \Big]^{p/q} \, dx \\ & \le (1+\sqrt{d})^{ap} \, \frac{1}{|Q|^{\tau p}} \Big[\sum_{\ell=0}^{j_Q-1} 2^{\ell s q} \Big(\frac{|Q|}{|Q_{\ell,m}|} \int_{Q_{\ell,m}} |f_\ell^*(y)|^p \, dy \Big)^{q/p} \Big]^{p/q} \\ & \le c_1 \sup_{0 \le \ell < j_Q} 2^{\ell s q} \Big(\frac{1}{|Q_{\ell,m}|^{\tau p}} \int_{Q_{\ell,m}} f_\ell^*(y)^p \, dy \Big)^{q/p} , \end{aligned}$$

since $0 \le \tau < 1/p$. As above we conclude

$$\|f\|^*_{F^{s,\tau}_{p,q}} \le c_2 (\|f\|_{F^{s,\tau}_{p,q}} + \|f\|_{B^{s,\tau}_{p,\infty}}) \le c_3 \, \|f\|_{F^{s,\tau}_{p,q}} ,$$

where c_3 does not depend on f. The reverse inequality is obvious. Also the needed modifications in case $q = \infty$ are obvious. □

Remark 10. Using atomic decompositions, Propositions 3.1 and 3.2 have been proved in [72].

Before we continue we need to recall the definition of the Morrey spaces (mainly to fix the notation).

Definition 6. *Let $0 < u \le p \le \infty$. The space $\mathcal{M}^p_u$ is defined to be the set of all u-locally Lebesgue-integrable functions f on $\mathbb{R}^d$ such that*

$$\|f\|_{\mathcal{M}^p_u} := \sup_B |B|^{1/p-1/u} \Big(\int_B |f(x)|^u \, dx \Big)^{1/u} < \infty ,$$

where the supremum is taken over all balls B in $\mathbb{R}^d$.

Remark 11. (i) Some of the basics of Morrey spaces may be found in the monograph of Kufner, John and Fučik [45] and in the survey paper of Peetre [61]. However, in [45] these authors consider a local version, i.e., they consider the supremum with respect to balls with volume ≤ 1 instead of all balls.
(ii) Obviously we have

$$\mathcal{M}^p_p = L_p \qquad \text{and} \qquad \mathcal{M}^\infty_u = L_\infty \, . \tag{3.18}$$

As a consequence of Hölder's inequality we conclude monotonicity with respect to u, i.e.,

$$\mathcal{M}^p_w \hookrightarrow \mathcal{M}^p_u \qquad \text{if} \qquad 0 < u \le w \le p \le \infty , \tag{3.19}$$

see [44]. There is no monotonicity with respect to p.
(iii) Another elementary but useful property of Morrey spaces is the following formula:

$$\| f(\lambda\,\cdot\,) \|_{\mathcal{M}^p_u} = \lambda^{-d/p} \| f(\,\cdot\,) \|_{\mathcal{M}^p_u} \,, \tag{3.20}$$

valid for all $\lambda > 0$ and all $f \in \mathcal{M}^p_u$. Hence, the spaces scale with p, independent of u.

According to (b) we shall introduce a new scale of spaces.

Definition 7. *Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity as defined in (2.5), (2.6). Let $s \in \mathbb{R}$, $0 < u \le p < \infty$ and $0 < q \le \infty$. Then $\mathcal{E}^s_{p,q,u}$ is defined to be the set of all $f \in \mathcal{S}'$ such that*

$$\|f\|_{\mathcal{E}^s_{p,q,u}} := \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)](x)|^q \Big)^{1/q} \Big\|_{\mathcal{M}^p_u} < \infty \,. \tag{3.21}$$

Remark 12. (i) With other words, the Lizorkin-Triebel-Morrey spaces $\mathcal{E}^s_{p,q,u}$ represent the Lizorkin-Triebel scale built on the Morrey space $\mathcal{M}^p_u$. This scale of spaces has been introduced by Tang and Xu [78] in the year 2005.
(ii) The definition of $\mathcal{E}^s_{p,q,u}$ does not make sense if $p = \infty$. This follows from (3.18) in combination with the comments at the beginning of Subsection 2.3.

Now we turn to (c) but restricted to the B-case. With τ as above this yields the following.

Definition 8. *Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity as defined in (2.5), (2.6). Let $s \in \mathbb{R}$, $0 < u \le p \le \infty$ and $0 < q \le \infty$.*
Then $\mathcal{N}^s_{p,q,u}$ is defined to be the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{\mathcal{N}^s_{p,q,u}} := \Big(\sum_{j=0}^{\infty} 2^{jsq} \,\|\mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)]\,\|^q_{\mathcal{M}^p_u} \Big)^{1/q} < \infty \,. \tag{3.22}$$

Remark 13. (i) The Nikol'skij-Besov-Morrey spaces $\mathcal{N}^s_{p,q,u}$ represent the Nikol'skij-Besov scale built on the Morrey space $\mathcal{M}^p_u$. Kozono, Yamazaki [44] in 1994 and later on Mazzucato [55] have been the first who investigated spaces of this type. In fact, they studied two, slightly different, types of spaces. The first modification consists in restricting the supremum within the definition of the Morrey norm to balls with volume ≤ 1, see Definition 11 below. For the second modification they used, instead of the nonhomogeneous smooth dyadic decomposition of unity, the homogeneous counterpart, see (2.1), which results in the scale of homogeneous Nikol'skij-Besov-Morrey spaces $\dot{\mathcal{N}}^s_{p,q,u}$.
(ii) By means of (3.18) and Definition 3 we obtain the identity $\mathcal{N}^s_{\infty,q,u} = B^s_{\infty,q}$.
(iii) Obviously we have the coincidence

$$\mathcal{B}^{s,\frac{1}{u}-\frac{1}{p}}_{u,q} = \mathcal{N}^s_{p,q,u} \,, \qquad 0 < u \le p \le \infty \,.$$

Comparing on the one side $F^{s,\tau}_{p,q}$ and $\mathcal{E}^s_{u,q,p}$ and on the other side $B^{s,\tau}_{p,q}$ and $\mathcal{N}^s_{u,q,p}$, we have an immediate conclusion, see Proposition 3.1, 3.2.

Proposition 3.3. *Let $s \in \mathbb{R}$ and $0 < p \le u \le \infty$.*
(i) For $0 < q < \infty$ we have the continuous embeddings

$$\mathcal{N}^s_{u,q,p} = \mathcal{B}^{s,\frac{1}{p}-\frac{1}{u}}_{p,q} \hookrightarrow B^{s,\frac{1}{p}-\frac{1}{u}}_{p,q} . \tag{3.23}$$

(ii) We have

$$\mathcal{N}^s_{u,\infty,p} = \mathcal{B}^{s,\frac{1}{p}-\frac{1}{u}}_{p,\infty} = B^{s,\frac{1}{p}-\frac{1}{u}}_{p,\infty}$$

in the sense of equivalent quasi-norms.
(iii) Let $0 < u < \infty$ and $0 < q \le \infty$. Then we have

$$\mathcal{E}^s_{u,q,p} = F^{s,\frac{1}{p}-\frac{1}{u}}_{p,q}$$

in the sense of equivalent quasi-norms.

Remark 14. (i) Proposition 3.3 has been proved in [72]. There the authors argued with atomic decompositions. In addition they have been able to show that the embedding in (i) is proper if $p < u$.
(ii) There is an interesting difference between the scales $B^{s,\tau}_{p,q}$, $F^{s,\tau}_{p,q}$ on the one side and $\mathcal{N}^s_{p,q,u}$, $\mathcal{E}^s_{p,q,u}$ on the other side. In fact, we have

$$\mathcal{N}^s_{p,u,u} = \mathcal{B}^{s,\,\frac{1}{u}-\frac{1}{p}}_{u,u} \hookrightarrow B^{s,\,\frac{1}{u}-\frac{1}{p}}_{u,u} = F^{s,\,\frac{1}{u}-\frac{1}{p}}_{u,u} = \mathcal{E}^s_{p,u,u} . \tag{3.24}$$

For $u < p < \infty$ it follows $\mathcal{N}^s_{p,u,u} \hookrightarrow \mathcal{E}^s_{p,u,u}$ and the embedding is strict.

Sawano has investigated the relations between $\mathcal{N}^s_{p,q,u}$ and $\mathcal{E}^s_{p,q,u}$, see [69]. He proved the following, compare with Lemma 3.2.

Lemma 3.5. *Let $0 < u \le p < \infty$, $0 < q, q_0, q_1 \le \infty$ and $s \in \mathbb{R}$.*
(i) Then

$$\mathcal{N}^s_{p,\min(q,u),u} \hookrightarrow \mathcal{E}^s_{p,q,u} \hookrightarrow \mathcal{N}^s_{p,\infty,u} . \tag{3.25}$$

The embedding $\mathcal{E}^s_{p,q_0,u} \hookrightarrow \mathcal{N}^s_{p,q_1,u}$ implies $q = \infty$.
(ii) Let $1 \le u \le p < \infty$. It holds

$$\mathcal{N}^0_{p,\min(q,u),u} \hookrightarrow \mathcal{M}^p_u \hookrightarrow \mathcal{N}^0_{p,\infty,u} . \tag{3.26}$$

Proof. As mentioned above, part (i) is due to Sawano [69]. If $u > 1$, then part (ii) follows from part (i) by taking $s = 0$ and $q = 2$, see Lemma 3.6 below. In case $u = 1$ we shall use (3.6) for getting the left part in (3.26) and the standard convolution inequality

$$\| \mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)] \|_{\mathcal{M}^p_u} \lesssim \| \mathcal{F}^{-1}\varphi_j \|_{L_1} \| f \|_{\mathcal{M}^p_u}$$

for deriving the second. □

3.3 Sobolev-Morrey spaces

Many times Sobolev spaces are more important than Nikol'skij-Besov spaces. For that reason we will have at least a short look onto the Sobolev type spaces in our framework.

