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OPERATORS IN MORREY TYPE SPACES AND APPLICATIONS

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Abstract. We consider partial differential equations with discontinuous coefficients and prove that, if the known term belongs to the Morrey space $L^{p,\lambda}$, the highest order derivatives of the solutions of the equations belong to the same space. As a consequence it is possible to obtain local Hölder continuity for the solutions. Moreover, are discussed some estimates for the derivatives of local minimizers of variational integrals.

1 Introduction

We study regularity properties of solutions of partial differential equations and systems. Preparatory to studying partial differential equations we shall discuss the action of some integral operators, that we extensively use. Then, the regularity properties of solutions of elliptic, parabolic and ultraparabolic equations of second order with discontinuous coefficients, and later of systems, will be discussed in depth.

To be more specific let us consider in the sequel a bounded open set $\Omega \subset \mathbb{R}^n$ with $\partial\Omega$ sufficiently smooth boundary, $f \in L^{p,\lambda}(\Omega)$, $1 < p < +\infty$, $0 < \lambda < n$, and the following equation

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} \;=\; f. \tag{1.1}$$

where a_{ij} are, in general, discontinuous functions.

Let us now recall the definition of the Morrey spaces $L^{p,\lambda}$ because in the sequel we are interested in Morrey regularity of the highest order derivatives of u in these spaces.

Definition 1. *([29]). Let $1 < p < \infty$ and $0 < \lambda < n$. A measurable function $f \in L^p_{loc}(\Omega)$ is in the Morrey class $L^{p,\lambda}(\Omega)$ if the following norm is finite*

$$\|f\|_{L^{p,\lambda}(\Omega)} = \sup_{x\in\Omega, 0<R<\operatorname{diam}\Omega} \left(\frac{1}{R^\lambda} \int\limits_{\Omega\cap B(x,R)} |f(y)|^p dy \right)^{\frac{1}{p}},$$

where $B(x,R)$ is the open ball centered at x of radius R.

Definition 2. *Let $f \in L^1(\Omega)$, we set the integral mean $f_{x,R}$ by*

$$f_{x,R}=\frac{1}{|\Omega\cap B(x,R)|}\int_{\Omega\cap B(x,R)} f(y)dy,$$

where $|\Omega \cap B(x,R)|$ is the Lebesgue measure of $\Omega \cap B(x,R)$.

If we are not interested in specifying which the center is, we write just f_R.

Let us now give the definition of functions of bounded mean oscillation (BMO) that appear at first in the note by John and Nirenberg [26].

Definition 3. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We say that f belongs to $BMO(\mathbb{R}^n)$ if the seminorm*

$$\|f\|_* \equiv \sup_{x\in\mathbb{R}^n,\, R>0} \frac{1}{|B(x,R)|}\int_{B(x,R)} |f(y)-f_{x,R}|dy$$

is finite

Let us recall the definition of the space of vanishing mean oscillation functions, given at first by Sarason in [44].

Definition 4. *Let $f \in BMO(\mathbb{R}^n)$, $R > 0$ and*

$$\eta(f,R) = \sup_{x\in\mathbb{R}^n,\, 0<\rho\le R} \frac{1}{|B(x,\rho)|}\int_{B(x,\rho)} |f(y)-f_\rho|dy$$

where $B(x,\rho)$ ranges over the class of the balls of $\mathbb{R}^n$ of radius ρ.

A function $f \in VMO(\mathbb{R}^n)$ if

$$\lim_{R\to 0} \eta(f,R) = 0.$$

2 State of the art

Let us consider, first, the following second order elliptic equation in nondivergence form

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} \;=\; f.$$

Regularity results for elliptic equations in nondivergence form with the right-hand side f in Morrey spaces $L^{p,\lambda}$ are obtained by Di Fazio and the author.

Let us consider the following second order differential operator

$$L\,u \equiv \sum_{i,j=1}^{m_0} \partial_{x_i}\left(a_{i,j}(x,t)\partial_{x_j}u\right) + \sum_{i,j=1}^{N} b_{i,j}x_i\partial_{x_j}u - \partial_t u,$$

where $z \;=\; (x,t) \in \mathbb{R}^{N+1}, 0 < m_0 \le N$.

This kind of linear operators of Fokker-Plank type are used in probability and in mathematical physics, for instance in the study of brownian motion of a particle in a fluid.

We point out that the natural geometry for the above operator is not euclidean but is given by a suitable groups structure.

