

THE ROLE OF MORREY SPACES IN THE STUDY  
OF NAVIER–STOKES AND EULER EQUATIONS

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**Abstract.** In this survey, we will pay a few words on the solution of the Cauchy problem for the 3D Navier-Stokes or Euler equations (with a focus on real harmonic analysis methods). Then we will highlight the role of Morrey spaces in other problems for the Navier-Stokes equations : uniqueness, weak-strong uniqueness, self-similar solutions, etc.

## 1 Euler and Navier–Stokes equations

The Euler or Navier–Stokes equations describe the motion of a fluid considered as a continuum. Let  $\rho(x, t)$  be the fluid density at time  $t \in \mathbb{R}$  and point  $x \in \mathbb{R}^3$  and let  $\vec{u}(t, x)$  be the velocity of the fluid. Applying Newton’s law on the conservation of momentum, we obtain the Cauchy momentum equations:

$$\rho(\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u}) = \operatorname{div} \sigma + \vec{f}, \quad (1.1)$$

where  $\sigma$  is the stress tensor and  $\vec{f}$  represents body forces (per unit volume) acting on the fluid. Besides, the conservation of mass gives the mass continuity equation

$$\partial_t \rho + \operatorname{div} (\rho \vec{u}) = 0. \quad (1.2)$$

We shall consider only the case of homogeneous incompressible fluids for which  $\rho$  is constant: the mass continuity equation then gives that  $\vec{u}$  is divergence-free:

$$\operatorname{div} \vec{u} = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0 \quad (1.3)$$

We shall consider only the case where the fluid is submitted only to stress, with no other forces:

$$\vec{f} = 0. \quad (1.4)$$

The stress tensor  $\sigma$  is usually split in the sum  $\sigma = -p Id + T$ , where  $p(t, x)$  is the pressure.  $-p Id$  is the isotropic part of the stress tensor  $\sigma$ : this part tends to change

the volume of the stressed body.  $T$  is the stress deviator tensor, which tends to distort the body. We shall consider the following cases for  $T$ :

**Case of an ideal fluid:** In the case of an ideal fluid, there is no viscous effect that distorts the fluid and we have  $T = 0$ . In that case, we obtain **the Euler equations**:

$$\begin{cases} \rho(\partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u}) = -\vec{\nabla} p \\ \operatorname{div} \vec{u} = 0 \end{cases} \quad (1.5)$$

**Case of a Newtonian fluid:** in the case of an incompressible Newtonian fluid with constant viscosity  $\nu > 0$ , we have  $\operatorname{div} T = \nu \Delta \vec{u}$ , so that we get **the Navier–Stokes equations** :

$$\begin{cases} \rho(\partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u}) = \nu \Delta \vec{u} - \vec{\nabla} p \\ \operatorname{div} \vec{u} = 0 \end{cases} \quad (1.6)$$

Finally, we shall make the following assumptions:

**Lack of border effects:** the fluid fills the entire space ( $x \in \mathbb{R}^3$  with no domain restriction) [for the physical irrelevance of the assumption, see the introduction of Tartar’s book [68)]

**Vanishing at infinity:** taking the divergence of (1.5) or (1.6), we get the following relationship between  $p$  and  $\vec{u}$

$$\Delta p = -\rho \sum_{i=1}^3 \sum_{j=1}^3 \partial_i \partial_j (u_i u_j). \quad (1.7)$$

Thus,  $p$  is determined by  $\vec{u}$  up to some harmonic correction. In order to define unambiguously the pressure  $p$ , we shall consider only solutions  $\vec{u}$  which vanish (in a loose sense [26], [45]) at infinity. In that case, we have:

$$\vec{\nabla} p = -\rho \sum_{i=1}^3 \sum_{j=1}^3 \vec{\nabla} \frac{1}{\Delta} \partial_i \partial_j (u_i u_j). \quad (1.8)$$

Let us stress that the operator  $T_{i,j,k} = \partial_k \frac{1}{\Delta} \partial_i \partial_j$  is a convolution operator with a kernel  $K_{i,j,k} \in \mathcal{E}' + L^1$ . The Leray projection operator  $\mathbb{P}$  is a projection on solenoidal (i.e. divergence-free) vector fields. Formally, we have

$$\mathbb{P} \vec{v} = \vec{v} - \vec{\nabla} \frac{1}{\Delta} \operatorname{div} \vec{v} \quad (1.9)$$

but the role of singular integrals in this definition makes it difficult to handle with. If  $\vec{v}$  is given as a divergence

$$\vec{v} = \sum_{i=1}^3 \partial_i \vec{w}_i \quad (1.10)$$

we find that

$$\mathbb{P} \vec{v} = \vec{v} - \sum_{i=1}^3 \sum_{j=1}^3 \vec{\nabla} \frac{1}{\Delta} \partial_i \partial_j (w_{i,j}). \quad (1.11)$$

This definition involves the convolution operators  $T_{i,j,k}$  which may be defined on a very large class of Banach spaces.

We may now formulate the equations we shall study in this survey:

**A) Cauchy initial value problem for the Euler equations:**

For some initial divergence-free vector field  $\vec{u}_0$ , find  $\vec{u}$  defined on  $(0, T) \times \mathbb{R}^3$  such that

$$\begin{cases} \partial_t \vec{u} + \mathbb{P} \operatorname{div} (\vec{u} \otimes \vec{u}) = 0 \\ \vec{u}|_{t=0} = \vec{u}_0 \end{cases} \quad (1.12)$$

**B) Cauchy initial value problem for the Navier–Stokes equations:**

For some initial divergence-free vector field  $\vec{u}_0$ , find  $\vec{u}$  defined on  $(0, T) \times \mathbb{R}^3$  such that

$$\begin{cases} \rho(\partial_t \vec{u} + \mathbb{P} \operatorname{div} (\vec{u} \otimes \vec{u})) = \nu \Delta \vec{u} \\ \vec{u}|_{t=0} = \vec{u}_0 \end{cases} \quad (1.13)$$

With no loss of generality, we may assume that  $\rho = \nu = 1$ . There is huge literature on the topic of fluid mechanics. Our main references in the rest of the text will be the books of Chemin and co-workers for the Euler equations [13], [2] and the books of Cannone [12] or Lemarié–Rieusset [45] for the Navier–Stokes equations.

## 2 Solutions to the Euler or Navier–Stokes equations

There are many ways to solve the Navier–Stokes equations. Classical solutions draw back to Oseen [62]. The modern way is to use an integral formulation of the Navier–Stokes equations and to look for a fixed–point of the associated integral transform (through Banach’s contraction principle). More precisely, we define the bilinear transform  $B$  with

$$B(\vec{u}, \vec{v}) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} (\vec{u}(s, \cdot) \otimes \vec{v}(s, \cdot)) ds \quad (2.1)$$

Then we have the following equivalence [26], [45]:  $\vec{u}$  will be a solution of (1.13) if and only if it is a solution of

$$\vec{u} = e^{t\Delta} \vec{u}_0 - B(\vec{u}, \vec{u}). \quad (2.2)$$

In order to find solutions to equations (1.13) or (2.2), we then have various strategies:

**a) mild solutions:** try to find a Banach space  $E_T$  such that  $(e^{t\Delta} \vec{u}_0)_{0 < t < T}$  belongs to  $E_T$  and  $B$  is bounded from  $E_T \times E_T$  to  $E_T$ ; then, if  $(e^{t\Delta} \vec{u}_0)_{0 < t < T}$  is small enough in  $E_T$ , the Picard–Duhamel iteration  $\vec{v}^{(0)} = e^{t\Delta} \vec{u}_0$ ,  $\vec{v}^{(n+1)} = e^{t\Delta} \vec{u}_0 - B(\vec{v}^{(n)}, \vec{v}^{(n)})$  will converge in  $E_T$  to a solution of (2.2). Following Browder [8] and Kato [33], such solutions are called mild solutions. Mild solutions were first described by Fujita and Kato in the context of Sobolev spaces [25]; Cannone proposed a systematic treatment in terms of Besov spaces in [12], and this treatment was generalized to many functional spaces in [45].

**b) weak solutions:** when the strategy of mild solutions cannot be applied, one uses arguments based on the control of energy (Leray’s energy inequality [50] or Scheffer’s local energy inequality [65], [9], [45]). We may soften the nonlinearity by choosing a bump function  $\omega \in \mathcal{D}(\mathbb{R}^3)$  (with  $\int \omega \, dx = 1$ ) and smoothen the quadratic term  $\vec{u} \otimes \vec{u}$  into  $(\omega_\epsilon * \vec{u}) \otimes \vec{u}$  (where  $\omega_\epsilon(x) = \epsilon^{-3}\omega(x/\epsilon)$ ). We may then solve the equations

$$\begin{cases} \partial_t \vec{u} + \mathbb{P} \operatorname{div} ((\omega_\epsilon * \vec{u}) \otimes \vec{u}) = \nu \Delta \vec{u} \\ \vec{u}|_{t=0} = \vec{u}_0 \end{cases} \quad (2.3)$$

or

$$\vec{u} = e^{t\Delta} \vec{u}_0 - B(\omega_\epsilon * \vec{u}, \vec{u}). \quad (2.4)$$

Usually, the contraction principle works on some time domain  $(0, T_\epsilon)$  which depends on  $\epsilon$ ; using energy, we then prove global existence of the solution and thereafter let  $\epsilon$  go to 0. A compactness argument (based on Rellich’s theorem) provides a weak solution.

**c) statistical solutions:** we shall pay a few words to the setting of statistical solutions, for which the individual energy inequality is not enough to grant existence of solutions. Existence of weak solutions is then provided by the existence of some characteristic function of a probability measure on trajectories, and individual existence is then assured only generically (for almost every initial value with respect to some initial distribution of random initial values). See [71], [24] or [4] for a modern treatment of such solutions.

In order to solve Euler equations, one does not deal directly with equations (1.12), where the transport term  $\vec{u} \cdot \vec{\nabla} \vec{u}$  has been altered by the use of Leray’s projection operator. One reintroduces the transport term in the equations and write (since  $\mathbb{P}\vec{u} = \vec{u}$ )

$$\begin{cases} \partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} = \vec{u} \cdot \vec{\nabla} \mathbb{P}\vec{u} - \mathbb{P} \operatorname{div} (\vec{u} \otimes \vec{u}) = \sum_{i=1}^3 [u_i, \mathbb{P}\partial_i] \vec{u} \\ \vec{u}|_{t=0} = \vec{u}_0 \end{cases} \quad (2.5)$$

The now classical way [13], [2] for solving (2.5) is to construct inductively approximations  $\vec{v}_n$  of the solution  $\vec{u}$  as solutions of the linear transport problem

$$\begin{cases} \partial_t \vec{v}_{n+1} + \vec{v}_n \cdot \vec{\nabla} \vec{v}_{n+1} = \sum_{i=1}^3 [v_{n,i}, \mathbb{P}\partial_i] \vec{v}_n \\ \vec{v}_{n+1}|_{t=0} = \vec{u}_0 \end{cases} \quad (2.6)$$

(with  $\vec{v}_0(t, x) = \vec{u}_0(x)$ ) but the intermediate solutions  $\vec{v}_n$  are not divergence-free. We shall see below why one should prefer divergence-free intermediate solutions. We therefore shall prefer the following scheme (as in [63], [16] and [47]): starting from  $\vec{f}_0 = \vec{u}_0$ , find a solution  $\vec{f}_{n+1}$  of the equation

$$\begin{cases} \partial_t \vec{f}_{n+1} + \vec{f}_n \cdot \vec{\nabla} \vec{f}_{n+1} = \sum_{i=1}^3 [f_{n,i}, \mathbb{P}\partial_i] \vec{f}_{n+1} \\ \vec{f}_{n+1}|_{t=0} = \vec{u}_0 \end{cases} \quad (2.7)$$

and check (by induction) that  $\vec{\nabla} \cdot \vec{f}_n = 0$ . In order to compute  $\vec{f}_{n+1}$ , we define inductively  $\vec{g}_{n,k}$  as  $\vec{g}_{n,0} = \vec{u}_0$  and  $\vec{g}_{n,k+1}$  as the solution of the linear transport problem

$$\begin{cases} \partial_t \vec{g}_{n,k+1} + \vec{f}_n \cdot \vec{\nabla} \vec{g}_{n,k+1} = \sum_{i=1}^3 [f_{n,i}, \mathbb{P}\partial_i] \vec{g}_{n,k} \\ \vec{g}_{n,k+1}|_{t=0} = \vec{u}_0 \end{cases} \quad (2.8)$$

The problem is then to prove the convergence of  $\vec{g}_{n,k}$  to  $\vec{f}_{n+1}$  (as  $k \rightarrow +\infty$ ) and of  $\vec{f}_n$  to  $\vec{u}$  (as  $n \rightarrow +\infty$ ).

### 3 Useful operators in order to deal with Euler or Navier–Stokes equations

Which operators do we need in the study of Euler or Navier–Stokes equations? In order to give some meaning to equations (1.12) or (1.13) we need to use the following operations:

**a) differentiation** (to compute  $\partial_t \vec{u}$  or  $\Delta \vec{u}$ ); this can be dealt with as differentiation in the sense of distributions; however, the use of a scale of Banach spaces defined in term of regularity may turn to be useful;

**b) pointwise product** (to compute  $\vec{u} \otimes \vec{u}$ ): this can be done if we look for locally (in time and space) square-integrable solutions. But the use of Banach spaces with more acute description of the pointwise product will be useful;

**c) convolution with  $L^1$  functions:** the operator  $\mathbb{P}\text{div}$  involves convolutions with distributions which are  $L^1$  outside from a compact neighbourhood of the origin. Thus, it will be useful to deal with Banach spaces which are stable under convolution with functions in  $L^1$ : it is more or less equivalent to ask that the norms of the distributions or the functions in those spaces are invariant through spatial translation of their argument (see [45]). Since the equations (1.12) and (1.13) are invariant through spatial translation (if  $\vec{u}(t, x)$  is a solution of (1.12) or (1.13) with initial value  $\vec{u}_0(x)$ , then  $\vec{u}(t, x - x_0)$  is still a solution of (1.12) or (1.13) with initial value  $\vec{u}_0(x - x_0)$ ), this requirement is quite natural.