Definition 9. *Let $m \in \mathbb{N}_0$ and $1 \le u \le p \le \infty$. Then the Sobolev-Morrey space $W^m\mathcal{M}^p_u$ is the collection of all functions $f \in \mathcal{M}^p_u$ such that all distributional derivatives $D^\alpha f$ of order $|\alpha| \le m$ belong to $\mathcal{M}^p_u$. We equip this space with the norm*

$$\|f\|_{W^m\mathcal{M}^p_u} := \sum_{|\alpha|\le m} \|D^\alpha f\|_{\mathcal{M}^p_u}\, .$$

The first result we wish to mention is the Littlewood-Paley characterization of Morrey spaces, see Mazzucato [54] and Sawano [70].

Lemma 3.6. *Let $1 < u \le p < \infty$. Then $\mathcal{M}^p_u = F^{0,\frac1u-\frac1p}_{u,2}$ in the sense of equivalent norm.*

Next we recall a characterization of $F^{s,\tau}_{p,q}$, due to Tang and Xu [78], in terms of lower order derivatives which is of interest for its own.

Lemma 3.7. *Let $m \in \mathbb{N}$, $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$ and $0 \le \tau < 1/p$. Then $f \in F^{s,\tau}_{p,q}$ if, and only if, the distribution f and its distributional derivatives $\frac{\partial^m f}{\partial x_j^m}$, $j = 1, \ldots, d$, belong to $F^{s-m,\tau}_{p,q}$. Furthermore, the quasi-norms $\|f\|_{F^{s,\tau}_{p,q}}$ and*

$$\|f\|_{F^{s-m,\tau}_{p,q}} + \sum_{j=1}^d \Big\|\frac{\partial^m f}{\partial x_j^m}\Big\|_{F^{s-m,\tau}_{p,q}}$$

are equivalent.

Remark 15. Tang and Yu [78] also proved such an assertion for the Besov-Morrey spaces $\mathcal{N}^s_{p,q,u}$.

As an immediate conclusion of these two lemma we obtain the identification of $F^{m,\frac1u-\frac1p}_{u,2}$ as Sobolev-Morrey space.

Theorem 3.1. *Let $m \in \mathbb{N}$ and $1 < u \le p < \infty$. Then $W^m\mathcal{M}^p_u = F^{m,\frac1u-\frac1p}_{u,2}$ in the sense of equivalent quasi-norms.*

3.4 The spaces $A^{s,\tau}_{p,q,\mathrm{unif}}$

Many times localized versions of the spaces introduced above are of interest. We recall a few notions with this respect.

Definition 10. *Let ψ be as in (2.5). Let E be a quasi-Banach space of distributions in $\mathcal{S}'$. Then E_{unif} is the collection of all distributions $f \in \mathcal{S}'$ such that*

$$\|f\|_{E_{\mathrm{unif}}} := \sup_{\lambda\in\mathbb{Z}} \|f\,\psi(\,\cdot\, - \lambda)\|_E < \infty\, .$$

Remark 16. In case $E = A^{s,\tau}_{p,q}$, $A \in \{B, F\}$ it is well-known that smooth functions are pointwise multipliers, see Theorem 3.10.1 below. As an immediate consequence of the inequality (3.10.1) we observe that $A^{s,\tau}_{p,q,\mathrm{unif}}$ does not depend on the particular choice of ψ (in the sense of equivalent quasi-norms).

There is an other way to proceed, compare with Definition 8.

Definition 11. *Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity as defined in (2.5), (2.6). Let $s \in \mathbb{R}$, $0 < u \le p \le \infty$ and $0 < q \le \infty$. Then $N^s_{p,q,u}$ is defined to be the set of all $f \in \mathcal{S}'$ such that*

$$\|f\|_{N^s_{p,q,u}} := \Big\{ \sum_{j=0}^{\infty} 2^{jsq} \Big[\sup_{|B|\le 1} |B|^{\frac{1}{p}-\frac{1}{u}} \Big(\int_B |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|^u dx \Big)^{1/u} \Big]^q \Big\}^{1/q} < \infty\,. \tag{3.27}$$

Here the supremum is taken with respect to all balls in $\mathbb{R}^d$ with volume ≤ 1.

Remark 17. These spaces have been considered, e.g., by Kozono and Yamazaki [44] and Mazzucato [55].

Here is the counterpart in case of Lizorkin-Triebel spaces, compare with Definition 7.

Definition 12. *Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity as defined in (2.5), (2.6). Let $s \in \mathbb{R}$, $0 < u \le p < \infty$ and $0 < q \le \infty$. Then $E^s_{p,q,u}$ is defined to be the set of all $f \in \mathcal{S}'$ such that*

$$\|f\|_{E^s_{p,q,u}} := \sup_{|B|\le 1} |B|^{\frac{1}{p}-\frac{1}{u}} \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|^q \Big)^{1/q} \Big\|_{L_u(B)} < \infty\,. \tag{3.28}$$

Also here the supremum is taken with respect to all balls in $\mathbb{R}^d$ with volume ≤ 1.

Problem 1. (a) *Under which conditions on the parameters s, u, p, q we have the coincidence*

$$E^s_{u,q,p} = F^{s,\frac{1}{p}-\frac{1}{u}}_{p,q,\mathrm{unif}} \qquad \textit{(see Proposition 3.3(iii))}\,. \tag{3.29}$$

(b) *Under which conditions on the parameters s, u, p, q we have the coincidence*

$$N^s_{u,q,p} = \mathcal{B}^{s,\frac{1}{p}-\frac{1}{u}}_{p,q,\mathrm{unif}} \qquad \textit{(see Proposition 3.3(i))}\,. \tag{3.30}$$

Some comments to this problem will be given below.

3.5 A first summary

Summarizing, one could ask the question: What is the best definition ? We do not know the answer ! But to give an answer one needs, first of all, a more precise question. This leads to the next question. What is a list of properties of our spaces we want to have? Here are some which are desirable:

1. Find more transparent descriptions of $A^{s,\tau}_{p,q}$, in particular, characterizations by differences and derivatives;
2. Key theorems (pointwise multipliers, diffeomorphisms, traces);
3. Embeddings;
4. Investigations of the scale properties (lifting, interpolation);
5. Boundedness of pseudo-differential operators;
6. Fourier multipliers;
7. Boundedness of singular integrals;
8. Discretization (wavelets, atoms);
9. Characterization by approximation;
10. Boundedness of extension operators for reasonable domains;
11. Inner descriptions for reasonable domains.

The last two are connected with the associated scales of spaces on domains (say, defined by restrictions). In what follows we shall collect some results with respect to a certain part of this list, e.g., the points 6., 7. and 11. are not touched.

3.6 A simplification

Here we would like to mention two remarkable results. The first one concerns the case $\tau = 1/p$.

Proposition 3.4. *Let $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Then $F^{s,1/p}_{p,q} = F^s_{\infty,q}$ in the sense of equivalent quasi-norms.*

Remark 18. (i) The identity, stated in Proposition 3.4, has been observed and proved by Frazier and Jawerth [26]. The most important special case is

$$\mathrm{bmo} = F^0_{\infty,2} = F^{0,1/p}_{p,2}\,, \qquad 0 < p < \infty\,,$$

see Proposition 2.4 and Lemma 3.3(i).
(ii) Let s, p and q as in Proposition 3.4. As a consequence of Definition 2, Proposition 3.4 and Theorem 3.10.4 below we obtain

$$F^{s,1/p}_{p,q} = F^s_{\infty,q} = F^s_{\infty,q,\mathrm{unif}} = F^{s,1/p}_{p,q,\mathrm{unif}}\,.$$

As it is classically known, the Nikol'skij-Besov spaces $B^s_{\infty,\infty}$ coincide with Hölder-Zygmund spaces if $s > 0$. To be more precise we recall the definition.

Definition 13. (i) *Let $s > 0$ and let s be not a natural number. Let $M \in \mathbb{N}_0$ such that $M < s < M + 1$. Then a continuous function f belongs to the Hölder-Zygmund space $\mathcal{Z}^s$ if*

$$\| f \|_{\mathcal{Z}^s} := \Big(\max_{|\alpha|\le M} \sup_{x\in\mathbb{R}^d} |D^\alpha f(x)| \Big) + \Big(\max_{|\alpha|=M} \sup_{x,y\in\mathbb{R}^d,\, x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{s-M}} \Big) < \infty\,.$$

(ii) *Let s be a natural number. Then a continuous function f belongs to the Hölder-Zygmund space $\mathcal{Z}^s$ if*

$$\begin{aligned} \| f \|_{\mathcal{Z}^s} \ := \ & \Big(\max_{|\alpha|\le s-1} \sup_{x\in\mathbb{R}^d} |D^\alpha f(x)| \Big) \\ + \ & \Big(\max_{|\alpha|=s-1} \sup_{x,h\in\mathbb{R}^d,\, h\neq 0} \frac{|D^\alpha f(x+2h) - 2\, D^\alpha f(x+h) + D^\alpha f(x)|}{|h|} \Big) < \infty\,. \end{aligned}$$

In case $s+d(\tau-1/p) > 0$ the Lizorkin-Triebel type spaces as well as the Besov-type spaces coincide with Hölder-Zygmund spaces.

Proposition 3.5. *Let $s \in \mathbb{R}$.*
(i) *Let $0 < p < \infty$. Let either $0 < q < \infty$ and $\tau > 1/p$ or $q = \infty$ and $\tau \ge 1/p$. Then*

$$F^{s,\tau}_{p,q} = B^{s+d(\tau-1/p)}_{\infty,\infty}$$

in the sense of equivalent quasi-norms.
(ii) *Let $0 < p \le \infty$. Let either $0 < q < \infty$ and $\tau > 1/p$ or $q = \infty$ and $\tau \ge 1/p$. Then*

$$B^{s,\tau}_{p,q} = B^{s+d(\tau-1/p)}_{\infty,\infty}$$

in the sense of equivalent quasi-norms.