Let us suppose that the matrix $A(z)$ is the $N \times N$ matrix

$$A(z) = \begin{pmatrix} A_0(z) & 0 \\ 0 & 0 \end{pmatrix}$$

where $A_0(z) = \left(a_{i,j}(z)\right)_{i,j=1,\dots,m_0}$ is symmetric and such that there exists $\Lambda > 0$ such that

$$\Lambda^{-1}|\xi|^2 \le \langle A_0(z)\xi, \xi\rangle \le \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^{m_0}, \forall z \in \mathbb{R}^{N+1}.$$

Suppose also that $B = (b_{i,j})$ is a suitable $N \times N$ constant real matrix.

Polidoro and Ragusa in [34] proved interior regularity for weak solutions to the following equation

$$\sum_{i,j=1}^{m_0} \partial_{x_i}\left(a_{i,j}(z)\partial_{x_j}u\right) + \sum_{i,j=1}^{N} b_{i,j}x_i\partial_{x_j}u - \partial_t u = \sum_{j=1}^{m_0}\partial_{x_j}F_j(z), \tag{2.1}$$

where F_j belong to a function space of Morrey type.

Local Hölder continuity of the solution u is also proved.

The authors considered $0 < m_0 \le N$ and $B = \left(b_{i,j}\right)_{i,j=1,\dots,N}$ a constant real matrix of the following form

$$B = \begin{pmatrix} 0 & B_1 & 0 & \dots & 0 \\ 0 & 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_r \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where each B_j is a $m_{j-1} \times m_j$ block matrix of rank m_j, with $j = 1, 2, ..., r$, and $m_0 \ge m_1 \ge ... \ge m_r \ge 1$ such that $m_0 + m_1 + ... + m_r = N$.

The study of this kind of operators arises in the stochastic theory, see the book [45] by Shiryayev, and in the theory of diffusion processes, see [6] by Chandrasekhar and [31] by Nguyen Dong An. Let us point out our attention to the operator

$$\mathcal{S} \equiv \sum_{j=1}^{n} \partial_{x_j}^2 + \sum_{j=1}^{n} x_j\partial_{x_{n+j}} - \partial_t u \tag{2.2}$$

for even N and $n = \frac{N}{2}$.

This is the linearized prototype of the Fokker-Plank operator that describes, under suitable conditions, the moving of brownian particles in a flow.

The operator $\mathcal{S}$ is of degenerate type because there are only $\frac{N}{2}$ second order derivatives. If we set $X_j = \partial_{x_j}$, $j = 1, \dots, n$, and

$$Y = \langle x, BD\rangle - \partial_t$$

the operator $\mathcal{S}$ takes the following form

$$\mathcal{S} = \sum_{j=1}^{n} X_j^2 + Y.$$

Let us observe that the equation we are studying is a linearized version of the Landau equation that in its turns is a simplified model for the Boltzmann equation, see the paper [24] by Landau.

We are interested in the interior regularity of weak solutions of the above equation (2.1). If A is a **constant matrix** and $F_j \in C^\infty$ then $u \in C^\infty$, this is proved in the note [23] by Lanconelli and Polidoro.

If a_{ij} are **Hölder continuous** the operator L considered in the above equation was studied by Polidoro in the papers [32], [33] and also by Manfredini in [27]. In the last mentioned paper interior Schauder estimates are also proved.

Some related results have been obtained by Lunardi and Da Prato in [25] and [9] in the setting of semigroup theory.

If a_{ij} are **not uniformly continuous** the problem has been less studied. In the note [2] by Bramanti, Cerutti and Manfredini the authors have studied interior regularity of strong solutions to nondivergence form of above equation while regularity results in the divergence case has been proved by Manfredini and Polidoro in [28].

In the study carried out by Polidoro and Ragusa in [34], the authors consider discontinuous coefficients a_{ij}, precisely a_{ij} in the Sarason class VMO_L of functions of vanishing mean oscillation, the subset of the John-Nirenberg class BMO_L. We remark that in the notation for the classes BMO_L and VMO_L we emphasize the role of the operator L, because they are naturally associated with the following group's structure.

Definition 5. *Let $(x,t),(\xi,\tau)$ be in $\mathbb{R}^{N+1}$. We set*

$$(x,t)\circ(\xi,\tau) = (\xi + E(\tau)x, t+\tau), \quad E(t) = \exp(-tB^T)$$

and

$$D(\lambda) = \operatorname{diag}\left(\lambda I_{m_0}, \lambda^3 I_{m_1}, \ldots, \lambda^{2r+1} I_{m_r}\right),$$

where I_{m_j} is the $m_j \times m_j$ identity matrix.