When we turn to the integral equation (2.2), we see that we may find some interest in studying the following operators:

**d) heat hernel:** when solving the Cauchy problem, the first task is to identify the space where  $e^{t\Delta} \vec{u}_0$  lives; the operator  $f \mapsto \int_0^t e^{(t-s)\Delta} f(s, \cdot) ds$  will be very important, as well;

**e) Riesz potentials:** factorizing  $e^{(t-s)\Delta} \mathbb{P}\text{div}$  for  $0 \leq \alpha < 1$  as

$$e^{(t-s)\Delta} \mathbb{P}\text{div} = e^{\frac{t-s}{2}\Delta} \mathbb{P}\text{div} (-\Delta)^{\alpha/2} \circ e^{\frac{t-s}{2}\Delta} \circ (-\Delta)^{-\alpha/2} \quad (3.1)$$

we find that this is a combination of a Riesz potential  $(-\Delta)^{-\alpha/2}$ , a heat kernel  $e^{\frac{t-s}{2}\Delta}$  and a convolution with an integrable kernel (with  $L^1$  norm proportional to  $(t-s)^{-\frac{1+\alpha}{2}}$ );

**f) Riesz transforms:** factorizing  $e^{(t-s)\Delta} \mathbb{P}\text{div}$  as

$$e^{(t-s)\Delta} \mathbb{P}\text{div} = e^{(t-s)\Delta} \Delta \circ \mathbb{P}(-\Delta)^{-1/2} \text{div} \circ (-\Delta)^{-1/2} \quad (3.2)$$

we find that this is a combination of a Riesz potential  $(-\Delta)^{-1/2}$ , a matrix  $\mathbb{P}(-\Delta)^{-1/2} \text{div}$  of singular integral operators (in the algebra generated by Riesz transforms) and the kernel  $e^{(t-s)\Delta} \Delta$  which appears when dealing with the well known maximal regularity property for the heat kernel;

The last operator will be very important in the study of the Navier–Stokes equations;

**g) Dilation operators:** Navier–Stokes equations satisfy a very useful scaling invariance property: when  $(\vec{u}, p)$  is a solution on  $(0, T) \times \mathbb{R}^3$  of the Cauchy problem (1.13) with initial value  $\vec{u}_0$ , then, for every  $R > 0$  the function  $\delta_R(\vec{u})(t, x) = \frac{1}{R} \vec{u}(\frac{t}{R^2}, \frac{x}{R})$  is a solution on  $(0, R^2 T) \times \mathbb{R}^3$  of the Cauchy problem with initial value  $\frac{1}{R} \vec{u}_0(\frac{x}{R})$ . This

scaling property has an important consequence: if  $X$  is a functional space on  $\mathbb{R}^3$  such that  $\|f(x/R)\|_X = R^\alpha \|f\|_R$  and  $Y$  is a functional space on  $(0, +\infty)$  such that  $\|g(t/R)\|_Y = R^\beta$ , and if

$$Z_T = Y_t((0, T), X_x(\mathbb{R}^3)) = \{f(t, x) / \| \mathbf{1}_{(0,T)}(t) \|f(t, \cdot)\|_X \|_Y < +\infty\} \tag{3.3}$$

then, since  $B(\delta_R(\vec{u}), \delta_R(\vec{v})) = \delta_R(B(\vec{u}, \vec{v}))$ , a necessary condition on  $\alpha$  and  $\beta$  for the bilinear operator  $B$  to be bounded from  $Z_T^3 \times Z_T^3$  to  $Z_T^3$  is  $\alpha + 2\beta \leq 1$  ( $T < +\infty$ ) or  $\alpha + 2\beta = 1$  ( $T = +\infty$ ).

For the resolution of the Euler equations, one has to analyze the operators linked to the transport equation (2.5), or its linearized version (2.7) (with  $\vec{f}_{n+1}$  as the unknown and  $\vec{f}_n$  as the parameter.

**h) Bi-lipschitzian homeomorphisms:** The advective term, when the advecting vector field is  $L_t^1 \text{Lip}_x$ , is treated via the characteristics lines associated to the flow. Moving along the characteristics generate bi-Lipschitzian homeomorphisms. Thus, we shall analyze the operator  $f \mapsto f \circ X$  where  $X$  is a bi-Lipschitzian homeomorphism.

**i) Commutators and singular integrals:** if  $T$  belongs to the algebra of singular integral operators generated by the Riesz transforms and if  $A \in \text{Lip}(\mathbb{R}^3)$ , then the commutators between  $T\partial_j$  ( $j = 1, \dots, 3$ ) and the pointwise multiplication operator  $M_A$  of pointwise multiplication by  $A$  are singular integral operators (they are no more convolution operators but they belong to the class of generalized Calderón–Zygmund operators [10], [17], [45]). But, since  $\vec{f}_n$  is divergence-free, the operators at stake in equations (2.5) belong to a smaller class : if  $P_{jk}$  are the coefficients of the matrix operator  $\mathbb{P}$ , and if  $T_{jk} = \sum_{i=1}^n [f_{n,i}, P_{jk}\partial_i]$ , then  $T_{jk}$  satisfies moreover  $T_{jk}(1) = -T_{jk}^*(1) = -P_{jk}(\text{div } \vec{f}_n) = 0$ , so that they belong to the algebra studied in [42]. While generalized Calderón–Zygmund operators are known to be bounded on  $L^p$  spaces for  $1 < p < +\infty$  (and some other spaces of measurable functions), the operators in the smaller class are also bounded on Besov spaces  $\dot{B}_{p,q}^s$   $0 < s < 1$  and  $1 \leq p \leq \infty$ ,  $1 \leq q \leq \infty$  [43] (or Triebel–Lizorkin spaces  $\dot{F}_{p,q}^s$  for  $0 < s < 1$  and  $1 \leq p < \infty$ ,  $1 \leq q < \infty$  as well [19]).

**j) Atomic decompositions:** When the advecting vector field is divergence-free, the associated homeomorphisms are measure-preserving for the Lebesgue measure. It is then obvious that if we take an atomic or a molecular decomposition in the sense of Coifman and Weiss [18], then it is transported by the flow to a new atomic or molecular decomposition, respecting the scale of the molecule and moving its center along the characteristics. Similarly, a generalized Calderón–Zygmund operator  $T$  such that  $T(1) = T^*(1) = 0$  will preserve the molecules, keeping their scales and their centers [42].

As a conclusion of this section, we see that in the study of both the Euler and the Navier–Stokes equations we may use many operators that are useful in real harmonic analysis, or for analysis in the setting of Besov, Triebel–Lizorkin spaces, and, in what is the main topic of this conference: Morrey spaces and generalizations.

## 4 Mild solutions for the Navier–Stokes equations in Lebesgue spaces and related spaces

In the formalism of mild solutions, we try to solve (2.2) by the fixed-point algorithm :  $\vec{u} = \lim_{n \rightarrow \infty} \vec{v}^{(n)}$  with  $\vec{v}^{(0)} = e^{t\Delta} \vec{u}_0$  and  $\vec{v}^{(n+1)} = e^{t\Delta} \vec{u}_0 - B(\vec{v}^{(n)}, \vec{v}^{(n)})$ . The resolution of this fixed-point problem is based on a general tool for bilinear equations in a Banach space:

**Lemma 4.1.** *Let  $E$  be a Banach space and  $B$  a bounded bilinear operator on  $E$*

$$\|B(x, y)\|_E \leq C_0 \|x\|_E \|y\|_E. \quad (4.1)$$

*Let  $x_0 \in E$  with  $\|x_0\|_E < \frac{1}{4C_0}$ . Then, the equation  $x = x_0 - B(x, x)$  has at least one solution. More precisely, it has one unique solution  $x \in E$  such that  $\|x\|_E \leq \frac{1}{2C_0}$ .*

In 1984, Kato [KAT 84] proved the existence of mild solutions in  $L^p$ ,  $p \geq 3$ . For  $p > 3$ , he used the estimate

$$\|e^{(t-s)\Delta} \mathbb{P} \vec{\nabla} \cdot (\vec{u} \otimes \vec{v})\|_p \leq C(t-s)^{-1/2} \|e^{\frac{t-s}{2}\Delta} (\vec{u} \otimes \vec{v})\|_p \leq C_p (t-s)^{-\frac{1}{2} - \frac{3}{2p}} \|\vec{u}\|_p \|\vec{v}\|_p \quad (4.2)$$

to prove the boundedness of  $B$  on  $L^\infty([0, T], (L^p)^3)$  :

$$\|B(\vec{u}, \vec{v})(t, \cdot)\|_p \leq C_p t^{\frac{1}{2} - \frac{3}{2p}} \sup_{0 < s < t} \|\vec{u}(s, \cdot)\|_p \sup_{0 < s < t} \|\vec{v}(s, \cdot)\|_p. \quad (4.3)$$

For the critical case  $p = 3$ , inequality (4.2) becomes

$$\|e^{(t-s)\Delta} \mathbb{P} \operatorname{div} (\vec{u} \otimes \vec{v})\|_3 \leq C \frac{1}{(t-s)} \|\vec{u}\|_3 \|\vec{v}\|_3. \quad (4.4)$$

This is a very inconvenient estimate for dealing with  $\vec{u}$  and  $\vec{v}$  in  $L^\infty([0, T], (L^3)^3)$ , since  $\int_0^t \frac{ds}{t-s}$  diverges at the endpoint  $s = t$ . Kato then used an idea of Weissler [72], namely to search for the existence of a solution in a smaller space of mild solutions ; indeed, whereas the bilinear operator  $B$  is unbounded on  $\mathcal{C}([0, T], (L^3(\mathbb{R}^3))^3)$  [61], it becomes bounded on the smaller space  $\{\vec{f} \in \mathcal{C}([0, T], (L^3(\mathbb{R}^3))^3) / \sup_{0 < t < T} \sqrt{t} \|\vec{f}(t, \cdot)\|_\infty < \infty\}$ . Thus, we replace the estimate (4.4) (which leads to a divergent integral) by the estimates

$$\|e^{(t-s)\Delta} \mathbb{P} \vec{\nabla} \cdot (\vec{u} \otimes \vec{v})\|_3 \leq C \frac{1}{\sqrt{t-s}\sqrt{s}} \|\vec{u}\|_3 \sqrt{s} \|\vec{v}\|_\infty \quad (4.5)$$

and

$$\|e^{(t-s)\Delta} \mathbb{P} \vec{\nabla} \cdot (\vec{u} \otimes \vec{v})\|_\infty \leq C \frac{1}{\sqrt{t-s}} \min\left(\frac{1}{(t-s)} \|\vec{u}\|_3 \|\vec{v}\|_3, \frac{1}{s} \sqrt{s} \|\vec{u}\|_\infty \sqrt{s} \|\vec{v}\|_\infty\right) \quad (4.6)$$

which lead to two convergent integrals. Now, we shall see how Kato's ideas are easily extended to the case of Morrey spaces. First, we fix the notations for Morrey spaces :

**Definition 1.** For  $1 < p \leq q < \infty$ , the homogeneous Morrey space  $\dot{M}^{p,q}(\mathbb{R}^3)$  is defined as the space of locally  $p$ -integrable functions  $f$  such that

$$\sup_{x_0 \in \mathbb{R}^3} \sup_{0 < R < \infty} R^{3(1/q-1/p)} \left( \int_{|x-x_0| < R} |f(x)|^p dx \right)^{1/p} < \infty; \quad (4.7)$$

For  $p = 1 \leq q < \infty$ , the homogeneous Morrey–Campanato space  $\dot{M}^{1,q}(\mathbb{R}^3)$  is defined as the space of locally bounded measures  $\mu$  such that

$$\sup_{x_0 \in \mathbb{R}^3} \sup_{0 < R < \infty} R^{3(\frac{1}{q}-1)} |\mu|(B(x_0, R)) < \infty; \quad (4.8)$$

The inhomogeneous Morrey spaces  $M^{p,q}$  have the same definitions, except that in (4.7) and (4.8) we take the supremum only on small balls (with radii  $R \in (0, 1)$ ).

Those spaces were introduced by Morrey [59]; they are usually written as  $M_{p,\lambda} = \dot{M}^{p,q}$  with  $\lambda = 3(1 - \frac{p}{q})$ . Our choice of notation is an easy reminder of the embedding  $L^q \subset \dot{M}^{p,q} \subset L^p_{loc}$  for  $1 < p \leq q$ . Existence of mild solution in  $L^\infty((0, T), \dot{M}^{p,q})$  has been proved by Kato [35] and Taylor [69] for  $q \geq 3$  ( $T$  small enough if  $q > 3$ ,  $T = +\infty$  if  $q = 3$  and  $\vec{u}_0$  is small enough). In the important case of  $\dot{M}^{2,3}$ , proof using the wavelet decomposition in  $\dot{M}^{2,3}$  was given by Federbush [21]; Cannone replaced the wavelet decomposition by the Littlewood–Paley decomposition [12]; both the proofs of Federbush and Cannone are much more intricate than Kato’s proof.