Remark 19. This remarkable result is due to Yang and Yuan [100]. The problem, under which restrictions on the parameters $B^{s,\tau}_{p,q}$ and $F^{s,\tau}_{p,q}$ coincide with a Hölder-Zygmund space, has been posed in [102, Remark 6.11(i)]. Their, in Subsection 6.3.2, also some results in this direction can be found.

As a consequence, from now on we will always consider the case $0 \le \tau \le 1/p$. We continue with a list of basic properties of the spaces. In almost all cases we shall treat the scales $F^{s,\tau}_{p,q}$, $B^{s,\tau}_{p,q}$, $\mathcal{E}^s_{p,q,u}$ and $\mathcal{N}^s_{p,q,u}$ parallel. Sometimes also comments to $B^{s,\tau}_{p,q,\mathrm{unif}}$, $F^{s,\tau}_{p,q,\mathrm{unif}}$, $E^s_{p,q,u}$ and $N^s_{p,q,u}$ will be given.

3.7 Pseudo-differential operators

We begin with recalling the following class of inhomogeneous symbols, which is a special case of the Hörmander class of symbols; see, for example, [33], [34] and [84, Chapter 6].

Definition 14. *Let $\mu \in \mathbb{R}$ and $0 \le \delta \le 1$. A smooth function a defined on $\mathbb{R}^d \times \mathbb{R}^d$ belongs to the class $\mathcal{S}^\mu_{1,\delta}(\mathbb{R}^d)$, if a satisfies the following set of differential inequalities: for all α, $\beta \in \mathbb{N}^d_0$ we have*

$$\sup_{x,\,\xi\in\mathbb{R}^d} (1+|\xi|)^{-\mu-\delta\,|\alpha|+|\beta|} |D^\alpha_x\, D^\beta_\xi\, a(x,\xi)| < \infty.$$

To each symbol a we associate the corresponding pseudo-differential operator

$$a(x,D)(f)(x) := \int_{\mathbb{R}^d} e^{ix\xi}\, a(x,\xi)\, \mathcal{F}f(\xi)\, d\xi\,, \qquad x\in\mathbb{R}^d\,, \quad f\in\mathcal{S}\,.$$

Recall, σ_p and $\sigma_{p,q}$ have been defined in (1.1). Boundedness of pseudo-differential operators of class $\mathcal{S}^{\mu}_{1,\delta}(\mathbb{R}^d)$ in the framework of the spaces $A^{s,\tau}_{p,q}$ has been investigated in [102, Chapter 5]. There the main result is the following.

Theorem 3.2. *Let $s\in\mathbb{R}$, $0<p,q\le\infty$ and $0\le\tau\le 1/p$. Let $\mu\in\mathbb{R}$, $a\in\mathcal{S}^{\mu}_{1,1}$ and $a(x,D)$ be the corresponding pseudo-differential operator.*
(i) If $s>\sigma_{p,q}$ ($s>\sigma_p$ if $A^{s,\tau}_{p,q}=B^{s,\tau}_{p,q}$), then $a(x,D)$ extends continuously to a linear continuous mapping of $A^{s+\mu,\tau}_{p,q}$ into $A^{s,\tau}_{p,q}$.
(ii) If $s\le\sigma_{p,q}$ ($s\le\sigma_p$ if $A^{s,\tau}_{p,q}=B^{s,\tau}_{p,q}$), assume further that its formal adjoint $a(x,D)^*$ satisfies

$$a(x,D)^*(x^\beta)\in\mathcal{P}$$

for all $\beta\in\mathbb{N}^d_0$,

$$|\beta|\le\max\Big(\sigma_{p,q}-s,0\Big) \qquad \Big(|\beta|\le\max\Big(\sigma_{p,q}-s,0\Big) \qquad \text{if} \quad A^{s,\tau}_{p,q}=B^{s,\tau}_{p,q}\Big).$$

Then $a(x,D)$ extends continuously to a linear continuous mapping of $A^{s+\mu,\tau}_{p,q}$ into $A^{s,\tau}_{p,q}$.

Remark 20. (i) For a proof of Theorem 3.2 we refer to [102, Theorem 5.1]. Let us mention that the proof given in [102] uses ideas of Grafakos and Torres [30], which itself has been based on [27, 24, 79, 80].
(ii) One can prove the estimate

$$\|a(x,D)|A^{s,\tau}_{p,q}\to A^{s,\tau}_{p,q}\| \lesssim \max_{|\alpha|,|\beta|\le M}\ \sup_{x,\xi}\,(1+|\xi|)^{-\mu-\delta\,|\alpha|+|\beta|}\,|D^\alpha_x\, D^\beta_\xi\, a(x,\xi)| \tag{3.31}$$

for some $M:=M(s,p,q,\tau)$, we refer to [102, Theorem 5.1].
(iii) The boundedness of pseudo-differential operators of the "exotic" class $\mathcal{S}^{\mu}_{1,1}$ has its own history. Here we only mention the contributions of Meyer [56] (boundedness on H^s_p, $s>0$, $1<p<\infty$), Bourdaud [6] (boundedness on $B^s_{p,q}$, $s>0$, $1\le p,q\le\infty$), Runst [66] and Torres [79]. The last two authors have dealt with the general case of Besov-Triebel-Lizorkin spaces including values of p and q less than 1.

Tang and Xu [78] have considered boundedness of pseudo-differential operators in the framework of the spaces $\mathcal{N}^s_{p,q,u}$ and $\mathcal{E}^s_{p,q,u}$.

Theorem 3.3. *Let $0<u\le p<\infty$, $0<q<\infty$ and $s\in\mathbb{R}$.*
(i) Let $a\in S^0_{1,\delta}$ with $0\le\delta<1$. Then $a(x,D)$ extends continuously to a linear continuous mapping of $\mathcal{N}^s_{p,q,u}$ ($\mathcal{E}^s_{p,q,u}$) into $\mathcal{N}^s_{p,q,u}$ ($\mathcal{E}^s_{p,q,u}$).
(ii) Let $a\in S^0_{1,1}$. Then $a(x,D)$ extends continuously to a linear continuous mapping of $\mathcal{N}^s_{p,q,u}$ ($\mathcal{E}^s_{p,q,u}$) into $\mathcal{N}^s_{p,q,u}$ ($\mathcal{E}^s_{p,q,u}$) if $s>d\Big(0,\frac{1}{u}-1,\frac{1}{q}-1\Big)$.

Remark 21. The boundedness of pseudo-differential operators with symbols in $S^0_{1,\delta}$, $0\le\delta\le1$, on the classes $N^s_{p,q,u}$ and $\dot{\mathcal{N}}^s_{p,q,u}$ has been investigated by Mazzucato [55].

As an immediate consequence of Theorems 3.2, 3.3 we have the following conclusion.

Corollary 3.1. *Let* $s \in \mathbb{R}$.
(i) Let $\gamma \in \mathbb{N}_0^d$, $0 < p, q \le \infty$ *and* $0 \le \tau \le 1/p$. *Then the operator* $\partial^\gamma : A_{p,q}^{s+|\gamma|,\tau} \to A_{p,q}^{s,\tau}$ *is continuous.*
(ii) Let $\gamma \in \mathbb{N}_0^d$, $0 < u \le p < \infty$, *and* $0 < q < \infty$. *Then the operator* $\partial^\gamma : \mathcal{N}_{p,q,u}^{s+|\gamma|} \to \mathcal{N}_{p,q,u}^{s}$ *is continuous.*

In addition, by Theorems 3.2, 3.3, we also obtain the so-called lifting properties for the spaces $A_{p,q}^{s,\tau}$ and $\mathcal{N}_{p,q,u}^{s}$. Let $\sigma \in \mathbb{R}$. Recall that the lifting operator I_σ is defined by

$$I_\sigma f := \mathcal{F}^{-1}[(1+|\cdot|^2)^{\sigma/2}\mathcal{F}f]\,, \qquad f \in \mathcal{S}'; \tag{3.32}$$

see, for example, [83, p. 58]. It is well known that I_σ is a one-to-one mapping from $\mathcal{S}'$ onto itself. Notice that

$$a(x,\xi) := (1+|\xi|^2)^{\sigma/2} \in \mathcal{S}_{1,0}^{\sigma}.$$

Applying Theorems 3.2, 3.3 we have the following result, see [78] and [102, Proposition 5.1].

Corollary 3.2. *Let* $\sigma, s \in \mathbb{R}$.
(i) Let $0 < p, q \le \infty$ *and* $0 \le \tau \le 1/p$. *Then the operator* I_σ *maps* $A_{p,q}^{s,\tau}$ *isomorphically onto* $A_{p,q}^{s-\sigma,\tau}$.
(ii) Let $0 < u \le p < \infty$ *and* $0 < q < \infty$. *Then the operator* I_σ *maps* $\mathcal{N}_{p,q,u}^{s}$ *isomorphically onto* $\mathcal{N}_{p,q,u}^{s-\sigma}$.

Remark 22. (i) Corollary 3.2 with $\tau = 0$, i.e., in the classic situation, has been proved at several places, see, e.g., [83, Theorem 2.3.8].
(ii) Fourier multipliers of Hörmander type for the spaces $A_{p,q}^{s,\tau}$ have been investigated in Yang, Yuan and Zhuo [101].

3.8 Discretization of the spaces

In recent times, more and more applications of Besov and Lizorkin-Triebel spaces are based on the possibility to discretize the spaces. Here we concentrate on characterizations by wavelets but making some remarks also to the decompositions into atoms and/or molecules.