We say that $\left(\mathbb{R}^{N+1},\circ\right)$ is the "translation group associated to L" and that $\left(D(\lambda),\lambda^2\right)_{\lambda>0}$ is the "dilation group associated to L".

Definition 6. *We call "homogeneous dimension" of $\mathbb{R}^{N+1}$ the integer $Q+2$, where*

$$Q = m_0 + 3m_1 + \ldots + (2r+1)m_r.$$

In the above mentioned note [34] the authors improve the results of Manfredini and Polidoro [28] assuming that the term $F = (F_1, F_2, \ldots, F_{m_0}, 0, \ldots, 0)$ is such that every F_j belongs the following Morrey space $L^{p,\lambda}(L,\Omega)$.

Definition 7. *Let Ω be a bounded open subset of R^{N+1}, $1<p<+\infty$ and $\lambda \in]0, Q+2[$, where $Q = m_0 + 3m_1 + \ldots + (2r+1)m_r$. A function $f \in L^p_{loc}(\Omega)$ is in $L^{p,\lambda}(L,\Omega)$ if*

$$\|f\|^p_{L^{p,\lambda}(L,\Omega)} = sup_{r>0, z\in\Omega} \frac{1}{r^\lambda} \int_{\Omega\cap B_r(z)} |f(w)|^p dw < +\infty.$$

The method used in [34] to obtain the results is inspired by the technique introduced by Chiarenza, Frasca and Longo in the papers [7] and [8].

It is based on explicit representation formulas for the first derivatives of the weak solutions of the above equation (1.1) and on the $L^{p,\lambda}$ estimates of singular integral operators and commutators of non convolution type with a Calderón-Zygmund kernel.

Let us now state the theorems obtained, using the notation

$$Yu = \langle x, BDu\rangle - \partial_t u,$$

which enables to equivalently write equation (2.1) in the following form

$$div\,(A(z)Du) + Yu = div(F). \tag{2.3}$$

The main results obtained by Polidoro and Ragusa [34] concerning the divergence case are contained in the following two theorems.

Theorem 2.1. *Let Ω be a bounded open subset of R^{N+1} and u be a weak solution in Ω of the equation*

$$div\,(A(x,t)Du) + Yu = div(F).$$

Let us suppose that the matrix B has the above considered structure. Let us also assume that $a_{ij} \in VMO_L$, $i,j = 1,\dots,m_0$, $u \in L^p(\Omega)$, $F_j \in L^{p,\lambda}(L,\Omega)$, $\forall j = 1,\dots,m_0$, $0 < \lambda < Q+2$, and $1 < p < \infty$.

Then, for any compact set $K \subset \Omega$, we have that $\partial_{x_j} u \in L^{p,\lambda}(L,K)$, $\forall j = 1,\dots,m_0$, for every $1 < p < \infty$ and $0 < \lambda < Q+2$.

Moreover there exists a positive constant c depending only on p,λ,K,Ω and L such that, $\forall j = 1,\dots,m_0$,

$$\|\partial_{x_j} u\|_{L^{p,\lambda}(L,K)} \le c\left(\sum_{k=1}^{m_0}\|F_k\|_{L^{p,\lambda}(L,\Omega)} + \|u\|_{L^p(\Omega)}\right). \tag{2.4}$$

Theorem 2.2. *Let Ω be a bounded open subset of R^{N+1} and u be a weak solution in Ω of equation* (2.1)*.*

Let us suppose that the operator L satisfies the same assumptions as in the above theorem. Let us also assume that $a_{ij} \in VMO_L$, $i,j = 1,\dots,m_0$, $u \in L^p(\Omega)$, $F_j \in L^{p,\lambda}(L,\Omega)$ $\forall j = 1,\dots,m_0$, $0 \le \lambda < Q+2$, and $p > Q+2-\lambda$.

Then, for any compact $K \subset \Omega$ there exists a positive constant c depending only on p,λ,K,Ω and L such that, $\forall z,\zeta \in K$, $z \ne \zeta$,

$$\frac{|u(z)-u(\zeta)|}{\|\zeta^{-1}\circ z\|^{1-\frac{Q+2}{p}+\frac{\lambda}{p}}} \le c\left(\sum_{k=1}^{m_0}\|F_k\|_{L^{p,\lambda}(L,\Omega)} + \|u\|_{L^p(\Omega)}\right). \tag{2.5}$$

Proof of Theorem 2.1. We give some definitions which will be useful in the sequel.