We may extend easily the proof to a large class of Banach spaces :

**Theorem 4.1.** *Let  $E$  be a Banach space which is continuously embedded into  $\mathcal{S}'(\mathbb{R}^3)$ . Assume that:*

- a) convolution is bounded from  $L^1 \times E$  to  $E$ ,
- b) for some  $\alpha \in [0, 1]$ , we have, for all  $f \in E$  and all  $R > 0$ ,  $\|f(x/R)\|_E = R^\alpha \|f\|_E$ ,
- c) if  $u$  and  $v$  belong to  $E \cap \mathcal{C}_b$  (where  $\mathcal{C}_b$  is the space of bounded continuous applications from  $\mathbb{R}^3$  to  $\mathbb{R}$ ), then  $e^\Delta(uv) \in E$  and

$$\|e^\Delta(uv)\|_E \leq C \|u\|_E^\gamma \|u\|_\infty^{1-\gamma} \|v\|_E^\gamma \|v\|_\infty^{1-\gamma} \quad (4.9)$$

for some constants  $C > 0$  and  $1/2 \leq \gamma \leq 1$  if  $\alpha < 1$ ,  $1/2 < \gamma < 1$  if  $\alpha = 1$ .

Let  $E_T = L_t^\infty((0, T), E)$  and  $F_T = \{f \in E_T / \sup_{0 < t < T} t^{\alpha/2} \|f(t, \cdot)\|_\infty < +\infty\}$  with norms  $\|f\|_{E_T} = \sup_{0 < t < T} \|f(t, \cdot)\|_E$  and  $\|f\|_{F_T} = \sup_{0 < t < T} \|f(t, \cdot)\|_E + \sup_{0 < t < T} t^{\alpha/2} \|f(t, \cdot)\|_\infty$ .

Then the bilinear operator  $f \mapsto (e^{t\Delta} f)_{0 < t < T}$  is bounded from  $E$  to  $E_T$  and to  $F_T$ .  $B$  defined by (2.1) is bounded from  $E_T^3 \times E_T^3$  to  $E_T^3$  for  $0 < T < +\infty$  if  $\alpha < 1$  and  $\gamma = 1$ , from  $F_T^3 \times F_T^3$  to  $F_T^3$  for  $0 < T < +\infty$  if  $\alpha < 1$  and  $1/2 \leq \gamma \leq 1$  and from  $F_T^3 \times F_T^3$  to  $F_T^3$  for  $0 < T \leq +\infty$  if  $\alpha = 1$  and  $1/2 \leq \gamma < 1$

**Remark 1.** assumption a) is roughly equivalent to the stability of the norm of  $E$  under spatial shifts of the argument ( $\sup_{x_0 \in \mathbb{R}^3} \|f(x - x_0)\|_E \leq C \|f\|_E$ ) [45].

*Proof.* Entirely similar to [34]. When  $\gamma = 1$  and  $\alpha < 1$ , one uses the estimate

$$\|e^{(t-s)\Delta} \mathbb{P}\vec{\nabla} \cdot (\vec{u} \otimes \vec{v})\|_E \leq C(t-s)^{-1/2} \|e^{\frac{t-s}{2}\Delta} (\vec{u} \otimes \vec{v})\|_E \leq C_E(t-s)^{-\frac{1}{2} - \frac{\alpha}{2}} \|\vec{u}\|_E \|\vec{v}\|_E \quad (4.10)$$



and thus:

$$\|B(\vec{u}, \vec{v})(t, \cdot)\|_{E_T} \leq C_E T^{\frac{1}{2} - \frac{\alpha}{2}} \|\vec{u}(s, \cdot)\|_{E_T} \|\vec{v}(s, \cdot)\|_{E_T}. \quad (4.11)$$

For  $1/2 \leq \gamma \leq 1$ , we write

$$\begin{aligned} \|e^{(t-s)\Delta} \mathbb{P} \vec{\nabla} \cdot (\vec{u} \otimes \vec{v})\|_E &\leq C(t-s)^{-1/2} \|e^{\frac{t-s}{2}\Delta} (\vec{u} \otimes \vec{v})\|_E \\ &\leq C_E (t-s)^{-\frac{1}{2} - \frac{\alpha(2\gamma-1)}{2}} \|\vec{u}\|_E^\gamma \|\vec{v}\|_E^\gamma \|\vec{u}\|_\infty^{1-\gamma} \|\vec{v}\|_\infty^{1-\gamma} \end{aligned} \quad (4.12)$$

$$\begin{aligned} \|e^{(t-s)\Delta} \mathbb{P} \vec{\nabla} \cdot (\vec{u} \otimes \vec{v})\|_\infty &\leq C_E (t-s)^{-(1+\alpha)/2} \|e^{\frac{t-s}{2}\Delta} (\vec{u} \otimes \vec{v})\|_E \\ &\leq C'_E (t-s)^{-\frac{1}{2} - \alpha\gamma} \|\vec{u}\|_E^\gamma \|\vec{v}\|_E^\gamma \|\vec{u}\|_\infty^{1-\gamma} \|\vec{v}\|_\infty^{1-\gamma} \end{aligned} \quad (4.13)$$

and

$$\|e^{(t-s)\Delta} \mathbb{P} \vec{\nabla} \cdot (\vec{u} \otimes \vec{v})\|_\infty \leq C(t-s)^{-1/2} \|e^{\frac{t-s}{2}\Delta} (\vec{u} \otimes \vec{v})\|_\infty \leq C(t-s)^{-\frac{1}{2}} \|\vec{u}\|_\infty \|\vec{v}\|_\infty \quad (4.14)$$

If  $(\alpha, \gamma) \neq (1, 1)$ , (4.12) gives

$$\|B(\vec{u}, \vec{v})(t, \cdot)\|_{E_T} \leq C_E T^{\frac{1}{2} - \frac{\alpha}{2}} \|\vec{u}(s, \cdot)\|_{F_T} \|\vec{v}(s, \cdot)\|_{F_T}. \quad (4.15)$$

If  $\alpha < 1$ , (4.14) gives

$$\sup_{0 < t < T} t^{\alpha/2} \|B(\vec{u}, \vec{v})\|_\infty \leq C_E T^{\frac{1}{2} - \frac{\alpha}{2}} \|\vec{u}(s, \cdot)\|_{F_T} \|\vec{v}(s, \cdot)\|_{F_T}. \quad (4.16)$$

If  $\alpha = 1$ , we use (4.13) for  $s < t/2$  and (4.14) for  $s > t/2$ , and we get (4.16).  $\square$

Thus, we may find existence of a solution in  $F_T$  :

**Theorem 4.2.** *Let  $E$  be a Banach space which is continuously embedded into  $\mathcal{S}'(\mathbb{R}^3)$ . Assume that :*

- a) *convolution is bounded from  $L^1 \times E$  to  $E$*
- b) *for some  $\alpha \in [0, 1]$ , we have, for all  $f \in E$  and all  $R > 0$ ,  $\|f(x/R)\|_E = R^\alpha \|f\|_E$ .*
- c) *if  $u$  and  $v$  belong to  $E \cap \mathcal{C}_b$  (where  $\mathcal{C}_b$  is the space of bounded continuous applications from  $\mathbb{R}^3$  to  $\mathbb{R}$ ), then  $e^\Delta(uv) \in E$  and*

$$\|e^\Delta(uv)\|_E \leq C \|u\|_E^\gamma \|u\|_\infty^{1-\gamma} \|v\|_E^\gamma \|v\|_\infty^{1-\gamma} \quad (4.17)$$

for some constants  $C > 0$  and  $1/2 \leq \gamma \leq 1$  if  $\alpha < 1$ ,  $1/2 < \gamma < 1$  if  $\alpha = 1$ .

Let  $E_T = L_t^\infty((0, T), E)$  and  $F_T = \{f \in E_T / \sup_{0 < t < T} t^{\alpha/2} \|f(t, \cdot)\|_\infty < +\infty\}$  with the norms

$$\|f\|_{E_T} = \sup_{0 < t < T} \|f(t, \cdot)\|_E, \quad \|f\|_{F_T} = \sup_{0 < t < T} \|f(t, \cdot)\|_E + \sup_{0 < t < T} t^{\alpha/2} \|f(t, \cdot)\|_\infty.$$

Let  $\vec{u}_0 \in E^3$  with  $\operatorname{div} \vec{u}_0 = 0$ .

Then: i) if  $\alpha < 1$ , the integral problem (2.2) has a solution  $\vec{u} \in F_T^3$  with  $T = O(\|\vec{u}_0\|_E^{-\frac{2}{1-\alpha}})$ .

ii) if  $\alpha = 1$ , there exists a positive  $\epsilon_E > 0$  such that, if  $\|\vec{u}_0\|_E < \epsilon_E$ , the integral problem (2.2) has a solution  $\vec{u} \in F_\infty^3$

iii) if  $\alpha = 1$  and if  $\vec{u}_0 = \vec{u}_1 + \vec{u}_2$  with  $\vec{u}_1 \in (E \cap L^\infty)^3$  and  $\|\vec{u}_2\|_E < \epsilon_E$ , then the integral problem (2.2) has a solution  $\vec{u} \in F_T^3$  with

$$T \leq C \min(\|\vec{u}_1\|_\infty^{-2}, \|\vec{u}_1\|_E^{-2(2\gamma-1)} \|\vec{u}_1\|_\infty^{-4(1-\gamma)}, \|\vec{u}_1\|_E^{-\frac{2\gamma}{1-\gamma}} \|\vec{u}_1\|_\infty^{-2}).$$

**Examples.**

a) Let us first notice that, if  $E$  satisfies a) and is stable under pointwise multiplication with bounded continuous functions then it satisfies c) with  $\gamma = 1/2$ . If  $E$  is stable under such pointwise multiplication, its elements are local bounded measures and we find (from a) and b)) that  $E \subset \dot{M}^{1,q}$  with  $q = 3/\alpha$ . If we want  $E$  to satisfy c) with  $\gamma = 1$ , we should ask the pointwise product to be defined on  $E \times E$ , and thus  $E \subset \dot{M}^{2,q}$  with  $q = 3/\alpha$ . This explains the special role played by the critical Morrey space  $\dot{M}^{2,3}$  [21], [12], [46].

b) **Lebesgue spaces:** Theorem 4.2 may be applied to  $L^p$  for  $3 < p \leq +\infty$  ( $\alpha = 3/p, \gamma = 1$ ) or  $p = 3$  ( $\alpha = 1, \gamma = 1/2$ ). This is Kato’s theorem [34].

c) **Lorentz spaces:** Theorem 4.2 may be applied to  $L^{p,q}$  for  $3 < p \leq +\infty$  and  $1 \leq q \leq +\infty$  ( $\alpha = 3/p, \gamma = 1$ ) or  $p = 3$  and  $1 \leq p \leq +\infty$  ( $\alpha = 1, \gamma = 1/2$ ). Solutions in the Lorentz space  $L^{3,\infty}$  have been considered by Barraza [3]; Meyer proved that the bilinear operator  $B$  is bounded on  $E_T$  in the case  $E = L^{3,\infty}$  (even if it corresponds to the forbidden case  $\alpha = \gamma = 1$ ) [56], [45].

d) **Morrey spaces:** Theorem 4.2 may be applied to  $\dot{M}^{p,q}$  for  $2 \leq p < +\infty$  and  $\max(p, 3) < q < +\infty$  ( $\alpha = 3/q, \gamma = 1$ ) or  $\dot{M}^{p,q}$  for  $1 \leq p < +\infty$  and  $\max(p, 3) \leq q < +\infty$  ( $\alpha = 3/q, \gamma = 1/2$ ). This has been proved by Kato [35] and Taylor [69].

e) **Lorentz–Morrey spaces:** One may replace the  $L^p$  norm in the definition of a Morrey space by a  $L^{p,q}$  Lorentz norm: define

$$\|f\|_{\dot{M}^{p,q,r}} = \sup_{x_0 \in \mathbb{R}^3} \sup_{0 < R < \infty} R^{3(1/r-1/p)} \|1_{|x-x_0| < R} f(x)\|_{L^{p,q}} \tag{4.18}$$

Theorem 4.2 may be applied to  $\dot{M}^{p,q,r}$  for  $2 \leq p < +\infty, 1 \leq q \leq +\infty$  and  $\max(p, 3) < r < +\infty$  ( $\alpha = 3/r, \gamma = 1$ ) or  $\dot{M}^{p,q,r}$  for  $1 \leq p < +\infty, 1 \leq q \leq +\infty$  and  $\max(p, 3) \leq r < +\infty$  ( $\alpha = 3/r, \gamma = 1/2$ ). Those spaces have been considered in [46]

f) **Multiplier spaces:** For  $1 < p < +\infty$  and  $0 < r < 3/p$ , let  $\dot{H}_p^r = (-\Delta)^{-r/2} L^p$  be the homogeneous Sobolev space and let  $\dot{X}_p^r = \mathcal{M}(\dot{H}_p^r \mapsto L^p)$  be the spaces of measurable functions whose pointwise product maps  $\dot{H}_p^r$  boundedly to  $L^p$  :

$$\|f\|_{\dot{X}_p^r} = \sup_{\|g\|_{\dot{H}_p^r} \leq 1} \|fg\|_p \tag{4.19}$$

Theorem 4.2 may be applied to  $\dot{X}_p^r$  for  $1 < p < +\infty$  and  $0 < r \leq 1$  and  $r < 3/p$  ( $\alpha = r, \gamma = 1/2$ ). The case  $p = 2$  has been discussed in [45]. Those multiplier spaces have been studied by Maz’ya [54].