3.8.1 Wavelet bases in L_2

Wavelet bases in Besov and Lizorkin-Triebel spaces are a well-developed concept. We refer to the monographs of Meyer [57], Wojtasczyk [90] and Triebel [86, 87] for the general d-dimensional case (for the one-dimensional case we refer to the books of Hernandez and Weiss [32], Kahane and Lemarie-Rieuseut [40] and the article of Bourdaud [7]). Let $\widetilde{\phi}$ be an orthonormal scaling function on $\mathbb{R}$ with compact support and of sufficiently high regularity. Let $\widetilde{\psi}$ be one corresponding orthonormal wavelet. Then the tensor product ansatz yields a scaling function ϕ and associated wavelets $\psi_1, \cdots, \psi_{2^d-1}$, all defined now on $\mathbb{R}^d$; see, e. g., [90, Proposition 5.2]. We suppose

$$\phi \in C^{N_1}(\mathbb{R}^d) \qquad \text{and} \qquad \operatorname{supp}\phi \subset [-N_2,\, N_2]^d \tag{3.33}$$

for certain natural numbers N_1 and N_2. This implies

$$\psi_i \in C^{N_1}(\mathbb{R}^d) \qquad \text{and} \qquad \operatorname{supp} \psi_i \subset [-N_3,\, N_3]^d\,, \qquad i = 1, \ldots, 2^d - 1 \tag{3.34}$$

for some $N_3 \in \mathbb{N}$. For $k \in \mathbb{Z}^d$, $j \in \mathbb{N}_0$ and $i = 1, \ldots, 2^d - 1$, we shall use the standard abbreviations in this context:

$$\phi_{j,k}(x) := 2^{jd/2}\phi(2^j x - k) \qquad \text{and} \qquad \psi_{i,j,k}(x) := 2^{jd/2}\psi_i(2^j x - k), \qquad x \in \mathbb{R}^d.$$

Furthermore, it is well known that

$$\int_{\mathbb{R}}^{d} \psi_{i,j,k}(x)\, x^\gamma\, dx = 0 \qquad \text{if} \qquad |\gamma| \le N_1$$

(see [90, Proposition 3.1]) and

$$\Psi := \{\phi_{0,k} :\ k \in \mathbb{Z}^d\} \cup \{\psi_{i,j,k} :\ k \in \mathbb{Z}^d,\ j \in \mathbb{N}_0,\ i = 1, \ldots, 2^d - 1\} \tag{3.35}$$

yields an orthonormal basis of $L^2(\mathbb{R}^d)$; see [57, Section 3.9] or [86, Section 3.1].

3.8.2 Wavelet bases of Besov type spaces

We need some more notation. Many times we shall work with $\widetilde{\chi}_Q$, the L_2-normalized characteristic function of the cube Q. i.e.,

$$\widetilde{\chi}_Q(x) := |Q|^{-1/2}\, \chi_Q(x)\,.$$

For $Q = Q_{jk} \in \mathcal{Q}$ and $m \in \mathbb{N}_0$ we put

$$\begin{aligned} J_Q &:= \big\{r \in \mathbb{Z}^d :\ |\operatorname{supp} \phi_{0,r} \cap Q| > 0\big\}\,, \\ I_{Q,m} &:= \big\{r \in \mathbb{Z}^d :\ \text{there exists}\, i \in \{1, \cdots, 2^d - 1\} \text{ such that } |\operatorname{supp} \psi_{i,m,r} \cap Q| > 0\big\}\,, \end{aligned}$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^d$. Let $|J_Q|$ and $|I_{Q,m}|$ denote the cardinalities of these sets. It is easy to check that there exists a positive constant $C = C(N_2, N_3)$ such that

$$|J_Q| \le C\, \max(1, |Q|) \qquad \text{and} \qquad |I_{Q,m}| \le C\, \max(1, 2^{md}|Q|)\,. \tag{3.36}$$

For $Q = Q_{jk}$ and $m \in \mathbb{N}_0$, we put

$$\mathcal{I}_{Q,m} := \bigcup_{|l-k| \le M} I_{Q_{jl},m} \qquad \text{and} \qquad \mathcal{J}_Q \equiv \bigcup_{|l-k| \le M} J_{Q_{jl}}.$$

The natural number M will be fixed later on. Finally, let

$$\begin{aligned} \|f\|_{\overset{\blacktriangle}{B}{}^{s,\tau}_{p,q}} \quad &:= \sup_{\{Q \in \mathcal{Q}:\, |Q| \ge 1\}} \frac{1}{|Q|^\tau} \left(\sum_{k \in \mathcal{J}_Q} |\langle f, \phi_{0,k}\rangle|^p \right)^{\frac{1}{p}} \\ &\qquad + \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^\tau} \left\{ \sum_{j=\max(j_Q,0)}^{\infty} 2^{j(s+d/2)q} \sum_{i=1}^{2^d-1} \left[\sum_{k \in \mathcal{I}_{Q,j}} 2^{-jd} |\langle f, \psi_{i,j,k}\rangle|^p \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}}. \end{aligned}$$

The functions $\phi_{0,k}$ and $\psi_{i,j,k}$ have compact support, but are not smooth. This means, the scalar products $\langle f, \phi_{0,k}\rangle$ and $\langle f, \psi_{i,j,k}\rangle$ with $f \in \mathcal{S}'$ require some interpretation. For the technicalities around this question we refer to [87, Theorem 1.20], where wavelet characterizations for $B^s_{p,q}$ and $F^s_{p,q}$ are discussed in full generality.

Theorem 3.4. *Let the generators ϕ and ψ of the wavelet system satisfy the conditions in (3.33), (3.34) with respect to N_1, N_2, $N_3 \in \mathbb{N}$. Let $0 < p$, $q \le \infty$,*

$$\sigma_p < s < N_1 \qquad \text{and} \qquad 0 \le \tau \le \frac{1}{p}\,.$$

Then $f \in B^{s,\tau}_{p,q}$ if, and only if, f is locally integrable and $\|f\|^{\blacktriangle}_{B^{s,\tau}_{p,q}} < \infty$. Further $\|f\|^{\blacktriangle}_{B^{s,\tau}_{p,q}}$ and $\|f\|_{B^{s,\tau}_{p,q}}$ are equivalent.

Remark 23. (i) A proof of Theorem 3.4 has been given in [102, Theorem 4.1].
(ii) For the case $q = \infty$ and $s \le \sigma_p$ we refer to the next subsection in view of the identity

$$\mathcal{N}^s_{u,\infty,p} = B^{s,\frac{1}{p}-\frac{1}{u}}_{p,\infty} \qquad \text{if} \qquad 0 < p \le u \le \infty\,.$$

(iii) A wavelet characterization of the classes $B^{s,\tau}_{p,q}$ for all admissible combinations of the parameters has been obtained recently in the paper Liang, Sawano, Ullrich, Yang, Yuan [47]. The homogeneous situation, i.e., the spaces $\dot{B}^{s,\tau}_{p,q}$, has been treated in Liang, Sawano, Ullrich, Yang, Yuan [46].

3.8.3 Wavelet bases of Lizorkin-Triebel-Morrey and Nikol'skij-Besov-Morrey spaces

Wavelet characterizations of the classes $\mathcal{N}^s_{p,q,u}$ and $\mathcal{E}^s_{p,q,u}$ for all admissible combinations of the parameters were derived in Sawano [69].

Theorem 3.5. *Let $0 < u \le p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let the generators ϕ and ψ of the wavelet system satisfy the conditions in (3.33), (3.34) with respect to N_1, N_2, $N_3 \in \mathbb{N}$ and suppose* $\min(N_1, N_2, N_3)$ *sufficiently large (depending on s, p, u, q). Then $f \in \mathcal{S}'$ belongs to $\mathcal{E}^s_{p,q,u}$ if, and only if, the following expression*

$$\|f\|^{\blacktriangle}_{\mathcal{E}^s_{p,q,u}} := \Big\| (\langle f, \phi_{0,k}\rangle)_{k\in\mathbb{Z}} \Big\|_{\ell_p} + \Big\| \Big(\sum_{i=1}^{2^d-1} \sum_{j=0}^{\infty} \Big| \sum_{k\in\mathbb{Z}} 2^{js} \langle f, \psi_{i,j,k}\rangle\, \widetilde{\chi}_{Q_{jk}} \Big|^q \Big)^{1/q} \Big\|_{\mathcal{M}^p_u}$$

is finite. Furthermore, $\|f\|^{\blacktriangle}_{\mathcal{E}^s_{p,q,u}}$ and $\|f\|_{\mathcal{E}^s_{p,q,u}}$ are equivalent.

Remark 24. (i) Recall the identity

$$\mathcal{E}^s_{u,q,p} = F^{s,\frac{1}{p}-\frac{1}{u}}_{p,q} \qquad \text{if} \qquad 0 < p \le u < \infty\,, \quad 0 < q \le \infty\,, \quad s \in \mathbb{R}\,.$$

Hence, Theorem 3.5 yields wavelet characterizations of $F^{s,\tau}_{p,q}$ in case $0 \le \tau < 1/p$.
(ii) Also in [102, Chapter 4] wavelet characterizations of the spaces $F^{s,\tau}_{p,q}$ were proved, but under the restriction $s > \sigma_{p,q}$ (see (1.1)). However, there $\tau = 1/p$ is admissible.
(iii) Again Liang, Sawano, Ullrich, Yang, Yuan [47] have been able to prove wavelet characterizations of the classes $F^{s,\tau}_{p,q}$ for all admissible combinations of the parameters. We also refer to [46] for the homogeneous spaces $\dot{F}^{s,\tau}_{p,q}$.

The counterpart for Nikol'skij-Besov-Morrey spaces, also proved by Sawano [69], reads as follows.