Let $r,s \in R$, with $0 < s < r$ and let $\phi \in C^\infty(\Omega)$ be a function such that $\phi(y) = 1$ for $0 \le y \le s$ and $\phi(y) = 0$ for $t \ge r$.

For every $\zeta \in \Omega$ and $r > 0$ such that $B_r = B_r(\zeta) \subset \Omega$ we set

$$\eta(z) = \phi(\|\zeta^{-1}\circ z\|). \tag{2.6}$$

Let u be a solution of equation (2.1), then u satisfies the equality

$$L(\eta u) = div\,(G) + g$$

where

$$G = \eta F + u\,A\,D\eta, \quad g = \langle ADu, D\eta\rangle - \langle F, D\eta\rangle + uY^*\eta, \tag{2.7}$$

and Y^* is the adjoint of the operator Y.

The proof is based on the following representation formula of the first derivatives of the function $v(z) = \eta(x)\,u(z)$ in terms of singular integral operators and commutators with a Calderón- Zygmund kernel.

For almost every $z \in \mathbb{R}^{N+1}$, we can write

$$\begin{aligned}\partial_{x_j}(\eta u)(z) = \quad & \textstyle\sum_{h,j=1}^{m_0} \lim_{\epsilon\to 0} \int\limits_{\|\zeta^{-1}\circ z\|\geq\epsilon} \Gamma_{jh}(z, \zeta^{-1}\circ z)\cdot \\ & \{(a_{hk}(z) - a_{hk}(\zeta))\partial_{x_k}(\eta u)(z) - G_h(\zeta)\}\,d\zeta + \\ & + \int\limits_{\mathbb{R}^{N+1}} \Gamma_j(z, \zeta^{-1}\circ z)\,g(\zeta)d\zeta + \textstyle\sum_{k=1}^{m_0} c_{jk}(z)G_k(z),\end{aligned}$$

where $c_{jk} = \int_{\|\zeta\|=1} \Gamma_j(z;\zeta)\nu_k(\zeta)\,d\sigma$ and $(\nu_1,\ldots,\nu_{N+1})$ is the outer normal of the set Σ_{N+1}.

Let us denote

$$T_j g(z) = \int\limits_{\mathbb{R}^{N+1}} \Gamma_j(z, \zeta^{-1}\circ z)g(\zeta)d\zeta.$$

We can express the $\partial_{x_j}v(z)$ in the following form

$$\partial_{x_j}v(z) = \sum_{h,k=1}^{m_0} C_{j,h}[a_{h,k}; v_k](z) - \sum_{h=1}^{m_0} T_{j,h}(G_h)(z) + T_j g(z) + \sum_{h=1}^{m_0} c_{j,h}G_h(z).$$

Then, we obtain

$$\begin{aligned}\|\partial_{x_j}v\|_{L^{p,\mu}(L,B_r)} \leq c\Big(& \textstyle\sum_{h,k=1}^{m_0} \|a_{h,k}\|_* \cdot \|\partial_{x_k}v\|_{L^{p,\lambda}(L,B_r)} + \\ & + \|G\|_{L^{p,\lambda}(L,B_r)} + \|g\|_{L^{p,\nu}(L,B_r)}\Big)\end{aligned}$$

where $0 \leq \nu \leq \lambda < Q+2$ and $\mu = min(\lambda, \nu + p)$.

Finally, using the definition of G and g we get the conclusion.

Proof of Theorem 2.2. Using the representation formula for $\eta\,u$ instead for that of $\partial_{x_j}v$, and some useful Sobolev Morrey embedding estimates we obtain the desired result.

Let us now study some estimates in Morrey spaces for the derivatives of local minimizers of variational integrals of the form

$$\mathcal{A}(u, \Omega) = \int_\Omega F(x, u, Du)dx$$

where Ω is a domain in $\mathbb{R}^m$, $u:\Omega\to\mathbb{R}$ and the integrand has the following form

$$F(x,u,Du) \,=\, A(x,u,\, g(x)\, h(u)\, |Du|^2).$$

The functions g and h will be specified later.

We are not assuming the continuity of A and g with respect to x. Let suppose that $A(\cdot,u,t)/(1+t)$ and $g(\cdot)$ are in the class $L^\infty\cap VMO$.

A "local minimizer"of the functional $\mathcal{A}$ is a function $u\in W^{1,p}_{loc}(\Omega,\mathbb{R}^n)$ which satisfies

$$\mathcal{A}(u;\, supp\,\varphi)\le \mathcal{A}(u+\varphi;\, supp\,\varphi)$$

for every $\varphi\in W^{1,p}_0(\Omega,\mathbb{R}^n)$.