## 5 Mild solutions for the Navier–Stokes equations in Besov spaces with negative regularity index

In 1972, Fabes, Jones and Rivière [20] proved the existence of mild solutions in  $F_T = L_t^p L_x^q, 2/p + 3/q \leq 1$  and  $2 < p < +\infty$ . They used the estimate

$$\|e^{(t-s)\Delta} \mathbb{P} \vec{\nabla} \cdot (\vec{u} \otimes \vec{v})\|_q \leq C_q (t-s)^{-\frac{1}{2} - \frac{3}{2q}} \|\vec{u}\|_q \|\vec{v}\|_q \tag{5.1}$$

and Young's inequality for convolution between Lorentz spaces (or the Hardy–Littlewood inequality) to prove the boundedness of  $B$  on  $L^p([0, T], (L^q)^3)$  : let  $\frac{1}{r} = \frac{1}{p} + \frac{1}{4}(1 - \frac{2}{p} - \frac{3}{q})$ ; we have  $2 < r \leq p$  and, since  $\frac{1}{2} + \frac{3}{2q} + \frac{2}{r} - 1 = \frac{1}{p}$ ,

$$\|B(\vec{u}, \vec{v})\|_{L^p L^q} \leq C_{p,q} \|\vec{u}\|_{L^r L^q} \|\vec{v}\|_{L^r L^q} \leq C_{p,q} T^{2(\frac{1}{r} - \frac{1}{p})} \|\vec{u}\|_{L^p L^q} \|\vec{v}\|_{L^p L^q}. \quad (5.2)$$

Thus, we find that, if  $e^{t\Delta} \vec{u}_0 \in (L^p((0, +\infty), L^q(\mathbb{R}^3)))^3$ , then the integral problem (2.2) has a local solution in  $(L^p((0, T), L^q(\mathbb{R}^3)))^3$  for  $T$  small enough ( $2/p + 3/q \leq 1$ ,  $2 < p < +\infty$ ) or a global solution ( $T = +\infty$ , if  $2/p + 3/q = 1$ ,  $2 < p < +\infty$  and  $\|e^{t\Delta} \vec{u}_0\|_{L^p L^q}$  is small enough). Let us notice that  $e^{t\Delta} f \in L^p((0, +\infty), L^q(\mathbb{R}^3))$  if and only if  $f$  belongs to the homogeneous Besov space  $\dot{B}_{q,p}^{-\frac{2}{p}}$ , where the homogeneous space is not defined modulo polynomials but as a subspace of  $\mathcal{S}'$  [47] :

$$\text{for } -\infty < \sigma < 3/q, f \in \dot{B}_{q,p}^\sigma \Leftrightarrow f = \sum_{j \in \mathbb{Z}} \Delta_j f \text{ in } \mathcal{S}' \text{ and } (2^{j\sigma} \|\Delta_j f\|_q) \in l^p \quad (5.3)$$

We can replace in the result of Fabes, Jones and Rivière the Lebesgue space  $L_t^p$  by the weak  $L^p$  space  $L^{p,*}$  of Marcinkiewicz, which is equal to the Lorentz space  $L^{p,\infty}$ . If  $2/p + 3/q \leq 1$  and  $2 < p < +\infty$ , then  $B$  is bounded on  $(L^{p,*}((0, T), L^q(\mathbb{R}^3)))^3$  (for every  $T < +\infty$  if  $2/p + 3/q \leq 1$ ; for  $T = +\infty$  if  $2/p + 3/q = 1$ ); more precisely, let  $\frac{1}{r} = \frac{1}{p} + \frac{1}{4}(1 - \frac{2}{p} - \frac{3}{q})$ ; we have

$$\|B(\vec{u}, \vec{v})\|_{L^{p,\infty} L^q} \leq C_{p,q} T^{2(\frac{1}{r} - \frac{1}{p})} \|\vec{u}\|_{L^{p,\infty} L^q} \|\vec{v}\|_{L^{p,\infty} L^q}. \quad (5.4)$$

Thus, we find that, if  $e^{t\Delta} \vec{u}_0 \in (L^{p,\infty}((0, +\infty), L^q(\mathbb{R}^3)))^3$ , then the integral problem (2.2) has a local solution in  $(L^{p,\infty}((0, T), L^q(\mathbb{R}^3)))^3$  for  $T$  small enough ( $2/p + 3/q < 1$ ,  $2 < p < +\infty$ ) or a global solution ( $T = +\infty$ , if  $2/p + 3/q = 1$ ,  $2 < p < +\infty$  and  $\|e^{t\Delta} \vec{u}_0\|_{L^{p,\infty} L^q}$  is small enough).

Let us notice that, since  $t \mapsto \|e^{t\Delta} f\|_p$  is non-increasing,  $e^{t\Delta} f \in L^p((0, +\infty), L^q(\mathbb{R}^3))$  if and only if we have  $\sup_{t>0} t^{-\frac{1}{p}} \|e^{t\Delta} f\|_q < +\infty$ . This is equivalent to the fact that  $f$  belongs to the homogeneous Besov space  $\dot{B}_{q,\infty}^{-\frac{2}{p}}$ . In that case, it is more convenient to work with  $G_T = \{f / \sup_{0 < t < T} t^{-\frac{1}{p}} \|e^{t\Delta} f\|_q\}$  instead of  $F_T = L_t^{p,\infty} L_x^q$ . We find that, for  $0 < t < T$ ,

$$\|B(\vec{u}, \vec{v})(t, \cdot)\|_q \leq C t^{-\frac{1}{p}} t^{\frac{1}{2}(1 - \frac{2}{p} - \frac{3}{q})} \|\vec{u}\|_{G_T} \|\vec{v}\|_{G_T} \quad (5.5)$$

and that

$$\left\| \frac{1}{\sqrt{-\Delta}} B(\vec{u}, \vec{v})(t, \cdot) \right\|_q \leq C t^{\frac{1}{2}} t^{-\frac{1}{p}} t^{\frac{1}{2}(1 - \frac{2}{p} - \frac{3}{q})} \|\vec{u}\|_{G_T} \|\vec{v}\|_{G_T} \quad (5.6)$$

so that

$$\|B(\vec{u}, \vec{v})(t, \cdot)\|_{\dot{B}_{q,1}^{-\frac{2}{p}}} \leq C \|B(\vec{u}, \vec{v})(t, \cdot)\|_q^{1 - \frac{2}{p}} \left\| \frac{1}{\sqrt{-\Delta}} B(\vec{u}, \vec{v})(t, \cdot) \right\|_q^{\frac{2}{p}} \leq C t^{\frac{1}{2}(1 - \frac{2}{p} - \frac{3}{q})} \|\vec{u}\|_{G_T} \|\vec{v}\|_{G_T} \quad (5.7)$$

Thus, we see that, if  $\vec{u}_0 \in X^3$  where  $X$  is a Banach space which is stable under convolution  $L^1$  and if  $\dot{B}_{q,1}^{-\frac{2}{p}} \subset X \subset B_{q,\infty}^{-\frac{2}{p}}$ , then the integral problem (2.2) will have a local solution in  $(G_T \cap L_t^\infty X)^3$  if  $2/p + 3/q < 1$  and  $T$  is small enough, or a global solution

in  $(G_\infty \cap L_t^\infty X)^3$  if  $2/p + 3/q = 1$  and  $\|\vec{u}_0\|_X$  is small enough. We can choose  $X = \dot{B}_{q,r}^{-\frac{1}{2p}}$  with  $1 \leq r \leq +\infty$  or  $X = \dot{F}_{q,r}^{-\frac{1}{2p}}$  with  $1 \leq r \leq +\infty$ .

Solutions in  $G_T$  were first studied by Cannone [12] and Planchon [64].

Those results are easily extended to the case of Besov–Morrey spaces. Such spaces have been considered by Kozono and Yamazaki [39].

**Theorem 5.1.** *Let  $1 \leq p \leq q < +\infty$  and  $1 \leq q \leq +\infty$ .  $\dot{B}_{M^{p,q},r}^\sigma$  be defined as*

$$\text{for } -\infty < \sigma < 3/q, f \in \dot{B}_{M^{p,q},r}^\sigma \Leftrightarrow f = \sum_{j \in \mathbb{Z}} \Delta_j f \text{ in } \mathcal{S}' \text{ and } (2^{j\sigma} \|\Delta_j f\|_{M^{p,q}}) \in l^r \quad (5.8)$$

Let  $E$  be a Banach space which is continuously embedded into  $\mathcal{S}'(\mathbb{R}^3)$ . Assume that:

- a) convolution is bounded from  $L^1 \times E$  to  $E$
- b) for some  $1 \leq p \leq q < +\infty$  ( $q > 3$ ) and some  $\sigma \in (0, 1 - \frac{3}{q}]$  we have the embeddings  $\dot{B}_{M^{p,q},1}^{-\sigma} \subset E \subset \dot{B}_{M^{p,q},\infty}^{-\sigma}$

Let  $G_T = \{f / \sup_{0 < t < T} t^{\sigma/2} \|f\|_{\dot{M}^{p,q}} < +\infty\}$  and  $H_T = \{f \in H_T / \sup_{0 < t < T} t^{\frac{(\sigma + \frac{3}{q})}{2}}\}$ . Let  $\vec{u}_0 \in E^3$  with  $\operatorname{div} \vec{u}_0 = 0$ . Then :

- i) if  $\sigma + \frac{3}{q} < 1$  and  $p > 2$ , the integral problem (2.2) has a solution  $\vec{u} \in (G_T \cap L^\infty(E))^3$  for  $T$  small enough.
- ii) if  $\sigma + \frac{3}{q} < 1$  and  $1 \leq p < +\infty$ , the integral problem (2.2) has a solution  $\vec{u} \in (H_T \cap L^\infty(E))^3$  for  $T$  small enough.
- iii) if  $\sigma + \frac{3}{q} = 1$  and if  $\|\vec{u}_0\|_E$  is small enough, the integral problem (2.2) has a global solution  $\vec{u} \in (H_\infty \cap L^\infty(E))^3$
- iv) there exist  $\epsilon_E > 0$  such that, if  $\sigma + \frac{3}{q} = 1$  and if  $\vec{u}_0 = \vec{u}_1 + \vec{u}_2$  with  $\vec{u}_1 \in (E \cap L^\infty)^3$  and  $\|\vec{u}_2\|_E < \epsilon_E$ , then the integral problem (2.2) has a solution  $\vec{u} \in (H_T \cap L^\infty(E))^3$  for  $T$  small enough.

**Example.** An interesting example is the case of  $E = (-\Delta)^{\sigma/2} \dot{M}^{p,q}$  which satisfies that it is stable under convolution with  $L^1$  and that  $\dot{B}_{M^{p,q},1}^{-\sigma} \subset E \subset \dot{B}_{M^{p,q},\infty}^{-\sigma}$ .

The limiting case  $\sigma = 1$  and  $q = \infty$  has been discussed by many authors. The space  $\dot{B}_{\infty,\infty}^{-1}$  is not well adapted to the Navier–Stokes equations. It was first stated in [58] for a model equation, then proved by Bourgain and Pavlovič [7] (and by Germain [29] and Yoneda [75] in the case  $\dot{B}_{\infty,q}^{-1}$  with  $q > 2$ ). The greatest (homogeneous) limit space in which one can work is the space  $BMO^{-1} = \sqrt{-\Delta} BMO$  was introduced by Koch and Tataru [36]:

**Lemma 5.1.** *Let  $f \in \mathcal{S}'(\mathbb{R}^3)$ . Then the following assertions are equivalent.*

- a)  $f = \sum_{j \in \mathbb{Z}} \Delta_j f$  in  $\mathcal{S}'$  and  $f$  belongs to the homogeneous Triebel–Lizorkin space  $f \in \dot{F}_{\infty,2}^{-1}$ .
- b) There exists  $g \in BMO$  such that  $f = \sqrt{-\Delta} g$ .
- c) There exists  $\vec{g} \in (BMO)^3$  such that  $f = \operatorname{div} \vec{g}$ .
- d)  $\sup_{t>0} \sup_{x_0 \in \mathbb{R}^3} t^{-3/2} \int_0^t \int_{|x-x_0| \leq \sqrt{t}} |e^{s\Delta} f(x)|^2 dx ds < +\infty$ .

$BMO$  is a limit case of Morrey spaces. Recall that  $\dot{M}^{p,q} = M_{p,\lambda}$  with  $\lambda = 3(1 - \frac{p}{q})$ . When  $1 \leq p \leq q < +\infty$ , we find that  $\lambda \in [0, 3)$ . We have

$$\int_{B(x_0, R)} |f|^p dx \leq \|f\|_{M_{p,\lambda}}^p R^\lambda \quad (5.9)$$

Now, if  $\lambda = 3$  we have  $M_{p,\lambda} = L^\infty$  and if  $\lambda > 3$  we have  $M_{p,\lambda} = \{0\}$ . Campanato [11] modified the definition of Morrey-spaces into

$$f \in \mathcal{M}_{p,\lambda} \Leftrightarrow \sup_{x_0 \in \mathbb{R}^3, R > 0} R^{-\lambda} \int_{B(x_0, R)} |f(x) - m_{B(x_0, R)} f|^p dx < +\infty \quad (5.10)$$

where  $m_{B(x_0, R)} f$  is the mean value of  $f$  on the ball  $B(x_0, R)$ . We have  $\mathcal{M}_{p,3} = BMO$ .

Koch and Tataru's result is the following one:

**Theorem 5.2.** *Let  $F_T$  be the space of functions on  $(0, T) \times \mathbb{R}^3$*

$$F_T = \left\{ f \mid \sup_{0 < t < T} \sup_{x_0 \in \mathbb{R}^3} t^{-3/2} \int_0^t \int_{|x-x_0| \leq \sqrt{t}} |f(t, x)|^2 dx ds < +\infty \right. \\ \text{and} \\ \left. \sup_{0 < t < T} \sup_{x \in \mathbb{R}^3} \sqrt{t} |f(t, x)| < +\infty \right\} \quad (5.11)$$

with

$$\|f\|_{F_T} = \left( \sup_{0 < t < T} \sup_{x_0 \in \mathbb{R}^3} t^{-3/2} \int_0^t \int_{|x-x_0| \leq \sqrt{t}} |f(t, x)|^2 dx ds \right)^{1/2} + \sup_{0 < t < T} \sup_{x \in \mathbb{R}^3} \sqrt{t} |f(t, x)| \quad (5.12)$$

Let  $\vec{u}_0 \in (BMO^{-1})^3$  with  $\operatorname{div} \vec{u}_0 = 0$ . Then:

i) if  $\|\vec{u}_0\|_{BMO^{-1}}$  is small enough, the integral problem (15) has a global solution  $\vec{u} \in (F_T)^3$

ii) there exists  $\epsilon > 0$  such that, if  $\vec{u}_0 = \vec{u}_1 + \vec{u}_2$  with  $\vec{u}_1 \in (BMO^{-1} \cap L^\infty)^3$  and  $\|\vec{u}_2\|_{BMO^{-1}} < \epsilon$ , then the integral problem (2.2) has a solution  $\vec{u} \in (F_T)^3$  for  $T$  small enough.