Theorem 3.6. *Let $0 < u \le p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let the generators ϕ and ψ of the wavelet system satisfy the conditions in (3.33), (3.34) with respect to N_1, N_2, $N_3 \in \mathbb{N}$ and suppose $\min(N_1, N_2, N_3)$ sufficiently large (depending on s, p, u). Then $f \in \mathcal{S}'$ belongs to $\mathcal{N}^s_{p,q,u}$ if, and only if, the following expression*

$$\|f\|^{\blacktriangle}_{\mathcal{N}^s_{p,q,u}} := \Big\| (\langle f, \phi_{0,k}\rangle)_{k\in\mathbb{Z}} \Big\|_{\ell_p} + \sum_{i=1}^{2^d-1} \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big\| \sum_{k\in\mathbb{Z}} \langle f, \psi_{i,j,k}\rangle \, \widetilde{\chi}_{Q_{jk}} \Big\|^q_{\mathcal{M}^p_u} \Big)^{1/q}$$

is finite. Furthermore, $\|f\|^{\blacktriangle}_{\mathcal{N}^s_{p,q,u}}$ and $\|f\|_{\mathcal{N}^s_{p,q,u}}$ are equivalent.

Remark 25. Because of

$$\mathcal{N}^s_{u,\infty,p} = B^{s,\frac{1}{p}-\frac{1}{u}}_{p,\infty} \qquad \text{if} \qquad 0 < p \le u \le \infty .$$

Theorem 3.6 yields wavelet characterizations of the classes $B^{s,\frac{1}{p}-\frac{1}{u}}_{p,\infty}$ for all s, supplementing Theorem 3.4 in this way.

The wavelet characterization of $F^s_{\infty,q}$

As a direct consequence of Theorem 3.4, Proposition 3.4 and the identity $F^{s,\tau}_{p,p} = B^{s,\tau}_{p,p}$ we obtain the following corollary.

Corollary 3.3. *Let $0 < q < \infty$ and*

$$d \max\Big(0, \frac{1}{q} - 1\Big) < s < \infty .$$

A tempered distribution $f \in \mathcal{S}'$ belongs to $F^s_{\infty,q}$ if, and only if,

$$\begin{aligned} |||f|||^{\blacktriangle}_{F^s_{\infty,q}} \quad &:= \sup_{k\in\mathbb{Z}} |\langle f, \phi_{0,k}\rangle| + \sup_{\{Q\in\mathcal{Q}:\, |Q|\le 1\}} \frac{1}{|Q|^{1/q}} \\ &\times \Bigg[\sum_{j=j_Q}^{\infty} \sum_{i=1}^{2^d-1} \sum_{k\in\mathcal{I}_{Q,j}} 2^{j(s+d(\frac{1}{2}-\frac{1}{q}))q} \, |\langle f, \psi_{i,j,k}\rangle|^q \Bigg]^{1/q} < \infty . \end{aligned} \tag{3.37}$$

Furthermore, $|||f|||^{\blacktriangle}_{F^s_{\infty,q}}$ and $\|f\|_{F^s_{\infty,q}}$ are equivalent.

Remark 26. (i) Wavelet characterizations of the homogeneous counterparts $\dot{F}^s_{\infty,2}$ have been obtained in [1].

(ii) Interesting limiting cases are bmo and BMO. Let $\widetilde{\psi}$ be a compactly supported, continuously differentiable wavelet on $\mathbb{R}$ and let $\psi_1, \ldots, \psi_{2^d-1}$ be the associated generators for a wavelet basis of $L^2(\mathbb{R}^d)$. Only here we shall use the convention

$\psi_{i,j,k}(x) := 2^{jd/2}\,\psi_i(2^j x - k)$ also for $j < 0$. Then a locally integrable function f belongs to BMO if, and only if

$$\sup_{P\in\mathcal{Q}} \frac{1}{|P|^{1/2}} \left[\sum_{j=j_P}^{\infty} \sum_{i=1}^{2^d-1} \sum_{k\in\mathcal{I}_{P,j}} |\langle f, \psi_{i,j,k}\rangle|^2 \right]^{1/2} < \infty\,;$$

see [57, 5.6] and [90, Ex. 8.8]. In the literature sometimes the convention

$$|\langle f, \psi_Q\rangle| \equiv \left(\sum_{i=1}^{d} |\langle f, \psi_{i,j,k}\rangle|^2 \right)^{1/2}$$

is used with $Q = Q_{j,k}$. In this language we obtain that a locally integrable function f belongs to BMO if, and only if

$$\sup_{P\in\mathcal{Q}} \left[\frac{1}{|P|} \sum_{Q\subset P} |\langle f, \psi_Q\rangle|^2 \right]^{1/2} < \infty\,,$$

The formula (3.37) remains to be true for bmo, i. e., if $s = 0$ and $q = 2$. For this result we refer to [1]. In addition we refer to the recent contributions by Liang, Sawano, Ullrich, Yang, Yuan [46, 47].

3.8.4 Discretization by means of atoms and molecules

Discretizations of $F^{s,\tau}_{p,q}$ and $B^{s,\tau}_{p,q}$ can be obtained also by means of atoms and molecules. Wavelet characterizations are just a special case of those characterizations. In fact, atoms and molecules allow much more flexible decompositions of distributions. We do not go into details, in particular, no definitions will be given. The aim is just to collect some references.

In the framework of Nikol'skij-Besov and Lizorkin-Triebel spaces Frazier and Jawerth [25, 26] have been the first who proved those characterizations. We also refer to Triebel [85, Section 13]. Hedberg and Netrusov [31] derived characterizations by atoms in their general axiomatic framework, which covers the scales $\mathcal{E}^s_{p,q,u}$ and $\mathcal{N}^s_{p,q,u}$. In case of Nikol'skij-Besov-type and Lizorkin-Triebel-type spaces we also refer to Sawano, Tanaka [71], Sawano, Yang, Yuan [72], Wang [89], Liang, Sawano, Ullrich, Yang, Yuan [47] and [102, Section 3.1].

Closely related to the characterization by atoms and molecules is the so-called φ-transform, see Frazier and Jawerth [25, 26] for Nikol'skij-Besov and Lizorkin-Triebel spaces. In case of the classes $B^{s,\tau}_{p,q}$ and $F^{s,\tau}_{p,q}$ the φ-transform has been investigated in [102, 2.1], see [97, 98] for the homogeneous case. In [48] Lin and Wang have introduced spaces $CMO^{\alpha}_{q,r}$ by means of the φ-transform and called them generalized Carleson measure spaces. The coincidence of these generalized Carleson measure spaces with elements of the scale $\dot{F}^{s,\tau}_{p,q}$ has been investigated in [48] and Yang, Yuan [100]. Let us mention that Theorem 1 in [48] is not correct without further restrictions, see the comments in [100].

Drihem [13] proved characterizations by means of maximal functions and local means for Nikol'skij-Besov-type spaces.

3.9 Characterization by differences

Characterizations by differences are the classical way to understand the smoothness and integrability requirements of those complicated spaces as Nikol'skij-Besov and Lizorkin-Triebel spaces are.

In our investigations the *Nikol'skij trick* playes an essential role, see [60, Section 5.2.1]. Starting point is our smooth cut-off function ψ, see (2.5). Now we define

$$\varphi_0(x) := (-1)^{N+1} \sum_{\ell=0}^{N-1} \binom{N}{\ell} (-1)^\ell \, \psi((N-\ell)x) \, .$$

This function φ_0 belongs to C_0^∞ and satisfies $\varphi_0(x) = 1$ if $|x| \le 1/N$ and $\varphi_0(x) = 0$ if $|x| \ge 3/2$. Elementary calculations for the Fourier transform yield the identity

$$\sum_{j=0}^{N} \mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x) \quad = \quad f(x) - \mathcal{F}^{-1}[\varphi_0(2^{-j}\xi)\, \mathcal{F}f(\xi)](x) \qquad (3.9.1)$$

$$= (2\pi)^{-n/2}(-1)^{N+1} \int \Big(\Delta^N_{2^{-j}y} f(x)\Big)\, \mathcal{F}^{-1}\psi(y)\, dy \, .$$

Here

$$\Delta_h^M f(x) := \sum_{j=0}^{M} (-1)^j \binom{M}{j} f(x + (M-j)h)$$

with $M \in \mathbb{N}$ and $x, h \in \mathbb{R}^d$. The formula (3.9.1) represents the bridge between the quasi-norm of the function f in $A^{s,\tau}_{p,q}$ with respect to the smooth dyadic decomposition of unity associated to φ_0, see (2.6), and the behaviour of quantities involving differences of f.

3.9.1 The characterization of Nikol'skij-Besov type spaces by differences

We shall work with quantities related to localized moduli of smoothness:

$$\|f\|^{\spadesuit}_{B^{s,\tau}_{p,q}} := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_0^{2\max(l(P),1)} t^{-sq} \sup_{t/2 \le |h| < t} \left(\int_P |\Delta_h^M f(x)|^p \, dx \right)^{q/p} \frac{dt}{t} \right\}^{1/q} .$$

Furthermore we shall need the space L^τ_p. By L^τ_p we denote the collection of all functions $f \in L_p^{\ell oc}$ such that

$$\|f\|_{L^\tau_p} := \sup \frac{1}{|P|^\tau} \left(\int_P |f(x)|^p \, dx \right)^{1/p} ,$$

where the supremum is taken over all dyadic cubes P with side length $l(P) \ge 1$. For technical reasons we have to distinguish the cases $p \ge 1$ and $0 < p < 1$.