Partial regularity for solutions of nonlinear elliptic systems was well studied in 1968-1969 by Morrey in [30], Giusti in [19], Giusti and Miranda [20] using an indirect argument similar to the one introduced by De Giorgi and Almgren in the regularity theory of parametrix minimal surfaces. New perturbation arguments were later considered by Giaquinta and Giusti in 1973 in [12], by Giaquinta and Modica in 1979 in [17] to study higher integrability of the gradient of the solutions. Using a perturbation method or direct argument, Tachikawa and Ragusa in [40]- [43] studied partial regularity for the minimizers of the variational integrals $\mathcal{A}(u;\Omega)$, where $u:\Omega\to\mathbb{R}^n$, $Du=(D_\alpha u^i)$, $\alpha=1,\dots,m,\ i=1,\dots,n$. In [43] the integrand has the following special form

$$F(x,u,Du)=A(x,u,g^{\alpha\beta}(x)h_{ij}D_\alpha u^i D_\beta u^j).$$

This kind of functionals arises as $p-$energy of maps between Riemannian manifolds. From this point of view, the geometric interest may occur for the above functionals. Moreover, let us observe that some methods of proofs of regularity for classes of nonlinear elliptic systems can also be applied to the equations of nonlinear Hodge theory, studied, for instance by L.M. Sibner and R.B. Sibner in 1970 in [46].

Also, we recall that in 1986 in the note [14] Giaquinta and Giusti considered the quadratic functionals

$$\int_\Omega g^{\alpha\,\beta}(x)\, h_{ij}(u)\, D_\alpha u^i\, D_\beta u^j\, dx,$$

where $g^{\alpha\,\beta}$ and h_{ij} are symmetric positive definite matrices having smooth entries.

We mention that later Giaquinta and Modica, in 1986, in the paper [18] studied partial regularity in the vector valued case and global regularity in the scalar case, for the minimizers of the variational integrals

$$\int_\Omega A(x,u,Du)\, dx$$

if the integrands has the special structure

$$A(x,u,|p|^2)$$

or, more generally,

$$A(x,u,a^{\alpha\beta}(x,u)b_{ij}(x,u)p^i_\alpha p^j_\beta)$$

where $a^{\alpha\beta}$ and b_{ij} are symmetric positive definite matrices and $g(x, u, t)$ is of class C^2 with respect to t.

A geometrically useful example is the following

$$\int_{B_1(0)} \frac{|Du|^2}{(1 + |u|^2)} dx$$

which is, in local coordinates, the energy of a map from the disk D^m to S^m.

Under assumptions similar to those in the previously mentioned paper by Giaquinta and Modica it is proved by Ivert, Giaquinta, Giusti and Modica in [13], [15] and [16] that minimizers have Hölder continuous derivatives in an open set Ω_0 contained in Ω such that $|\Omega \backslash \Omega_0| = 0$.

The hypothesis considered by Tachikawa and Ragusa has been inspired by [4] and [5], where Campanato obtained deep Hölder regularity results in $\mathcal{L}^{p,\lambda}$ spaces for solutions of elliptic systems having nonlinearity greater than or equal to 2. In these notes the coefficients of second order elliptic differential operators are supposed to be continuous.

The VMO assumption is a more recent idea. Let us focus our attention on the note [41] where the authors investigated partial regularity of the minimizers of quadratic functionals, whose integrands have VMO coefficients, using some majorizations for the functionals, rather than the well known Euler's equation associated to it. The functional is

$$\int_\Omega \left\{ A^{\alpha\beta}_{ij}(x, u) D_\alpha u^i D_\beta u^j + g(x, u, Du) \right\} dx,$$

where $\Omega \subset \mathbb{R}^m$, $n \geq 3$, is a bounded open set, $u \colon \Omega \to \mathbb{R}^n$, $n > 1$, $u(x) = (u^1(x), \ldots, u^n(x))$, $Du = (D_\alpha u^i)$, $D_\alpha = \partial/\partial x_\alpha$, $\alpha = 1, \ldots, m$, $i = 1, \ldots, n$. Let us assume that $A^{\alpha\beta}_{ij}$ are bounded functions on $\Omega \times \mathbb{R}^n$ which satisfy the following conditions

1. $A^{\alpha\beta}_{ij} = A^{\beta\alpha}_{ji}$;

2. for every $u \in \mathbb{R}^n$, $A^{\alpha\beta}_{ij}(\cdot, u) \in VMO(\Omega)$;

3. for every $x \in \Omega$ and $u, v \in \mathbb{R}^n$

   $$\left| A^{\alpha\beta}_{ij}(x, u) - A^{\alpha\beta}_{ij}(x, v) \right| \leq \omega(|u - v|^2)$$

   for some monotonically increasing concave function ω with $\omega(0) = 0$;

4. there exists a positive constant ν such that

   $$\nu|\xi|^2 \leq A^{\alpha\beta}_{ij}(x, u) \xi^i_\alpha \xi^j_\beta$$

   for almost all $x \in \Omega$, for all $u \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^{mn}$.