In order to generalize Theorem 5.2, May and Xiao both considered solutions in  $F_T^\sigma$  defined for  $0 < \sigma < 1$  as

$$F_T^\sigma = \left\{ f \mid \sup_{0 < t < T} \sup_{x_0 \in \mathbb{R}^3} t^{-3/2} \int_0^t \int_{|x-x_0| \leq \sqrt{t}} |f(t, x)|^2 \left(\frac{s}{t}\right)^\sigma dx ds < +\infty \right. \\ \text{and} \\ \left. \sup_{0 < t < T} \sup_{x \in \mathbb{R}^3} \sqrt{t} |f(t, x)| < +\infty \right\} \quad (5.13)$$

Xiao published his result [74] and explained the connection between  $F_T^\sigma$  and  $Q$ -spaces [73], May did not publish this part of his thesis [51] since the condition  $e^{t\Delta} f \in F_T^\sigma$  reduces to  $f \in (-\Delta)^{\sigma/2} \dot{M}^{2,q}$  with  $q = \frac{3}{1-\sigma}$ .

## 6 Mild solutions for the Navier–Stokes equations in Besov spaces of positive regularity index

The resolution of the Navier–Stokes equations can be performed in Besov spaces of positive regularity index [12], [64] or of null regularity index [56]. In [45] a general framework is presented which can be applied to Besov–Morrey spaces.

The key estimates for studying mild solutions for initial values in such spaces are the following ones :

$$\text{for } 0 < \sigma < 3/q, \quad \|fg\|_{\dot{B}_{MP,q,r}^\sigma} \leq C(\|f\|_{\dot{B}_{MP,q,r}^\sigma} \|g\|_\infty + \|g\|_{\dot{B}_{MP,q,r}^\sigma} \|f\|_\infty) \quad (6.1)$$

$$\|fg\|_{\dot{B}_{MP,q,\infty}^0} \leq C(\|f\|_{\dot{B}_{MP,q,r}^\sigma} \|g\|_{\dot{B}_{\infty,1}^0} + \|g\|_{\dot{B}_{MP,q,r}^\sigma} \|f\|_{\dot{B}_{\infty,1}^0}) \quad (6.2)$$

## 7 Weak solutions : the role of the $L^2$ norm

In Leray's theory [50], a *weak solution* of equations (1.13) is a solution  $\vec{u} \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$  defined on  $(0, +\infty) \times \mathbb{R}^3$  which satisfies the energy inequality

$$\|\vec{u}(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{u}\|_2^2 ds \leq \|\vec{u}_0\|_2^2 \quad (7.1)$$

where the initial value  $\vec{u}_0$  is a square-integrable divergence-free vector field. The associated pressure  $p(t, x)$  belongs to  $L_t^2 L_x^{3/2}$  and can be recovered from  $\vec{u}$  by the formula

$$p = - \sum_{i=1}^3 \sum_{j=1}^3 \frac{1}{\Delta} \partial_i \partial_j (u_i u_j). \quad (7.2)$$

In particular, a Leray solution  $\vec{u}$  belongs to  $L_t^{8/3} L_x^4$ . If we assume more regularity on the solution  $\vec{u}$  ( $\vec{u} \in L_t^4 L_x^4$ ), then the inequality (7.1) becomes an equality. Indeed, in that case  $p \in L_t^2 L_x^2$  and thus  $\partial_t \vec{u} \in L_t^2 \dot{H}_x^{-1}$ . Thus, we may write  $\partial_t |\vec{u}|^2 = 2\partial_t \vec{u} \cdot \vec{u}$ . and find :

$$\partial_t |\vec{u}|^2 + 2|\vec{\nabla} \otimes \vec{u}|^2 = \sum_{i=1}^3 \partial_i (2\vec{u} \cdot \partial_i \vec{u} - (|\vec{u}|^2 + 2p)u_i) + R \quad (7.3)$$

where

$$R = (|\vec{u}|^2 + 2p) \vec{\nabla} \cdot \vec{u} = 0 \quad (7.4)$$

When  $\vec{u}$  is a Leray solution but does not belong to  $L_t^4 L_x^4$ , we cannot write  $\partial_t |\vec{u}|^2 = 2\partial_t \vec{u} \cdot \vec{u}$ . Energy equality is not fulfilled (or, at least, is not known to be fulfilled). If  $\vec{u}_\epsilon$  is the solution of the mollified equations, we have

$$\partial_t |\vec{u}_\epsilon|^2 + 2|\vec{\nabla} \otimes \vec{u}_\epsilon|^2 = \sum_{i=1}^3 \partial_i (2\vec{u}_\epsilon \cdot \partial_i \vec{u}_\epsilon - |\vec{u}_\epsilon|^2 \omega_\epsilon * u_{\epsilon,i} - 2p_\epsilon u_{\epsilon,i}) + R_\epsilon \quad (7.5)$$

where

$$R_\epsilon = |\vec{u}_\epsilon|^2 \omega_\epsilon * (\vec{\nabla} \cdot \vec{u}_\epsilon) + 2p_\epsilon \vec{\nabla} \cdot \vec{u}_\epsilon = 0 \quad (7.6)$$

By a compactness argument based on Rellich's theorem (for details, we refer to [45] chapters 13 and 14), there is a sequence  $\epsilon_k \rightarrow 0$  and a distribution  $\vec{u} \in L_t^\infty L^2 \cap L_t^2 \dot{H}_x^1$  such that  $\vec{u}_{\epsilon_k}$  converges to  $\vec{u}$  weakly in  $L_t^2 \dot{H}_x^1$  and strongly in  $L^2$  norm on every compact subset of  $(0, +\infty) \times \mathbb{R}^3$ . Thus, we have (in  $\mathcal{D}'((0, +\infty) \times \mathbb{R}^3)$ )

$$\lim_{\epsilon_k \rightarrow 0} \partial_t \vec{u}_{\epsilon_k} + (\omega_{\epsilon_k} * \vec{u}_{\epsilon_k}) \cdot \vec{\nabla} \vec{u}_{\epsilon_k} - \Delta \vec{u}_{\epsilon_k} + \vec{\nabla} p_{\epsilon_k} = \partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} - \Delta \vec{u} + \vec{\nabla} p = 0 \quad (7.7)$$

and

$$\begin{aligned} \lim_{\epsilon_k \rightarrow 0} \partial_t |\vec{u}_{\epsilon_k}|^2 + \sum_{i=1}^3 \partial_i (-2\vec{u}_{\epsilon_k} \cdot \partial_i \vec{u}_{\epsilon_k} + |\vec{u}_{\epsilon_k}|^2 \omega_{\epsilon_k} * u_{\epsilon_k, i} + 2p_{\epsilon_k} u_{\epsilon_k, i}) \\ = \partial_t |\vec{u}|^2 + \sum_{i=1}^3 \partial_i (-2\vec{u} \cdot \partial_i \vec{u} + |\vec{u}|^2 u_i + 2p u_i) \end{aligned} \quad (7.8)$$

However, there is no reason that  $|\vec{\nabla} \otimes \vec{u}_{\epsilon_k}|^2$  should converge to  $|\vec{\nabla} \otimes \vec{u}|^2$ . The best we can get is

$$\lim_{\epsilon_k \rightarrow 0} |\vec{\nabla} \otimes \vec{u}_{\epsilon_k}|^2 = |\vec{\nabla} \otimes \vec{u}|^2 + \mu \quad (7.9)$$

where  $\mu$  is a non-negative distribution on  $(0, +\infty) \times \mathbb{R}^3$  (hence a locally finite non-negative measure). This gives

$$\partial_t |\vec{u}|^2 + 2|\vec{\nabla} \otimes \vec{u}|^2 = \Delta |\vec{u}|^2 - \vec{\nabla} \cdot (|\vec{u}|^2 + 2p)\vec{u} - \mu \quad (7.10)$$

This is Scheffer's local energy inequality [65]. This inequality plays an important part in the study of partial regularity of weak solutions [9]. Solutions which satisfy the local energy inequality (7.10) are called suitable.

Inequality (7.10) is a key tool to develop a theory of weak solutions for initial values  $\vec{u}_0$  with infinite energy ( $\|\vec{u}_0\|_2 = +\infty$ ). In [44] [45] a theory has been developed to exhibit suitable weak solutions associated to an initial value  $\vec{u}_0$  which is uniformly locally square integrable (i.e.  $\sup_{x_0 \in \mathbb{R}^3} \int_{|x-x_0| \leq 1} |\vec{u}_0(x)|^2 dx < +\infty$  or equivalently  $\vec{u}_0 \in (M^{2,2})^3$ ). The basic idea of the proof is to consider the mollified equations (2.3) and to compute the  $L^2_{uloc}$  norm of  $\vec{u}_\epsilon$  as

$$\|\vec{u}_\epsilon\|_{L^2_{uloc}} = \sup_{x_0 \in \mathbb{R}^3} \|\varphi_0(x - x_0)\vec{u}_\epsilon\|_2 \quad (7.11)$$

for some  $\varphi_0 \in \mathcal{D}(\mathbb{R}^3)$  (with  $\varphi_0 \neq 0$ ). In contrast with the finite-energy case,  $p_\epsilon$  cannot be computed as  $p_\epsilon = -\sum_{i=1}^3 \sum_{j=1}^3 \frac{1}{\Delta} \partial_i \partial_j (\omega_\epsilon * u_{\epsilon, i} u_{\epsilon, j})$  since the kernel of the convolution operator  $\frac{1}{\Delta} \partial_i \partial_j$  has slow decay at infinity, hence is not defined on  $L^p_{uloc}$ . But  $\vec{\nabla} p_\epsilon$  is well defined: the kernel of  $\partial_k \frac{1}{\Delta} \partial_i \partial_j$  has enough decay at infinity to operate on  $L^p_{uloc}$ . Then formulas (7.5) and (7.6) remain true. Carefully integrated against test functions  $\varphi(x) = \varphi_0^2(x - x_0)$ , they give a control independent of  $\epsilon$ : we start from the identity

$$\int \varphi(x) |\vec{u}_\epsilon(t, x)|^2 dx + 2 \int_0^t \int \varphi(x) |\vec{\nabla} \otimes \vec{u}(s, x)|^2 dx ds = \int \varphi(x) |\vec{u}_0(x)|^2 dx + I_\epsilon(t) \quad (7.12)$$

with

$$I_\epsilon(t) = \int \int_0^t |\vec{u}_\epsilon(s, x)|^2 \Delta \varphi(x) dx ds + \int_0^t \int |\vec{u}_\epsilon|^2 (\omega_\epsilon * \vec{u}_\epsilon) * \vec{\nabla} \varphi dx ds + 2 \int \int_0^t p_\epsilon \vec{u}_\epsilon \cdot \vec{\nabla} \varphi dx ds \quad (7.13)$$

and defining

$$\alpha_\epsilon(t) = \sup_{x_0 \in \mathbb{R}^3} \int \varphi_0^2(x - x_0) |\vec{u}_\epsilon(t, x)|^2 dx \quad \text{and} \quad \beta_\epsilon(t) = \sup_{x_0 \in \mathbb{R}^3} \int_0^t \int \varphi_0^2(x - x_0) |\vec{\nabla} \otimes \vec{u}(s, x)|^2 dx ds \quad (7.14)$$

we find that

$$I_\epsilon(t) \leq C \left( \int_0^t \alpha_\epsilon(s) ds + \left( \int_0^t \alpha_\epsilon^3(s) ds \right)^{1/4} (\beta_\epsilon(t) + \int_0^t \alpha_\epsilon(s) ds)^{3/4} \right) \quad (7.15)$$

In [45], we show that inequalities (7.12) and (7.15) provide a control uniform in  $\epsilon$  on a time interval  $(0, T)$  with  $T = O(\min(1, \|\vec{u}_0\|_{L^2_{uloc}}^{-2}))$ . Then the same compactness argument as in the case of finite-energy initial values allows us to show that :

**Theorem 7.1.** *Let  $\vec{u}_0 \in (L^2_{uloc}(\mathbb{R}^3))^3$  be such that  $\vec{\nabla} \cdot \vec{u}_0 = 0$ . Then, there exists a positive constant  $C_0$  (which does not depend on  $\vec{u}_0$ ) such that, defining  $T_0 = \frac{1}{C_0^4 \sup(1, \|\vec{u}_0\|_{L^2_{uloc}}^2)}$ , the equations (1.13) have a suitable solution  $\vec{u}$  on  $(0, T_0) \times \mathbb{R}^3$  such that for all  $0 < t < T_0$  we have*

$$\|\vec{u}(t, \cdot)\|_{L^2_{uloc}} \leq \sqrt{C_0} \|\vec{u}_0\|_{L^2_{uloc}} \left(1 - \frac{t}{T_0}\right)^{-1/4} \quad (7.16)$$

and

$$\sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{|x-x_0|<1} |\vec{\nabla} \otimes \vec{u}(s, x)|^2 dx ds \leq C_0 \|\vec{u}_0\|_{L^2_{uloc}}^2 \left(1 - \frac{t}{T_0}\right)^{-1/2}. \quad (7.17)$$