Theorem 3.9.1. *Let $1 \le p \le \infty$, $0 < q \le \infty$, $0 \le \tau \le 1/p$, $M \in \mathbb{N}$, and $0 < s < M$. Then $f \in B^{s,\tau}_{p,q}$ if, and only if, $f \in L^\tau_p$ and $\|f\|^{\spadesuit}_{B^{s,\tau}_{p,q}} < \infty$. Furthermore, $\|f\|_{L^\tau_p} + \|f\|^{\spadesuit}_{B^{s,\tau}_{p,q}}$ and $\|f\|_{B^{s,\tau}_{p,q}}$ are equivalent.*

Remark 27. (i) For a proof we refer to [102, Theorem 4.7].

(ii) There are many references for the case $\tau = 0$. We refer to [60, Section 4.3], [5, Section 18], [3, Theorem 6.2.5], and [83, Section 2.5.12]. Let us mention that in [60], [5] the spaces are introduced by differences and the equivalence to other characterizations, like these in terms of Nikol'skij representations, are established afterwards.

(iii) Also Drihem [14, 15] has given characterizations of $B^{s,\tau}_{p,q}$ in terms of differences.

In the case $0 < p < 1$ and and $0 \le \tau < 1/p$ we meet a technical difficulty. We have to add an additional term involving $\|f\|_{B^{s_0}_{p,\infty}(2P)}$ for $P \in \mathcal{Q}$, $|P| \ge 1$.

Theorem 3.9.2. *Let $0 < p < 1$, $0 < q \le \infty$, $M \in \mathbb{N}$, $\sigma_p < s < M$ and $0 \le \tau < 1/p$. Let $\sigma_p < s_0 < s$. Then $f \in B^{s,\tau}_{p,q}$ if, and only if*

$$f \in L^\tau_p\,, \qquad \sup_{\{P\in\mathcal{Q},\,|P|\ge 1\}} \frac{\|f\|_{B^{s_0}_{p,\infty}(2P)}}{|P|^\tau} < \infty\,, \qquad \textit{and} \qquad \|f\|^{\spadesuit}_{B^{s,\tau}_{p,q}} < \infty\,.$$

Further

$$\sup_{\{P\in\mathcal{Q},\,|P|\ge 1\}} \frac{\|f\|_{B^{s_0}_{p,\infty}(2P)}}{|P|^\tau} + \|f\|_{L^\tau_p} + \|f\|^{\spadesuit}_{B^{s,\tau}_{p,q}}$$

and $\|f\|_{B^{s,\tau}_{p,q}}$ are equivalent.

Remark 28. (i) A proof has been given in [102, Theorem 4.8].

(ii) In case of the Nikol'skij type spaces $B^{s,\tau}_{p,\infty}$ the general approach of Hedberg and Netrusov [31] yields a slightly different characterization in view of the identity $B^{s,\frac{1}{p}-\frac{1}{u}}_{p,\infty} = \mathcal{N}^s_{u,\infty,p}$, see Theorem 3.9.4 below.

(iii) For $\tau = 0$ we refer to Triebel [84, Section 3.5.3].

3.9.2 The characterization of Lizorkin-Triebel-Morrey spaces by ball means of differences

The Lizorkin-Triebel-Morrey spaces $\mathcal{E}^s_{p,q,u}$ are special realizations of the general class of Lizorkin-Triebel spaces considered in Hedberg and Netrusov [31], see also part II of this survey [76]. There Hedberg and Netrusov developed an axiomatic approach to function spaces of Nikol'skij-Besov-Lizorkin-Triebel type including characterizations by atoms and differences. They work with ball means of differences. We shall use the abbreviations $B(x,r) := \{y \in \mathbb{R}^d :\ |x-y| < r\}$, $x \in \mathbb{R}^d$, $r > 0$, and

$$b_{v,t}f(x) := \left(\frac{1}{t^n}\int_{|x-h|<t} |\Delta^M_h f(x)|^v\,dh\right)^{1/v}\,, \qquad t > 0\,, \quad x \in \mathbb{R}^d\,.$$

The outcome is the following, we refer to [102, Section 4.5] for all details.

Theorem 3.9.3. *Let $0 < v < \infty$, $0 < q \le \infty$, $0 < u \le p < \infty$, and $M \in \mathbb{N}$ such that*

$$0 < r < \min(u\,q) \qquad \textit{and} \qquad d\,\max\left\{\frac{1}{r}-1,\,\frac{1}{r}-\frac{1}{v}\right\} < s < M\,.$$

Then the following assertions are equivalent for functions in $L^{\ell oc}_r$:

(i) $f \in \mathcal{E}^s_{p,q,u} = F^{s,\frac{1}{u}-\frac{1}{p}}_{u,q}$;

(ii) $f \in L^{\ell oc}_v$ *and*

$$\|f\|^*_{\mathcal{E}^s_{p,q,u}} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\frac{1}{u}-\frac{1}{p}}} \left(\int_Q \left[\int_{B(x,1)} |f(y)|^v\, dy\right]^{u/v} dx\right)^{1/u} + \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\frac{1}{u}-\frac{1}{p}}} \left(\int_Q \left[\int_0^1 t^{-sq}\,(b_{v,t}f)^q(x)\,\frac{dt}{t}\right]^{u/q} dx\right)^{1/u} < \infty\,.$$

The quasi-norms $\|f\|_{\mathcal{E}^s_{p,q,u}}$ *and* $\|f\|^*_{\mathcal{E}^s_{p,q,u}}$ *are equivalent.*

Remark 29. (i) In [102, 4.3.1] we derived a number of different characterizations of $F^{s,\tau}_{p,q}$ in terms of differences. Also the arguments in the proof slightly differ. In addition, $\tau = 1/p$ is admissible there. In this context we also have to mention Drihem [15] who proved some characterizations by differences without taking ball means.

(ii) For $\tau = 0$ we refer to Seeger [73] and Triebel [84, 3.5.3]

These characterization by differences allow also some conclusions for the classes $\mathcal{E}^s_{p,q,u,\mathrm{unif}}$, see Definition 10

Corollary 3.9.1. *Let* $0 < v < \infty$, $0 < q \le \infty$, $0 < u \le p < \infty$, *and* $M \in \mathbb{N}$ *such that*

$$0 < r < \min(u\,q) \qquad \text{and} \qquad d\,\max\left\{\frac{1}{r}-1,\, \frac{1}{r}-\frac{1}{v}\right\} < s < M\,.$$

Then $\mathcal{E}^s_{p,q,u,\mathrm{unif}}$ *is the collection of all* $f \in L^{\ell oc}_u$ *such that*

$$\|\, f\,\|^*_{\mathcal{E}^s_{p,q,u,\mathrm{unif}}} := \sup_{\substack{Q\in\mathcal{Q}\\ |Q|\le 1}} \frac{1}{|Q|^{\frac{1}{u}-\frac{1}{p}}} \left(\int_Q \left[\int_{B(x,1)} |f(y)|^v\, dy\right]^{u/v} dx\right)^{1/u} + \sup_{\substack{Q\in\mathcal{Q}\\ |Q|\le 1}} \frac{1}{|Q|^{\frac{1}{u}-\frac{1}{p}}} \left(\int_Q \left[\int_0^1 t^{-sq}\,(b_{v,t}f)^q(x)\,\frac{dt}{t}\right]^{u/q} dx\right)^{1/u} < \infty\,.$$

Furthermore, $\|\, f\,\|_{\mathcal{E}^s_{p,q,u,\mathrm{unif}}}$ *and* $\|\, f\,\|^*_{\mathcal{E}^s_{p,q,u,\mathrm{unif}}}$ *are equivalent.*

Also Nikol'skij-Besov-Morrey spaces can be characterized by differences in a similar way. The Hedberg-Netrusov approach yields the following.

Theorem 3.9.4. *Let* $0 < v \le \infty$, $0 < r < u \le p \le \infty$, $M \in \mathbb{N}$ *and*

$$d\,\max\left\{\frac{1}{r}-1,\, \frac{1}{r}-\frac{1}{v}\right\} < s < M\,.$$

Then the following assertions are equivalent for functions in $L^{\ell oc}_r$:

(i) $f \in \mathcal{N}^s_{p,q,u}$;

(ii) $f \in L_v^{\ell oc}$ *and*

$$\|f\|^*_{\mathcal{N}^s_{p,q,u}} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\frac{1}{u}-\frac{1}{p}}} \left(\int_Q \left[\int_{B(x,1)} |f(y)|^v\, dy\right]^{u/v} dx\right)^{1/u} + \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\frac{1}{u}-\frac{1}{p}}} \left(\int_0^1 t^{-sq} \left[\int_Q |b_{v,t}f(x)|^u dx\right]^{q/u} \frac{dt}{t}\right)^{1/q} < \infty\, .$$

The quasi-norms $\|f\|_{\mathcal{N}^s_{p,q,u}}$ *and* $\|f\|^*_{\mathcal{N}^s_{p,q,u}}$ *are equivalent.*

Remark 30. In view of the the identity $B^{s,\,1/u-1/p}_{u,\infty} = \mathcal{N}^s_{p,\infty,u}$ Theorem 3.9.4 supplements the results of Theorems 3.9.1, 3.9.2.

Similar as in the $\mathcal{E}$-case we can derive a conclusion about $\mathcal{N}^s_{p,q,u,\mathrm{unif}}$.