We should mention that since C^0 is a proper subset of VMO, the continuity of $A^{\alpha\beta}_{ij}(x, u)$ with respect to x is not assumed.

It is also assumed that the function g is a Charathéodory function and has growth less than quadratic.

Theorem 2.3. *Let $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ be a minimizer of the above defined functional. Suppose that the above assumptions on $A_{ij}^{\alpha\beta}(x,u)$ and $g(x,u,Du)$ are satisfied.*

Then, for $\lambda = m(1-\frac{2}{p})$, we have

$$Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^{mn})$$

where

$$\Omega_0 = \{x \in \Omega :\ \liminf_{R\to 0} \frac{1}{R^{m-2}} \int_{B(x,R)} |Du(y)|^2 dy \ =\ 0\}.$$

As a corollary, for any $\alpha \in (0,1)$,

$$u \in C^{0,\alpha}(\Omega_0, \mathbb{R}^n).$$

We recall that, for linear systems, regularity results assuming that $A_{ij}^{\alpha\beta}$ are constants or are in $C^0(\overline{\Omega})$, have been obtained by Campanato in [3]. As for the case when the continuity of the coefficients is not assumed let us mention the note by Acquistapace [1] where the author refines Campanato's results assuming that the coefficients $A_{ij}^{\alpha\beta}$ belong to a class neither containing nor contained in $C^0(\overline{\Omega})$ hence, in general, discontinuous.

Moreover we recall the study carried out by Huang [22] where he proved regularity results for weak solutions of linear elliptic systems with coefficients in the class *VMO*.

Therefore, it seems to be natural to expect partial regularity results under the condition that the coefficients of the principal terms $A_{ij}^{\alpha\beta} \in VMO$, even for nonlinear cases.

Daněček and Viszus in [10] treated regularity of the minimizers for the functional

$$\int_\Omega \left\{ A_{ij}^{\alpha\beta}(x) D_\alpha u^i D_\beta u^j + g(x,u,Du) \right\} dx,$$

where $g(x,u,Du)$ is a lower order term which satisfies

$$|g(x,u,z)| \le f(x) + L|z|^\gamma,$$

where $f \in L^p(\Omega), 2 < p \le \infty$, $f \ge 0$ almost everywhere in Ω, L is a nonnegative constant, and $0 \le \gamma < 2$.

They obtained Hölder regularity of a minimizer assuming that $A_{ij}^{\alpha\beta} \in VMO$.

Tachikawa and Ragusa have extended both the results by Huang and Daněček and Viszus because they treat the functional whose integrand contains the term $g(x,u,Du)$ and has coefficients $A_{ij}^{\alpha\beta}$ dependent not only on x but also on u. Let us now formulate the regularity results proved in [40]-[42]. Let $\mu \ge 0,\ p \ge 2$. Let the integrand function $A(x,u,t)$ be defined on $\Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$, and following assumptions:

(A-1) there exist positive constants $C, \lambda, \Lambda,\ \lambda \le \Lambda$ such that

$$\lambda(\mu^2 + t)^{\frac{p}{2}} \le A(x,u,t) \le \Lambda(\mu^2 + t)^{\frac{p}{2}},$$

$$\lambda(\mu^2 + t)^{\frac{p}{2}-1} \le |A_t(x,u,t)| \le \Lambda(\mu^2 + t)^{\frac{p}{2}-1},$$

$$\lambda(\mu^2 + t)^{\frac{p}{2}-2} \le A_{tt}(x,u,t) \le \Lambda(\mu^2 + t)^{\frac{p}{2}-2},$$

for all $(x,u,t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$;