The Morrey norm in  $\dot{M}^{2,3}$  may be viewed as a non-scaled version of the norm in  $L^2_{uloc}$  :

$$\|f\|_{L^2_{uloc}} = \sup_{x_0 \in \mathbb{R}^3} \|1_{B(x_0, 1)} f\|_2 \quad \text{and} \quad \|f\|_{\dot{M}^{2,3}} = \sup_{R>0} R \|f(Rx)\|_{L^2_{uloc}} \quad (7.18)$$

A direct consequence of (7.18) is that, when  $\vec{u}_0 \in \dot{M}^{2,3}$  the Proof of Theorem 7.1 can be adapted to any scale, hence will provide a solution on any time interval  $(0, T)$ , and finally (through a diagonal extraction process) a global solution [46]:

**Theorem 7.2.** *Let  $\vec{u}_0 \in (\dot{M}^{2,3}(\mathbb{R}^3))^3$  be such that  $\vec{\nabla} \cdot \vec{u}_0 = 0$ . Then, there exists a positive constant  $C_0$  (which does not depend on  $\vec{u}_0$ ) such that, defining  $T_0 = \frac{1}{C_0^4 \sup(1, \|\vec{u}_0\|_{\dot{M}^{2,3}}^2)}$ , equations (1.13) have a suitable solution  $\vec{u}$  on  $(0, +\infty) \times \mathbb{R}^3$  such that*

$$\sup_{x_0 \in \mathbb{R}^3, R>0, t>0} \frac{1}{R + \sqrt{\frac{t}{T_0}}} \int_{|x-x_0|<R} |\vec{u}(t, x)|^2 dx \leq C_0 \|\vec{u}_0\|_{\dot{M}^{2,3}}^2 \quad (7.19)$$

and

$$\sup_{x_0 \in \mathbb{R}^3, t>0} \sqrt{\frac{T_0}{t}} \int_0^t \int_{|x-x_0|<\sqrt{\frac{t}{T_0}}} |\vec{\nabla} \otimes \vec{u}(s, x)|^2 dx ds \leq C_0 \|\vec{u}_0\|_{\dot{M}^{2,3}}^2. \quad (7.20)$$



## 8 Homogeneous statistical solutions

Statistical solutions for three dimensional Navier–Stokes equations were introduced by Hopf [32] as a mathematical model for the statistical theory of turbulent flows. More precisely, we consider spatially homogeneous statistical solutions (which correspond to homogeneous turbulence), in the sense of Vishik and Fursikov [71], [24].

We consider the Cauchy problem (1.13) for the Navier–Stokes equations, with (divergence free) initial value  $\vec{u}_0$  being a random variable  $\omega \mapsto \vec{u}_0(\omega)$  defined on some probability space  $(\Omega, P)$  with values in  $(L^2_{loc})^3$ . We define  $\mu_0$  the distribution law of  $\vec{u}_0$ : it is a Borel measure on  $(L^2_{loc})^3$  defined by

$$\mu_0(A) = P(\{\omega \in \Omega / \vec{u}_0 \in A\}) \quad (8.1)$$

For  $x_0 \in \mathbb{R}^3$ , we define  $\tau_{x_0}$  the map  $f \in L^2_{loc} \mapsto \tau_{x_0} f \in L^2_{loc}$  by  $\tau_{x_0} f(x) = f(x - x_0)$ . We define, for  $A \subset (L^2_{loc})^3$ ,  $\tau_{x_0} A = \{\tau_{x_0}(\vec{f}) / \vec{f} \in A\}$ . We shall say that  $\mu_0$  is spatially homogeneous if, for every  $x_0 \in \mathbb{R}^3$  and every Borel set  $A \subset (L^2_{loc})^3$ ,  $\mu_0(\tau_{x_0} A) = \mu_0(A)$ . We shall be interested in solving (1.13) for a random  $\vec{u}_0$ , where  $\vec{u}_0$  is locally square-integrable and divergence-free, and where the distribution law  $\mu_0$  will be spatially homogeneous and satisfy the following energy estimate:

$$\int_{\Omega} \left( \int_{B(0,1)} |\vec{u}_0(\omega, x)|^2 dx \right) dP(\omega) < +\infty \quad (8.2)$$

A statistical solution of (1.13) on  $(0, T) \times \mathbb{R}^3$  is a random variable  $\vec{u}$  on  $(\Omega, P)$  with values in  $(L^2_{loc}([0, T] \times \mathbb{R}^3))^3$  such that

- a) for P-almost every  $\omega$ , for every  $N > 0$ ,  $\partial_t \vec{u}(\omega, t, x) \in (L^2((0, T), H^{-3}(B(0, N))))^3$
- b) for P-almost every  $\omega$ ,  $\vec{u}(\omega, t, x)$  is a solution of the Cauchy problem (13) with initial value  $\vec{u}(\omega, 0, x)$
- c) for P-almost every  $\omega$ ,  $\vec{u}(\omega, 0, x) \in (L^2_{loc}(\mathbb{R}^3))^3$  and the variable  $\omega \mapsto \vec{u}(\omega, 0, x)$  has the same law as  $\omega \mapsto \vec{u}_0(\omega, x)$

Property a) allows us to give some meaning to the mapping  $\omega \mapsto \vec{u}(\omega, 0, x)$ : if  $\theta \in \mathcal{D}(\mathbb{R})$  satisfies  $\theta(0) = 1$  and  $\theta(t) = 0$  for  $t > T/2$ , we define  $\vec{u}(\omega, 0, x)$  as  $\vec{u}(\omega, 0, x) = \int_0^T \partial_t(\theta(t)\vec{u}(\omega, t, x)) dx$ . (See [4] for a more precise description of those conditions).

The main trouble with spatial homogeneity is the fact that the initial value  $\vec{u}_0$ , if not identically equal to 0, cannot be in  $L^2$ , so that it forbids the use of Leray’s energy inequality. More precisely, if  $w$  is a positive continuous weight on  $\mathbb{R}^3$ , then we have the following properties:

- i) if  $\int w(x) dx = +\infty$ , then  $P(\{\omega \in \Omega / \vec{u}_0(\omega, \cdot) \neq 0 \text{ and } \int |\vec{u}_0(\omega, x)|^2 w(x) dx < +\infty\}) = 0$
- ii) if (8.2) is fulfilled, then, if  $\int w(x) dx < +\infty$ , for P-almost every  $\omega$ , we have  $\int |\vec{u}_0(\omega, x)|^2 w(x) dx < +\infty$ .

In order to use energy estimates, one may consider only periodical solutions. Leray’s method works in the periodical setting as well and provides weak solutions to the Navier–Stokes equations.

However, one may consider non-periodical solutions. For instance, Vishik and Fursikov describe spatially homogeneous random initial values  $\vec{u}_0$  that are (non-periodical) trigonometric polynomials  $\vec{u}_0 = \sum_{k=1}^N e^{i\lambda_k \cdot x} \vec{a}_k(\omega)$ . Vishik and Fursikov “solve” (1.13)

in the space  $L^2(\frac{1}{(1+|x|)^{d+\epsilon}} dx)$  with  $\epsilon > 0$ : as a matter of fact, one cannot solve (1.13) in a deterministic way in such a space, but Vishik and Fursikov can construct a random variable  $\vec{u}$  which solves (1.13) for P-almost every  $\omega$ . If we want to be able to exhibit a solution for one special initial value, one has to consider a smaller space of initial values. A space that is larger than periodical functions but smaller than  $L^2(\frac{1}{(1+|x|)^{d+\epsilon}} dx)$  is the space  $L^2_{uloc}$ .

In dimension 2, the space  $L^2_{uloc}$  is well adapted to the problem (1.13): to any divergence free initial value in  $(L^2_{uloc})^3$ , one may associate one unique global regular solution to the Cauchy problem [5]. In case of  $\vec{u}_0$  being a (non-periodical) trigonometric polynomial, or being almost-periodic in the sense of Stepanov (i.e. belonging to the closure of trigonometric polynomials in the  $L^2_{uloc}$  norm [6]), we have a solution  $\vec{u}$  that is almost-periodic in the sense of Bohr for every positive  $t$ .

In dimension 3, we can construct only local solutions for initial values in  $L^2_{uloc}$ . In case of  $\vec{u}_0$  being almost-periodic in the sense of Stepanov, we don't even know whether one may find an almost-periodic solution (since the space of almost-periodic functions in the sense of Stepanov is not closed in  $L^2_{uloc}$  for the weak-\* topology).

### 9 Serrin's uniqueness criterion

Leray [50] studied the Cauchy initial value problem for equations (1.13) with a square-integrable initial value. He proved the existence of weak solutions, which satisfy moreover energy inequality (7.1). An easy consequence of inequality (7.1) is then the strong continuity at  $t = 0$  :

$$\lim_{t \rightarrow 0^+} \|\vec{u} - \vec{u}_0\|_2 = 0. \tag{9.1}$$

But it is still not known whether we have continuity for all time  $t$  and whether we have uniqueness in the class of Leray solutions. Serrin's theorem [66] gives a criterion for uniqueness:

**Proposition 9.1.** *(Serrin's uniqueness theorem) Let  $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$  with  $\text{div } \vec{u}_0 = 0$ . Assume that there exists a solution  $\vec{u}$  of the Navier-Stokes equations on  $(0, T) \times \mathbb{R}^3$  (for some  $T \in (0, +\infty]$ ) with initial value  $\vec{u}_0$  such that:*

- i)  $\vec{u} \in L^\infty((0, T), (L^2(\mathbb{R}^3))^3)$ ;*
- ii)  $\vec{u} \in L^2((0, T), (\dot{H}^1(\mathbb{R}^3))^3)$ ;*
- iii) For some  $r \in [0, 1)$ ,  $\vec{u}$  belongs to  $(L^\sigma((0, T), L^{3/r}))^3$  with  $2/\sigma = 1 - r$ .*

*Then,  $\vec{u}$  satisfies the Leray energy inequality and it is the unique Leray solution associated to  $\vec{u}_0$  on  $(0, T)$ .*

The limit case  $r = 1$  is dealt with Sohr and Von Wahl's theorem [76] ( $\vec{u}$  belongs to  $(\mathcal{C}([0, T], L^3))^3$ ), or Kozono and Sohr's theorem [37] ( $\vec{u} \in (L^\infty([0, T], L^3))^3$ ). Serrin's theorem is quite easy to prove. We sketch the proof for  $r < 1$ . Let  $\vec{v}$  be another solution associated to  $\vec{u}_0$  on  $(0, T)$  (with associated pressure  $q$ ) such that  $\vec{v} \in L^\infty((0, T), (L^2(\mathbb{R}^3))^3) \cap L^2((0, T), (\dot{H}^1(\mathbb{R}^3))^3)$  and  $\vec{v}$  satisfies the energy inequality (7.1).  $\vec{u}$  is regular enough to justify the formulae  $\partial_t(|\vec{u}|^2) = 2\vec{u} \cdot \partial_t \vec{u}$  and  $\partial_t(\vec{u} \cdot \vec{v}) = \vec{u} \cdot \partial_t \vec{v} + \vec{v} \cdot \partial_t \vec{u}$ . We then write

$$\|\vec{u} - \vec{v}\|_2^2 = \|\vec{v}\|_2^2 - \|\vec{u}_0\|_2^2 + \int_0^t \partial_t(|\vec{u}|^2) ds - 2 \int_0^t \partial_t(\vec{u} \cdot \vec{v}) ds. \tag{9.2}$$

Now,  $\vec{v}$  satisfies the Leray inequality

$$\|\vec{v}(t)\|_2^2 - \|\vec{u}_0\|_2^2 \leq -2 \int_0^t \|\vec{\nabla} \otimes \vec{v}\|_2^2 ds, \quad (9.3)$$

so that we get the following inequality for  $\vec{u} - \vec{v}$ :

$$\|\vec{u}(t, \cdot) - \vec{v}(t, \cdot)\|_2^2 \leq -2 \int_0^t \int_{\mathbb{R}^3} |\vec{\nabla} \otimes (\vec{u} - \vec{v})|^2 dx ds - 2 \int_0^t \int_{\mathbb{R}^3} \vec{u} \cdot ((\vec{u} - \vec{v}) \cdot \vec{\nabla})(\vec{v} - \vec{u}) dx ds. \quad (9.4)$$

We then write

$$\begin{aligned} & \left| \int_\tau^t \int_{\mathbb{R}^3} \vec{u} \cdot ((\vec{u} - \vec{v}) \cdot \vec{\nabla})(\vec{v} - \vec{u}) dx ds \right| \\ & \leq C_r \left( \int_\tau^t \|\vec{u}\|_{L^{3/r}}^{\frac{2}{1-r}} ds \right)^{(1-r)/2} \left( \int_0^t \|\vec{v} - \vec{u}\|_{\dot{H}^1}^2 ds \right)^{1/2} \left( \int_\tau^t \|\vec{v} - \vec{u}\|_{\dot{H}^r}^{\frac{2}{r}} ds \right)^{\frac{r}{2}} \\ & \leq C'_r \left( \int_\tau^t \|\vec{u}\|_{L^{3/r}}^{\frac{2}{1-r}} ds \right)^{(1-r)/2} \left( \int_0^t \|\vec{v} - \vec{u}\|_{\dot{H}^1}^2 ds \right)^{(1+r)/2} \left( \sup_{\tau < s < t} \|\vec{v} - \vec{u}\|_2^2 \right)^{\frac{(1-r)}{2}} \end{aligned} \quad (9.5)$$

With help of the Young inequality, we find that

$$\sup_{0 < s \leq t} \|\vec{u} - \vec{v}\|_2^2 \leq C_r \left( \int_\tau^t \|\vec{u}\|_{L^{3/r}}^{\frac{2}{1-r}} ds \right)^{1/2} \sup_{0 < s \leq t} \|\vec{u} - \vec{v}\|_2^2 \quad (9.6)$$

and we may conclude locally (for  $t$  small enough) and then globally by bootstrap.