Corollary 3.9.2. *Let* $0 < v \le \infty$, $0 < r < u \le p \le \infty$, $M \in \mathbb{N}$ *and*

$$d\, \max\left\{\frac{1}{r} - 1,\, \frac{1}{r} - \frac{1}{v}\right\} < s < M\, .$$

Then $\mathcal{N}^s_{p,q,u,\mathrm{unif}}$ *is the collection of all* $f \in L_u^{\ell oc}$ *such that*

$$\| f \|^*_{\mathcal{N}^s_{p,q,u,\mathrm{unif}}} := \sup_{\substack{Q\in\mathcal{Q}\\ |Q|\le 1}} \frac{1}{|Q|^{\frac{1}{u}-\frac{1}{p}}} \left(\int_Q \left[\int_{B(x,1)} |f(y)|^v\, dy\right]^{u/v} dx\right)^{1/u} + \sup_{\substack{Q\in\mathcal{Q}\\ |Q|\le 1}} \frac{1}{|Q|^{\frac{1}{u}-\frac{1}{p}}} \left(\int_0^1 t^{-sq} \left[\int_Q |b_{v,t}f(x)|^u dx\right]^{q/u} \frac{dt}{t}\right)^{1/q} < \infty\, .$$

Furthermore, $\| f \|_{\mathcal{N}^s_{p,q,u,\mathrm{unif}}}$ *and* $\| f \|^*_{\mathcal{N}^s_{p,q,u,\mathrm{unif}}}$ *are equivalent.*

3.9.3 The classes $B^{s,\tau}_{p,p}$ and their relations to Q spaces

In recent years, independent of the existing literature on Nikol'skij-Besov and Lizorkin-Triebel spaces, there were a lot of interest in Q_α spaces.

Definition 15. *Let* $\alpha \in \mathbb{R}$*. The space* Q_α *is defined to be the collection of all* $f \in L_2^{\ell oc}$ *such that*

$$\|f\|_{Q_\alpha} := \sup_Q \left\{ \frac{1}{|Q|^{1-\frac{2\alpha}{d}}} \int_Q \int_Q \frac{|f(x) - f(y)|^2}{|x-y|^{d+2\alpha}}\, dx\, dy \right\}^{1/2} < \infty,$$

where Q *ranges over all cubes in* $\mathbb{R}^d$*.*

Remark 31. The history of Q_α spaces (or simply Q spaces) started in 1995 with a paper by Aulaskari, Xiao and Zhao [2]. Originally they were defined as spaces of holomorphic functions on the unit disk, which are geometric in the sense that they transform naturally under conformal mappings (see [2], [93]). Following earlier contributions of Essén and Xiao [20] and Janson [36] on the boundary values of these functions on the unit circle, Essén, Janson, Peng and Xiao [19] extended these spaces to the d-dimensional Euclidean space $\mathbb{R}^d$. There is a rapidly increasing literature devoted to this subject, we refer, e.g., to [2, 93, 19, 20, 36, 11, 12, 91, 92, 94, 95, 103].

Mainly as a consequence of Theorem 3.9.1 one can prove the following characterization of $B^{s,\tau}_{p,p}$, see [102, 4.3.3].

Corollary 3.9.3. *Let $1 \le p \le \infty$, $0 < s < 1$ and $0 \le \tau \le 1/p$. Then $f \in B^{s,\tau}_{p,p}$ if, and only if, $f \in L^\tau_p$ and*

$$|||f|||^{\diamondsuit}_{B^{s,\tau}_{p,p}} := \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^\tau} \left\{ \int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{sp+n}}\, dx\, dy \right\}^{1/p} < \infty\,. \tag{3.9.2}$$

Furthermore, $\|f\|_{L^\tau_p} + |||f|||^{\diamondsuit}_{B^{s,\tau}_{p,p}}$ and $\|f\|_{B^{s,\tau}_{p,p}}$ are equivalent.

Remark 32. (i) With other words: the spaces $B^{s,\tau}_{2,2}$ and $Q_\alpha \cap L^\tau_2$ coincide in the sense of equivalent norms as far as $0 < s = \alpha < 1$ and $\tau = \frac{1}{2} - \alpha/d \ge 0$.

(ii) Originally Dafni and Xiao [11] posed the question on the relation of Q spaces and Nikol'skij-Besov-Lizorkin-Triebel spaces. In fact, it holds

$$\dot{B}^{\alpha,\frac{1}{2}-\frac{\alpha}{d}}_{2,2} = Q_\alpha$$

if $\alpha \in (0,1)$ $(d \ge 2)$, see Yang and Yuan [97, 98]. Here $\dot{B}^{\alpha,\frac{1}{2}-\frac{\alpha}{d}}_{2,2}$ denotes the homogeneous counterpart of $B^{\alpha,\frac{1}{2}-\frac{\alpha}{d}}_{2,2}$.

3.10 Key theorems

Key theorems are those which are needed to establish a corresponding theory for function spaces on smooth domains, see Triebel's monograph [84]. We focus on pointwise multipliers, diffeomorphisms and traces.

3.10.1 Pointwise multipliers

Pointwise multiplication in Besov and Triebel-Lizorkin spaces has been studied extensively in the last 30 years; see, for example, [65], [83], [52], [84], [68] and [53]. The two monographs [52], [53] by Maz'ya and Shaposnikova are completely devoted to this subject. However, the authors restrict their interest essentially to the Sobolev and Bessel-potential spaces $F^s_{p,2}$, $1 < p < \infty$, and the Slobodeckij spaces $B^s_{p,p}$, $1 \le p \le \infty$.

Let X and Y be two quasi-Banach spaces of functions (distributions). Then the basic question consists in descriptions of the associated multiplier space $M(X,Y)$ given by

$$M(X,Y) \equiv \{f \,:\, f \cdot g \in Y \text{ for all } g \in X\}\,.$$

This space is equipped with the induced quasi-norm

$$\|f\|_{M(X,Y)} := \sup_{\|g\|_X \le 1} \|f \cdot g\|_Y\,.$$

Here, in this survey, we will be concerned with the easier problem of proving embeddings into $M(X) := M(X,X)$ with $X = A^{s,\tau}_{p,q}$.

The first nontrivial result we want to present is the fact that some finite Hölder-Zygmund regularity of a function is sufficient to be a pointwise multiplier for a space $A^{s,\tau}_{p,q}$, see [102, Theorem 6.1].

Theorem 3.10.1. *Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$ and $0 \le \tau \le 1/p$. If $m \in \mathbb{N}$ is sufficiently large, then there exists a positive constant c such that for all $g \in \mathcal{Z}^m$ and all $f \in A^{s,\tau}_{p,q}$,*

$$\| g \cdot f \|_{A^{s,\tau}_{p,q}} \le c \, \|g\|_{\mathcal{Z}^m} \, \|f\|_{A^{s,\tau}_{p,q}} \, . \tag{3.10.1}$$

In case of the F-spaces a more precise estimate can be given.

Theorem 3.10.2. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$ and $0 \le \tau < 1/p$. Suppose*

$$\varrho > \max \left\{ |s|, \, \frac{d}{p} - d - s \right\} . \tag{3.10.2}$$

Then the embedding $\mathcal{Z}^\varrho \subset M(F^{s,\tau}_{p,q})$ holds.

Remark 33. (i) For $\tau = 0$ this is a well-known result, we refer to [83, Corollary 2.8.2], [26] and [68, Section 4.7.1].

(ii) The proof can be found in [102, 6.1.2.3]. It uses paramultiplication, very much in the spirit of [83, Corollary 2.8.2]. Further tools are Marschall's pointwise inequality for certain convolutions, see [50], [39] and [102, 6.1.2.1], and Nikol'skij type characterizations, see [83, 2.5.2], [96], [68, 2.3.2] and [102, 6.1.2.1]. Probably these arguments carry over the case of Besov type spaces. But we did not check all details.

Multiplication Algebras

This time we study the question under which conditions we have the embedding $X \subset M(X)$. Just for having a simple reference at hand we concentrate on the F-case. Essentially the same methods as used in the proof of Theorem 3.10.2 apply, see [102, 6.1.2.4].

Theorem 3.10.3. *Let $0 < p < \infty$, $0 < q \le \infty$, $0 \le \tau < 1/p$ and $s > \sigma_{p,q}$. Then there exists a positive constant c such that for all $f, g \in F^{s,\tau}_{p,q} \cap L_\infty$,*

$$\|f \cdot g\|_{F^{s,\tau}_{p,q}} \le c \left(\|f\|_{L_\infty} \|g\|_{F^{s,\tau}_{p,q}} + \|g\|_{L_\infty} \|f\|_{F^{s,\tau}_{p,q}} \right) . \tag{3.10.3}$$

Remark 34. (i) The estimate (3.10.3) implies that the spaces $F^{s,\tau}_{p,q} \cap L_\infty$ are algebras with respect to pointwise multiplication.

(ii) For $\tau = 0$ we refer to Runst [67] and [68, Theorem 4.6.4/2].

(iii) Such Moser-type estimates have been proved also for the spaces $N^s_{p,q,u}$, see Mazzucato [55].

Combining Theorem 3.10.3 with some embeddings, see Lemma 3.2, combined with Corollary 2.2 and Proposition 2.6 in [102], we get the following conclusion concerning the algebra properties of $F^{s,\tau}_{p,q}$.

Corollary 3.10.1. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$ and $0 \le \tau < 1/p$ such that*

$$s > d \, \max \left\{ \frac{1}{p} - \tau, \, \frac{1}{q} - 1 \right\} .$$

Then $F^{s,\tau}_{p,q}$ is an algebra with respect to pointwise multiplication.

Remark 35. (i) For $\tau = 0$ this question had some history. For the Bessel potential spaces $H^s_p = F^{s,0}_{p,2}$, $p \in (1,\infty)$, it was settled by Strichartz [77]. This was extended by Triebel in [82, Section 2.6.2], Kalyabin [41, 42] and Franke [23]; see also [68, Theorem 4.6.4/1].

(ii) Characterizations of $M(W^m_p)$, H^s_p and $M(B^s_{p,p})$ can be found in the monographs of Maz'ya and Shaposnikova [52, 53]. For a characterization of $M(F^s_{p,q})$, $s > d/p$, we refer to Franke [23] and [68, Theorem 4.9.1/1].