(A-2) for every $(u,t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $A(\cdot,u,t) \in VMO(\Omega)$ and the mean oscillation of $A(\cdot,u,t)/(\mu^2+|t|)^{(p/2)}$ vanishes uniformly with respect to u,t in the following sense: there exist a positive number ρ_0 and a function $\sigma(z,\rho) : \mathbb{R}^m \times [0,\rho_0[\to [0,+\infty[$ with

$$\lim_{r\to 0} \sup_{\rho<r} \frac{1}{|Q(x,\rho)\cap\Omega|} \int_{Q(x,\rho)\cap\Omega} \sigma(z,\rho)dz = 0,$$

such that $A(\cdot,u,t)$ satisfies, for every $x \in \Omega$ and $y \in Q(x,\rho_0)\cap\Omega$, the inequality

$$\left|A(y,u,t) - A_{x,\rho}(u,t)\right| \leq \sigma(x-y,\rho)(\mu^2 + t)^{\frac{p}{2}},$$

for all $(u,t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, where

$$A_{x,\rho}(u,t) = \frac{1}{|Q(x,\rho)\cap\Omega|} \int_{Q(x,\rho)\cap\Omega} A(z,u,t)dz;$$

(A-3) for every $x \in \Omega$, $t \in \mathbb{R}^{mn}$ and $u,v \in \mathbb{R}^n$,

$$\left|A(x,u,t) - A(x,v,t)\right| \leq \omega(|u-v|^2)(\mu^2 + t)^{\frac{p}{2}}$$

where ω is some monotonically increasing concave function with $\omega(0) = 0$;

(A-4) for almost all $x \in \Omega$ and all $u \in \mathbb{R}^n$, $A(x,u,\cdot) \in C^2(\mathbb{R}^{mn})$;

(A-5) there exist constants $\lambda_0,\ \Lambda_0,\ \lambda_1,\ \Lambda_1$,

$$\lambda_0|\zeta|^2 \leq g^{\alpha\beta}(x)\zeta_\alpha\zeta_\beta \leq \Lambda_0|\zeta|^2,$$

$$\lambda_1|\eta|^2 \leq h_{ij}(u)\eta^i\eta^j \leq |\eta|^2,$$

for all $x \in \Omega$, $u,\zeta \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^n$;

(A-6) for every $u,v \in \mathbb{R}^n$

$$|\,h(u) - h(v)\,| \leq \omega(|\,u - v\,|^2)$$

where ω is, as in (A-2), some monotonically increasing concave function with $\omega(0) = 0$;

(A-7) the function g is in the class $L^\infty \cap VMO(\Omega)$.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Let also $u \in W^{1,p}(\Omega,\mathbb{R}^n)$, $p \geq 2$, be a minimizer of the functional*

$$\mathcal{A}(u,\Omega) = \int_\Omega F(x,u,Du)\,dx$$

with the integrand of the form

$$F(x,u,Du) = A(x,u,g^{\alpha\beta}(x)h_{ij}(u)D_\alpha u^i D_\beta u^j).$$

Suppose that $A(x,u,t)$ satisfies assumptions (A-1) – (A-7) .

Then there exists an open set $\Omega_0 \subset \Omega$ such that $u \in C^{0,\alpha}(\Omega_0)$ for any $\alpha \in (0,1)$.

Let us now give the idea of the proof. The theorem is proved proceeding as in the note by Giaquinta and Modica [18]. Let $x_0 \in \Omega,\ R > 0, Q(2R) = Q(x_0, 2R) \subset\subset \Omega$. For every $(u,t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$ let us set

$$A_R(u,t) = \frac{1}{|Q(R)|}\int_{Q(R)} A(y,u,t)dy, \quad u_R = \frac{1}{|Q(R)|}\int_{Q(R)} u(y)dy,$$

$$g_R = \frac{1}{|Q(R)|}\int_{Q(R)} g(y)\,dy,$$

and

$$A_0(\zeta) = A_R(u_R,\, g_R\, h(u_R)\,|\zeta|^2).$$

Here and below, we omit indices α, β, i, j when there is no possibility of confusion.

Now, let us consider the following "frozen functional"

$$\mathcal{A}_0(u) = \int_{Q(R)} A_0(Du)dx = \int_{Q(R)} A_R(u_R, g_R h(u_R)|Du|^2)dx.$$

Let also $\sqsubseteq \in \mathcal{H}^{\infty,}\surd(\mathcal{Q}(\mathcal{R}))$ be a minimizer of $\mathcal{A}_0(\mathcal{V},\, Q(R))$ in the set of functions

$$\{\mathcal{V} \in H^{1,p}(Q(R))\ ;\ u - \mathcal{V} \in H_0^{1,p}(Q(R))\}$$

and $w = u - v$.