Thus, the main tool in proving Proposition 9.1 is the fact that when  $f \in L^\infty L^2 \cap L^2 \dot{H}^1$ , then  $f$  belongs to  $L^{2/r} \dot{H}^r$  and that the pointwise product is bounded from  $\dot{H}^r \times L^{3/r}$  to  $L^2$ . Considering the space  $\dot{X}^r$  of pointwise multipliers from  $\dot{H}^r$  to  $L^2$  then gives a direct generalization of Proposition 1, as it has been observed in [45], [49]:

**Theorem 9.1.** *Let  $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$  with  $\operatorname{div} \vec{u}_0 = 0$ . Assume that there exists a solution  $\vec{u}$  of the Navier-Stokes equations on  $(0, T) \times \mathbb{R}^3$  (for some  $T \in (0, +\infty]$ ) with initial value  $\vec{u}_0$  such that:*

- i)  $\vec{u} \in L^\infty((0, T), (L^2(\mathbb{R}^3))^3)$ ;*
- ii)  $\vec{u} \in L^2((0, T), (\dot{H}^1(\mathbb{R}^3))^3)$ ;*
- iii) For some  $r \in [0, 1)$ ,  $\vec{u}$  belongs to  $(L^\sigma((0, T), \dot{X}_r))^3$  with  $2/\sigma = 1 - r$ .*

*Then,  $\vec{u}$  satisfies the Leray energy inequality and it is the unique Leray solution associated to  $\vec{u}_0$  on  $(0, T)$ .*

*A similar results holds for  $r = 1$  when iii) is replaced by*

*iii')  $\vec{u}$  belongs to  $(\mathcal{C}([0, T], \tilde{X}_1))^3$ , where  $\tilde{X}_1$  is the closure of the space  $\mathcal{D}$  of smooth test functions in  $\dot{X}_1$ .*

The spaces  $\dot{X}_r$  have been characterized by Maz'ya [53] in terms of Sobolev capacities. A weaker result establishes a comparison between the spaces  $\dot{X}_r$  and the Morrey–Campanato spaces  $\dot{M}^{2,p}$ : for  $2 < p \leq 3/r$  and  $0 < r$  we have

$$\dot{M}^{p,3/r} \subset \dot{X}_r \subset \dot{M}^{2,3/r}, \quad (9.7)$$

a result first noticed by Fefferman and Phong in the setting of Schrödinger equation [22], [55], [45].

There are many ways to outperform Theorem 9.1. The point is to estimate

$$I(\vec{f}, \vec{g}, \vec{h}) = \int_0^t \int \vec{f} \cdot (\vec{g} \cdot \vec{\nabla} \vec{h}) \, ds \quad (9.8)$$

with  $\vec{f} = \vec{u}$ ,  $\vec{g} = \vec{h} = \vec{u} - \vec{v}$ . What information can we use on  $\vec{g}$  and  $\vec{h}$  and what information do we need on  $\vec{f}$ ?

A) **Sobolev regularity of  $\vec{g}$** : we use  $\vec{h} \in (L^2 \dot{H}^1)^3$  and  $\vec{g} \in (L^{\frac{2}{r}} \dot{H}^r)^3$  ( $0 \leq r \leq 1$ ). In that case, we shall need some pointwise multiplication that maps  $\dot{H}^r$  to  $L^2$ , hence we shall need  $\vec{f} \in (L^{\frac{2}{1-r}} \dot{X}^r)^3$  if  $0 < r < 1$ ,  $f \in (\mathcal{C}([0, T], \tilde{X}_1))^3$  if  $r=1$  (Theorem 7) or  $f \in (L^2 L^\infty)^3$  if  $r = 0$  (Proposition 1).

B) **Solenoidality of  $\vec{g}$** : we use  $\vec{h} \in (L^{\frac{2}{r}} \dot{H}^r)^3$  ( $0 \leq r \leq 1$ ),  $\vec{g} \in (L^\infty L^2)^3$  and  $\operatorname{div} \vec{g} = 0$ . In that case, we shall need to replace direct estimates on pointwise products by estimates on paraproducts : we split  $\vec{g} \cdot \vec{\nabla} \vec{h}$  into three terms and write (for  $0 < r < 1$ )

$$\left\{ \begin{array}{l} \|\sum_{j \in \mathbb{Z}} \Delta_j \vec{g} \cdot \vec{\nabla} S_j \vec{h}\|_{\dot{B}_{1,1}^{r-1}} \leq C_r \|\vec{g}\|_2 \|\vec{h}\|_{\dot{H}^r} \\ \|\operatorname{div} (\sum_{j \in \mathbb{Z}} \sum_{k=-2}^2 \Delta_j \vec{g} \otimes \Delta_{j+k} \vec{h})\|_{\dot{B}_{1,1}^{r-1}} \leq C_r \|\vec{g}\|_2 \|\vec{h}\|_{\dot{H}^r} \\ \|\sum_{j \in \mathbb{Z}} S_j \vec{g} \cdot \vec{\nabla} \Delta_j \vec{h}\|_{\dot{F}_{1,2}^{r-1}} \leq C_r \|\vec{g}\|_2 \|\vec{h}\|_{\dot{H}^r} \end{array} \right. \quad (9.9)$$

Thus, we obtain the following theorem of Kozono and Taniuchi ( $r = 1$ ) and Germain ( $0 \leq r < 1$ ) [38], [28]:

**Theorem 9.2.** *Let  $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$  with  $\operatorname{div} \vec{u}_0 = 0$ . Assume that there exists a solution  $\vec{u}$  of the Navier-Stokes equations on  $(0, T) \times \mathbb{R}^3$  (for some  $T \in (0, +\infty]$ ) with initial value  $\vec{u}_0$  such that:*

i)  $\vec{u} \in L^\infty((0, T), (L^2(\mathbb{R}^3))^3)$ ;

ii)  $\vec{u} \in L^2((0, T), (\dot{H}^1(\mathbb{R}^3))^3)$ ;

iii) For some  $r \in [0, 1)$ ,  $(-\Delta)^{\frac{1-r}{2}} \vec{u}$  belongs to  $(L^\sigma((0, T), BMO)^3$  with  $2/\sigma = 2 - r$ .

Then,  $\vec{u}$  satisfies the Leray energy inequality and it is the unique Leray solution associated to  $\vec{u}_0$  on  $(0, T)$ .

A similar results holds for  $r = 1$  when iii) is replaced by

iii')  $\vec{\nabla} \otimes \vec{u}$  belongs to  $(L^1((0, T), L^\infty))^9$ .

C) **Besov regularity of  $\vec{g}$** : we use  $\vec{h} \in (L^2 \dot{H}^1)^3$  and  $\vec{g} \in (L^{\frac{2}{r}} \dot{B}_{2,1}^r)^3$  ( $0 < r < 1$ ). In that case, we shall need some pointwise multiplication that maps  $\dot{B}_{2,1}^r$  to  $L^2$ , hence we shall need  $\vec{f} \in (L^{\frac{2}{1-r}} \dot{Y}^r)^3$  with  $\dot{Y}^r = \mathcal{M}(\dot{B}_{2,1}^r \mapsto L^2)$ . While the structure of the multiplier spaces  $\dot{X}_r$  is not easy to describe, the multiplier space is much more simple : for  $0 < r < 3/2$ , we have  $\dot{Y}^r = \dot{M}^{2,3/r}$ . This is easily proved through a wavelet decomposition for  $\dot{B}_{2,1}^r$  [46]. Theorem 9.1 now turns into the following one:

**Theorem 9.3.** *Let  $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$  with  $\operatorname{div} \vec{u}_0 = 0$ . Assume that there exists a solution  $\vec{u}$  of the Navier-Stokes equations on  $(0, T) \times \mathbb{R}^3$  (for some  $T \in (0, +\infty]$ ) with initial value  $\vec{u}_0$  such that:*

i)  $\vec{u} \in L^\infty((0, T), (L^2(\mathbb{R}^3))^3)$ ;

ii)  $\vec{u} \in L^2((0, T), (\dot{H}^1(\mathbb{R}^3)^3))$ ;

iii) For some  $r \in [0, 1)$ ,  $\vec{u}$  belongs to  $(L^\sigma((0, T), \dot{M}^{2,3/r}))^3$  with  $2/\sigma = 1 - r$ .

Then,  $\vec{u}$  satisfies the Leray energy inequality and it is the unique Leray solution associated to  $\vec{u}_0$  on  $(0, T)$ .

Theorem 9.3 does not include the limit case  $r = 1$ , which is still an open question: we don't know whether a similar results holds for  $r = 1$  when iii) is replaced by

iii')  $\vec{u}$  belongs to  $(\mathcal{C}([0, T], \tilde{M}^{2,3}))^3$  where  $\tilde{M}^{2,3}$  is the closure of the test functions  $\mathcal{D}$  in  $M^{2,3}$ .

D) **Besov regularity of  $\vec{h}$  and solenoidality of  $\vec{g}$** : we use  $h \in (L^{\frac{2}{r}} \dot{B}_{2,1}^r)^3$  ( $0 < r < 1$ ),  $\vec{g} \in (L^\infty L^2)^3$  and  $\operatorname{div} \vec{g} = 0$ . We write

$$\|\operatorname{div}(\sum_{j \in \mathbb{Z}} S_j \vec{g} \otimes \Delta_j h)\|_{\dot{B}_{1,1}^{r-1}} \leq C_r \|\vec{g}\|_2 \|\vec{h}\|_{\dot{B}_{2,1}^r} \quad (9.10)$$

Theorem 9.2 now turns into the following theorem of Chen, Miao and Zhang [15]:

**Theorem 9.4.** *Let  $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$  with  $\operatorname{div} \vec{u}_0 = 0$ . Assume that there exists a solution  $\vec{u}$  of the Navier-Stokes equations on  $(0, T) \times \mathbb{R}^3$  (for some  $T \in (0, +\infty]$ ) with initial value  $\vec{u}_0$  such that:*

i)  $\vec{u} \in L^\infty((0, T), (L^2(\mathbb{R}^3)^3))$ ;

ii)  $\vec{u} \in L^2((0, T), (\dot{H}^1(\mathbb{R}^3)^3))$ ;

iii) For some  $r \in [0, 1)$ ,  $\vec{u}$  belongs to  $(L^\sigma((0, T), \dot{B}_{\infty,\infty}^{1-r}))^3$  with  $2/\sigma = 2 - r$ .

Then,  $\vec{u}$  satisfies the Leray energy inequality and it is the unique Leray solution associated to  $\vec{u}_0$  on  $(0, T)$ .

E) **Regularity of  $\vec{u} - \vec{v}$** : We may now consider what happens if  $\vec{u}$  is controlled in a norm of negative regularity. Following an idea of Chemin [14] and Lemarié–Rieusset [48], Chen, Miao and Zhang [15] and May [52] proved the following theorem:

**Theorem 9.5.** *Let  $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$  with  $\operatorname{div} \vec{u}_0 = 0$ . Assume that there exist solutions  $\vec{u}$  and  $\vec{v}$  of the Navier-Stokes equations on  $(0, T) \times \mathbb{R}^3$  (for some  $T \in (0, +\infty]$ ) with the same initial value  $\vec{u}_0$  such that:*

i)  $\vec{u}, \vec{v} \in L^\infty((0, T), (L^2(\mathbb{R}^3)^3))$ ;

ii)  $\vec{u}, \vec{v} \in L^2((0, T), (\dot{H}^1(\mathbb{R}^3)^3))$ ;

iii) For some  $r \in (0, 1)$ ,  $\vec{u}$  belongs to  $(L^\sigma((0, T), \dot{B}_{\infty,\infty}^{-r}))^3$  with  $2/\sigma = 1 - r$ .

iv) For some  $\rho \in (0, 1)$ ,  $\vec{v}$  belongs to  $(L^\tau((0, T), \dot{B}_{\infty,\infty}^{-\rho}))^3$  with  $2/\tau = 1 - \rho$ .

Then,  $\vec{u} = \vec{v}$ .

One may be tempted to drop the assumption on  $\vec{v}$  by noticing that  $\vec{u} - \vec{v}$  is more regular than  $\vec{u}$  and  $\vec{v}$  [23], [45]

$$\vec{u} - \vec{v} \in (L^1((0, T), \dot{B}_{2,1}^{3/2}))^3 \quad (9.11)$$

One would get the following result:

**Theorem 9.6.** *Let  $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$  with  $\operatorname{div} \vec{u}_0 = 0$ . Assume that there exists a solution  $\vec{u}$  of the Navier-Stokes equations on  $(0, T) \times \mathbb{R}^3$  (for some  $T \in (0, +\infty]$ ) with initial value  $\vec{u}_0$  such that:*

- i)  $\vec{u} \in L^\infty((0, T), (L^2(\mathbb{R}^3))^3)$ ;
- ii)  $\vec{u} \in L^2((0, T), (\dot{H}^1(\mathbb{R}^3))^3)$ ;
- iii) For some  $r \in (0, 1/2)$ ,  $\vec{u}$  belongs to  $(L^\sigma((0, T), \dot{B}_{\infty, \infty}^{-r})^3)$  with  $2/\sigma = 1 - 2r$ .

Then,  $\vec{u}$  satisfies the Leray energy inequality and it is the unique Leray solution associated to  $\vec{u}_0$  on  $(0, T)$ .