(iii) Mazzucato [55] proved that the classes $N^s_{p,q,u}$ are algebras with respect to pointwise multiplication if either $1 \le u \le p < \infty$, $s > d/p$ and $1 < q \le \infty$ or $1 \le u \le p < \infty$, $s \ge d/p$ and $q = 1$.

The case $\tau = 1/p$ can be treated separately, see [102, Theorem 6.4].

Theorem 3.10.4. *Let $0 < q \le \infty$ and $\sigma_{1,q} < s < \infty$. Then $M(F^s_{\infty,q}) = F^s_{\infty,q}$ in the sense of equivalent quasi-norms.*

Remark 36. (i) The assertion of Theorem 3.10.4 does not extend to $s = 0$. E.g, if $q = 2$, the correct description of $M(\mathrm{bmo})$ was found by Janson [35]. For a description of $M(B^0_{\infty,\infty})$ we refer to [43].

(ii) Theorem 3.10.4 implies that the spaces $F^s_{\infty,q}$ are algebras with respect to pointwise multiplication, at least, if $s > \sigma_{1,q}$. For $q \ge 1$ a different proof of this fact can be found in Marschall [49].

A characterization of $M(F^s_{p,q})$, $s < d/p$

As said above, in case $\tau = 0$ much more is known; see, for example, [74]. Of certain relevance for this survey is the description of $M(F^s_{p,q})$, $0 < p < 1$, $\sigma_{p,q} < s < d/p$, given by Netrusov [58].

Theorem 3.10.5. *Let $0 < p \le 1$, $0 < q \le \infty$ and $\sigma_{p,q} < s < d/p$. Then $f \in M(F^s_{p,q})$ if, and only if, $f \in L_\infty$, f can be represented in $\mathcal{S}'$ in the form*

$$f = \sum_{j=0}^{\infty} f_j \, , \qquad \operatorname{supp} \mathcal{F} f_j \subset \{\xi : \quad 2^{j-1} \le |\xi| \le 2^{j+1}\} \, , \quad j \in \mathbb{N} \, ,$$

$\operatorname{supp} \mathcal{F} f_0 \subset \{\xi : \quad |\xi| \le 2\}$, *such that*

$$M(f) := \sup_{j \in \mathbb{N}_0} \sup_{x \in \mathbb{R}^d} 2^{j(\frac{d}{p} - s)} \left(\int_{B(x,2^{-j})} \left[\sum_{k=j}^{\infty} 2^{ksq} |f_k(x)|^q \right]^{p/q} dx \right)^{1/p} < \infty \, .$$

Remark 37. (i) Clearly, if either $f \in F^{s,\frac{1}{p}-\frac{s}{d}}_{p,q}$ or $f \in E^s_{u,q,p}$, $u = d/s$ (see Proposition 3.3(iii) and Definition 11), then $M(f) < \infty$, i.e.,

$$L_\infty \cap F^{s,\frac{1}{p}-\frac{s}{d}}_{p,q} \hookrightarrow L_\infty \cap E^s_{d/s,q,p} \hookrightarrow M(F^{s,\tau}_{p,q}) \qquad (3.10.4)$$

if $\sigma_{p,q} < s < d/p$.

(ii) Netrusov [58] did not publish a proof of this remarkable result. A proof under more restrictive conditions can be found in [75].

For the special case $p = q = 1$ some more simple characterizations of $M(F^s_{1,1})$ have been found by Maz'ya and Shaposnikova; see [52, 3.4.2].

Theorem 3.10.6. *Let $s = m + \sigma$, where m is a nonnegative integer and σ is a real number with $\sigma \in (0,1)$. Then $f \in M(F^s_{1,1})$ if, and only if, $f \in L_\infty$ and*

$$\sup_{0<r<1} \sup_{x\in\mathbb{R}^d} \quad r^{s-d} \sum_{|\alpha|\le m} \bigg(\int_{B(x,r)} |D^\alpha f(y)|\, dy + \int_{B(x,r)} \int_{B(x,r)} \frac{|D^\alpha f(y) - D^\alpha f(x)|}{|y-x|^{d+\sigma}} dy\, dx \bigg) < \infty\,. \tag{3.10.5}$$

We would like to reformulate Theorem 3.10.6. Therefore we recall that $A^{s,\tau}_{p,q,\mathrm{unif}}$ has been defined in Definition 10. Let $m = 0$ in Theorem 3.10.6. Then, as an immediate conclusion of Corollary 3.9.3, we obtain the following.

Corollary 3.10.2. *Let $0 < s < 1$. Then $f \in M(F^s_{1,1})$ if, and only if $f \in L_\infty \cap F^{s,\tau}_{1,1,\mathrm{unif}}$, where $\tau = 1 - s/d$.*

3.10.2 Diffeomorphisms

By C we denote the collection of all complex-valued bounded and continuous functions on $\mathbb{R}^d$. We begin with recalling the notion of diffeomorphisms; see, for example, [84, p. 206].

Definition 16. *(i) Let $m \in \mathbb{N}$. A one-to-one mapping $y = \psi(x)$ of $\mathbb{R}^d$ onto $\mathbb{R}^d$ is called a m-diffeomorphism if the components ψ_j of $\psi := (\psi_1, \cdots, \psi_d)$ have classical derivatives up to order m with $D^\alpha\psi_j \in C$ if $0 < |\alpha| \le m$, and $|\det \psi_*(x)| \ge c > 0$ for some positive constant c and all $x \in \mathbb{R}^d$, where ψ_* stands for the Jacobian matrix of ψ.*
(ii) The mapping ψ is called a diffeomorphism if it is a m-diffeomorphism for any $m \in \mathbb{N}$.

We remark that if ψ is a m-diffeomorphism, then its inverse ψ^{-1} is also a m-diffeomorphism. Further, if ψ is a diffeomorphism then the mapping

$$D_\psi : \quad f \longrightarrow f \circ \psi\,, \qquad f \in \mathcal{S}'\,,$$

makes sense. If ψ is only a m-diffeomorphism and $f \in A^{s,\tau}_{p,q}$, the composition $f \circ \psi$ can be defined via smooth atoms for $A^{s,\tau}_{p,q}$. We do not go into details at this technical point. Based on the smooth atomic decomposition of $A^{s,\tau}_{p,q}$ in [102, 3.1] we have the following conclusion.

Theorem 3.10.7. *Let $m \in \mathbb{N}$, ψ be an m-diffeomorphism. Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and $0 \le \tau \le 1/p$. If $m \in \mathbb{N}$ is sufficiently large, then D_ψ is an isomorphic mapping of $A^{s,\tau}_{p,q}$ onto itself ($p < \infty$ if $A^{s,\tau}_{p,q} = F^{s,\tau}_{p,q}$).*

Remark 38. For the case $\tau = 0$ we refer to [84, Proposition 4.3.1, Remark 4.3.1, Theorem 4.3.2]. However, in special situations much more is known. In case of Sobolev spaces the group around Reshetnyak, Gol'dstein and Vodop'yanov worked on this topic, see, e.g., [29]. Also Maz'ya [51] has dealt with this topic. In case of Besov spaces we refer to Vodop'yanov [88] and to Bourdaud and Sickel [8].

3.10.3 Traces

We are interested in properties of the trace operator

$$\mathrm{Tr}: \quad f(x', x_d) \to f(x', 0), \qquad x' := (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}. \tag{3.10.6}$$

For $\tau = 0$ such problems have been treated extensively; see the remarks below. Clearly, (3.10.6) makes sense for all continuous functions f and therefore, for smooth atoms. Frazier and Jawerth [25, 26] were the first which have shown that the use of atomic decompositions in the context of trace problems is a very good and successful idea. By employing the method of Frazier and Jawerth Sawano, Yang and Yuan in [72] have been able to characterize the images of the homogeneous spaces $\dot{A}^{s,\tau}_{p,q}$ under the mapping Tr. Essentially by the same arguments the nonhomogeneous case has been treated in [102, Theorem 6.8].

Theorem 3.10.8. *Let $d \geq 2$, $0 < p, q \leq \infty$, $0 \leq \tau < 1/p$ and*

$$s > \frac{1}{p} + (d-1)\Big(\frac{1}{\min(1, p)} - 1\Big)$$

Then Tr *is a linear, continuous and surjective operator from $B^{s,\tau}_{p,q}$ onto $B^{s-\frac{1}{p},\frac{d\tau}{d-1}}_{p,q}(\mathbb{R}^{d-1})$ and from $F^{s,\tau}_{p,q}$ onto $F^{s-\frac{1}{p},\frac{d\tau}{d-1}}_{p,p}(\mathbb{R}^{d-1})$ ($p < \infty$).*

Remark 39. (i) For $\tau = 0$ we are back in the classical case. It is interesting to notice that the mapping Tr does not lead to a change of the smoothness s only, but also to a change of the Morrey parameter τ.

(ii) For the classical trace theorems for Nikol'xsij-Besov spaces and Triebel-Lizorkin spaces, i.e., the case $\tau = 0$, we refer to Nikol'skij [59], [60], Besov, Iljin and Nikol'skij [4], Jawerth [37], Frazier and Jawerth [25, 26], and Triebel [83, Section 2.7.2], [84, Section 4.4].

(iii) Limiting situations for $\tau = 0$, i.e. $s = \frac{1}{p} + (d-1)\max\{0, 1/p - 1\}$, are investigated in Peetre [63], Burenkov and Gol'dman [10], Frazier and Jawerth [26], Triebel [84, Section 4.4.3] and Farkas, Johnsen and S. [21].

(iv) Frazier and Jawerth proved in [26, Theorem 11.2] that in case $s > 0$, $0 < q \leq \infty$, the operator Tr extends to a linear, continuous and surjective mapping of $F^s_{\infty,q}(\mathbb{R}^d)$ onto $F^s_{\infty,q}(\mathbb{R}^{d-1})$. In this context we also refer to Marschall [49].

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