Moreover, as in the paper [18], let us put

$$H(\xi) = (\mu^2 + |\xi|^2)^{\frac{p}{2}}.$$

In the sequel we use a regularity theorem by Uhlenbeck, see [47], for the minimizers of the functionals of the form

$$\mathcal{F}(v) = \int F(Dv)dx.$$

According to it, for $r < \frac{R}{2}$, we have :

$$\int_{Q(R)} H(Dv)dx \le c\left(\frac{r}{R}\right)^m \int_{Q_{R/2}} H(Dv)dx$$

where c does not depend on r, R, x_0.

Using formula (4.8) in [18] (or, for $p = 2$, formula (2.9) in [13]), we have

$$\int_{Q(R)} |Dw|^p dx \le \{\mathcal{A}_0(u) - \mathcal{A}_0(v)\} =$$

$$= \int_{Q(R)} \left[A_R(u_R, g_R h(u_R)|Du|^2) - A_R(u_R, g_R h(u_R)|Dv|^2)\right]dx.$$

Adding and subtracting the terms

$$A(x, u_R, g_R\, h(u_R)|Du|^2),\ A(x, u, g_R\, h(u_R)|Du|^2)$$

$$A(x,u,g(x)\,h(u_R)|Du|^2),\ A(x,u,g(x)\,h(u)|Du|^2)$$
$$A(x,u,g(x)\,h(v)|Dv|^2),\ A(x,v,g(x)\,h(v)|Dv|^2)$$
$$A(x,v,g(x)\,h(u_R)|Dv|^2),\ A(x,v,g_R\,h(u_R)|Dv|^2)$$

we obtain four kind of integrals.

So we obtain, using the assumptions on A :

$$\int_{Q(R)} |Dw|^p\,dx \le c\int_{Q(R)} H(Du)dx\left[\left(\frac{1}{|Q(R)|}\int_{Q(R)}\sigma(x-x_0,R)^{q'}\,dx\right)^{\frac{1}{q'}}\right.$$
$$+\left(\frac{1}{|Q(R)|}\int_{Q(R)}\omega(|u_R-u|^2)^{q'}\,dx\right)^{\frac{1}{q'}}+\left(\frac{1}{|Q(R)|}\int_{Q(R)}\omega(|u_R-v|^2)^{q'}\,dx\right)^{\frac{1}{q'}}$$
$$\left.+\left(\frac{1}{|Q(R)|}\int_{Q(R)}|g_R-g(x)|^{q'}dx\right)^{\frac{1}{q'}}\right] = I+II+III+IV.$$

where $q>p$ and q' is its conjugate (such that $\frac{1}{q}+\frac{1}{q'}=1$). Let us use the above mentioned regularity theorem by Uhlenbeck in the first part of I, in II and III Hölder's inequality and also Jensen's and Poincare's inequality and in IV the assumption $(A-7)$ on g, we have

$$\int_{Q(R)} |Du|^p\,dx \le C\left\{\left(\frac{r}{R}\right)^{\lambda}+\left(\frac{1}{|Q(R)|}\int_{Q(R)}\sigma(x,R)dx\right)^{\frac{q-1}{q}}+\right.$$
$$\left.+\omega\left(R^{p-m}\int_{Q(R)}|Du|^p dx\right)^{\frac{q-1}{q}}+\eta(g,R)\right\}\cdot\int_{Q(2R)}H(Du)dx.$$

Furthermore recalling the VMO assumption we have

$$\frac{1}{|B(R)|}\int_{B(R)}\sigma(x,R)dx\to 0,\quad ,\quad \eta(g,R)\to 0\quad \text{as}\quad R\to 0.$$

Finally, applying an useful lemma contained in [11], the proof is completed.

Morrey spaces will be an object of future research of the author in cooperation with V. Shakhmurov, more specifically on embedding theorems for vector valued Morrey spaces and on separable differential operators.

3 Open Problems

1. Extend the paper by Polidoro and Ragusa [34] up to the boundary.

2. It is possible to start a new study of the results obtained in [35] inspired by new definitions of modified Morrey spaces given by J. J. Hasanov, V. Guliyev and Y. Zeren in [21].

3. Let us consider Herz spaces studied, e. g. in [37], [38]. I suggest to study the nondivergence elliptic and parabolic case.

4. In [39] new classes of functions, $\mathcal{R}^{(p,q,\lambda)}$, are defined. These spaces generalize Lorentz spaces and give a refinement of Lebesgue spaces L^p, of weak$-L^p$ spaces and of Morrey spaces $L^{p,\lambda}$. Some embeddings between these new classes are also proved and some others could be proved.

5. The author suggest to study Vanishing-Morrey spaces and related properties and continue the work statrted in [36].

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