However, Theorem 9.6 is neither new nor optimal. In Theorems 9.1 to 9.6, we are dealing with conditions  $\vec{u} \in (L_t^\sigma X_x)^3$  where the norm of  $X$  is homogeneous :  $\|f(x/R)\|_X = R^\alpha \|f\|_X$ . While in Theorems 9.1 to 9.5, the indexes  $\sigma$  and  $\alpha$  respect the critical scaling condition  $\alpha + \frac{2}{\sigma} = 1$ , in Theorem 9.6 we need a subcritical scaling condition:  $\alpha + \frac{2}{\sigma} = 1 - r < 1$ . But this subcritical condition is not optimal: if  $u \in L^2 \dot{H}^1 \cap L^\sigma((0, T), \dot{B}_{\infty, \infty}^{-r})$  with  $\sigma = \frac{4}{2-3r}$  [thus,  $\alpha + \frac{2}{\sigma} = 1 - \frac{r}{2}$ ], then, using the generalized Sobolev inequalities of Gérard–Meyer–Oru [27], we find that  $u \in L^p L^q$  with  $1/q = \frac{r}{r+1} \frac{1}{2}$  and  $1/p = \frac{r}{r+1} \frac{1}{2} + \frac{1}{r+1} \frac{1}{\sigma} = \frac{1}{2} \frac{1}{r+1} (1 - \frac{r}{2})$  so that  $2/p + 3/q = 1$  and we may apply Serrin’s theorem ...

## 10 Uniqueness of mild solutions

In 1984, Kato [34] proved the existence of mild solutions to problem (1.13) when  $\vec{u}_0 \in (L^3)^3$ . However, the fixed–point algorithm did not work in the space  $\mathcal{C}([0, T], (L^3)^3)$ , but in a smaller space (one required that  $\sup_{0 < t < T} \sqrt{t} \|\vec{u}(t, \cdot)\|_\infty < +\infty$  and that  $\lim_{t \rightarrow 0} \sqrt{t} \|\vec{u}(t, \cdot)\|_\infty = 0$ ). In 1997, Furioli, Lemarié–Rieusset and Terraneo [26] proved uniqueness of mild solutions in  $\mathcal{C}([0, T^*), (L^3)^3)$ . They extended their proof to the case of Morrey–Campanato spaces by using the Besov spaces over Morrey–Campanato spaces described by Kozono and Yamazaki [39] and found that uniqueness holds as well in the class  $\mathcal{C}([0, T^*), (\tilde{M}^{p,3})^3)$  for  $p > 2$ , where  $\tilde{M}^{p,3}$  is the closure of the smooth compactly supported functions in the Morrey–Campanato space  $\dot{M}^{p,3}$ . In his thesis dissertation, May [51], [49] proved a slightly more general result by extending the approach of Monniaux [57] (i.e. by using the maximal  $L^p L^q$  property of the heat kernel):

**Theorem 10.1.** *If  $\vec{u}$  and  $\vec{v}$  are two weak solutions of the Navier–Stokes equations on  $(0, T^*) \times \mathbb{R}^3$  such that  $\vec{u}$  and  $\vec{v}$  belong to  $\mathcal{C}([0, T^*), (\tilde{X}_1)^3)$  and have the same initial value, then  $\vec{u} = \vec{v}$ .*

May’s result generalizes the results of Furioli, Lemarié–Rieusset and Terraneo, but leaves open the limit case of  $\dot{M}^{2,3}$  :

### Open question:

*Does uniqueness holds in  $(\mathcal{C}([0, T^*), \tilde{M}^{2,3})^3)$ ?*

In [46], we considered the following problem of uniqueness:

**Definition 2.** (Regular critical space) A regular critical space is a Banach space  $X$  such that we have the continuous embeddings  $\mathcal{D}(\mathbb{R}^3) \subset X \subset L_{loc}^2(\mathbb{R}^3)$  and such that moreover:

- (a) for all  $x_0 \in \mathbb{R}^3$  and for all  $f \in X$ ,  $f(x - x_0) \in X$  and  $\|f\|_X = \|f(x - x_0)\|_X$ .
- (b) for all  $\lambda > 0$  and for all  $f \in X$ ,  $f(\lambda x) \in X$  and  $\lambda \|f(\lambda x)\|_X = \|f\|_X$ .
- (c)  $\mathcal{D}(\mathbb{R}^3)$  is dense in  $X$ .

We have the obvious embedding result for a regular critical space  $X$ :  $X$  is continuously embedded in  $\tilde{M}^{2,3}$ . The uniqueness problem is then the following one:

**Uniqueness problem:**

Let  $X$  be a regular critical space. If  $\vec{u}$  and  $\vec{v}$  are two weak solutions of the Navier–Stokes equations on  $(0, T^*) \times \mathbb{R}^3$  such that  $\vec{u}$  and  $\vec{v}$  belong to  $\mathcal{C}([0, T^*), X^3)$  and have the same initial value, then do we have  $\vec{u} = \vec{v}$  ?

In order to deal with that problem, we modified the notions of adapted space for the Navier–Stokes equations that were introduced by Cannone [12] Meyer and Muschetti [56] or Auscher and Tchamitchian [1]:

**Definition 3.** (Fully adapted critical space) A fully adapted critical Banach space for the Navier–Stokes equations is a Banach space  $E$  such that we have the continuous embeddings  $\mathcal{D}(\mathbb{R}^3) \subset E \subset L_{loc}^2(\mathbb{R}^3)$  and such that moreover:

- (a) for all  $x_0 \in \mathbb{R}^3$  and for all  $f \in E$ ,  $f(x - x_0) \in E$  and  $\|f\|_E = \|f(x - x_0)\|_E$ .
- (b) for all  $\lambda > 0$  and for all  $f \in E$ ,  $f(\lambda x) \in E$  and  $\lambda \|f(\lambda x)\|_E = \|f\|_E$ .
- (c) The closed unit ball of  $E$  is a metrizable compact subset of  $\mathcal{S}'(\mathbb{R}^3)$ .
- (d)  $e^\Delta$  maps boundedly  $E$  to the space  $\mathcal{M}$  of pointwise multipliers of  $E$
- (e) Let  $F$  be the Banach space

$$F = \left\{ f \in L_{loc}^1 / \exists (f_n), (g_n) \in E^{\mathbb{N}} \text{ s.t. } f = \sum_{n \in \mathbb{N}} f_n g_n \text{ and } \sum_{n \in \mathbb{N}} \|f_n\|_E \|g_n\|_E < \infty \right.$$

(normed with  $\|f\|_F = \min_{f = \sum_{n \in \mathbb{N}} f_n g_n} \sum_{n \in \mathbb{N}} \|f_n\|_E \|g_n\|_E$ ). There exists a Banach space of tempered distributions  $G$  such that

- i)  $e^\Delta$  maps boundedly  $F$  to  $G$
- ii) the real interpolation space  $[F, G]_{1/2, \infty}$  is continuously embedded into  $E$
- iii) for all  $\lambda > 0$  and for all  $f \in G$ ,  $f(\lambda x) \in G$  and  $\|f(\lambda x)\|_G = \|f\|_G$ .

Hypothesis (c) (together with (a)) shows that  $E$  is invariant under convolution with an integrable kernel :

$$\forall f \in E \forall g \in L^1 \quad f * g \in E \text{ and } \|f * g\|_E \leq \|f\|_E \|g\|_1. \quad (10.1)$$

This hypothesis (c) is fulfilled in the case where  $E$  is the dual space of a separable Banach space containing  $\mathcal{S}$  as a dense subspace.

The following proposition shows why those spaces are called adapted to the Navier–Stokes equations :

**Proposition 10.1.** Let  $E$  be a fully adapted critical space and let  $\mathcal{M} = \mathcal{M}(E \mapsto E)$  be the space of pointwise multipliers of  $E$ . For  $T \in (0, +\infty)$ , let  $\mathcal{A}_T$  and  $\mathcal{B}_T$  be the spaces defined by

$$f \in \mathcal{A}_T \Leftrightarrow f \in L_{loc}^2((0, T) \times \mathbb{R}^3), \quad \sup_{0 < t < T} \|f(t, \cdot)\|_E < \infty$$

and

$$f \in \mathcal{B}_T \Leftrightarrow f \in L_{loc}^1((0, T) \times \mathbb{R}^3), \quad \sup_{0 < t < T} t^{1/2} \|f(t, \cdot)\|_{\mathcal{M}} < \infty.$$

Then  $B$  is bounded from  $(\mathcal{A}_T)^3 \times (\mathcal{A}_T)^3$  to  $(\mathcal{A}_T)^3$  and from  $(\mathcal{A}_T)^3 \times (\mathcal{B}_T)^3$  or  $(\mathcal{B}_T)^3 \times (\mathcal{A}_T)^3$  to  $(\mathcal{A}_T)^3$ . More precisely, there exists a constant  $C_E$  such that, for all  $T \in (0, +\infty]$ , all  $\vec{u}_0 \in E^3$ , all  $\vec{f}, \vec{g} \in (\mathcal{A}_T)^3$  and all  $\vec{h} \in (\mathcal{A}_T)^3$  we have

$$\sup_{t>0} \sqrt{t} \|e^{t\Delta} \vec{u}_0\|_{\mathcal{M}} \leq C_E \|\vec{u}_0\|_E \quad (10.2)$$

$$\|B(\vec{f}, \vec{g})\|_{\mathcal{A}_T} \leq C_E \|\vec{f}\|_{\mathcal{A}_T} \|\vec{g}\|_{\mathcal{A}_T} \quad (10.3)$$

and

$$\|B(\vec{f}, \vec{h})\|_{\mathcal{A}_T} + \|B(\vec{h}, \vec{f})\|_{\mathcal{A}_T} \leq C_E \|\vec{f}\|_{\mathcal{A}_T} \|\vec{h}\|_{\mathcal{B}_T}. \quad (10.4)$$

The basic idea in Furioli, Lemarié-Rieusset and Terraneo [26] is to split the solutions in tendency and fluctuation, and to use different estimates on each term. More precisely, we consider two mild solutions  $\vec{u} = e^{t\Delta} \vec{u}_0 - B(\vec{u}, \vec{u}) = e^{t\Delta} \vec{u}_0 - \vec{w}_1$  and  $\vec{v} = e^{t\Delta} \vec{u}_0 - B(\vec{v}, \vec{v}) = e^{t\Delta} \vec{u}_0 - \vec{w}_2$  in  $\mathcal{C}([0, T^*), X^3)$  and write  $\vec{w} = \vec{u} - \vec{v} = \vec{w}_2 - \vec{w}_1 = -B(\vec{w}, \vec{v}) - B(\vec{u}, \vec{w})$ , and finally

$$\vec{w} = B(\vec{w}_1, \vec{w}) + B(\vec{w}, \vec{w}_2) - B(e^{t\Delta} \vec{u}_0, \vec{w}) - B(\vec{w}, e^{t\Delta} \vec{u}_0). \quad (10.5)$$

Combining (10.5) and Proposition 10.1, we easily get the following uniqueness result:

**Theorem 10.2.** *If  $X$  is a regular critical space such that  $X$  is boundedly embedded into a fully adapted critical space  $E$ , then uniqueness holds in  $(\mathcal{C}([0, T^*), X))^3$ .*

**Examples of fully adapted spaces:**

i) the space of Le Jan and Sznitman [40]

$$E = \dot{B}_{PM}^{2,\infty} = \{f \in \mathcal{S}'(\mathbb{R}^3) / \hat{f} \in L_{loc}^1 \text{ and } \xi^2 \hat{f}(\xi) \in L^\infty\}$$

with  $F \subset \dot{B}_{PM}^{1,\infty}$  and  $G = \dot{B}_{PM}^{3,1}$

ii) the homogeneous Besov space

$$E = \dot{B}_p^{3/p-1,\infty} \text{ where } 1 \leq p < 3$$

with  $F \subset \dot{B}_p^{3/p-2,\infty}$  and  $G = \dot{B}_p^{3/p,1}$

iii) the Lorentz space

$$E = L^{3,\infty}$$

with  $F = L^{3/2,\infty}$  and  $G = L^\infty$

iv) the homogeneous Morrey–Campanato spaces based on Lorentz spaces :

$$E = \dot{M}_*^{p,3} \text{ where } 2 < p \leq 3$$

with  $F = \dot{M}_*^{p/2,3/2}$  and  $G = L^\infty$ . The space  $\dot{M}_*^{p,q}(\mathbb{R}^3)$  is defined for  $1 < p \leq q < \infty$  as the space of locally integrable functions  $f$  such that

$$\sup_{x_0 \in \mathbb{R}^3} \sup_{0 < R < \infty} R^{3(1/q-1/p)} \|1_{B(x_0,R)} f\|_{L^{p,\infty}} < \infty;$$

the predual of  $\dot{M}_*^{p,q}(\mathbb{R}^3)$  is then the space of functions  $f$  which may be decomposed as a series  $\sum_{n \in \mathbb{N}} \lambda_n f_n$  with  $f_n$  supported by a ball  $B(x_n, R_n)$  with  $R_n > 0$ ,  $f_n \in L^{p/(p-1),1}$ ,



$\|f_n\|_{L^{p/(p-1),1}} \leq R_n^{3(1/q-1/p)}$  and  $\sum_{n \in \mathbb{N}} |\lambda_n| < \infty$ . All those examples however give

no new information on the uniqueness problem, since we have the embeddings (for  $2 \leq p < 3$  and  $2 < q \leq 3$ )

$$\dot{B}_{PM}^{2,\infty} \subset \dot{B}_p^{3/p-1,\infty} \subset L^{3,\infty} \subset \dot{M}_*^{q,3} \subset \dot{X}_1 \quad (10.6)$$

and thus uniqueness may be dealt with by using May's theorem (Theorem 10.1).

We finish this section with an example of a regular space where uniqueness holds but which cannot be dealt with by using either Theorem 10.1 or Theorem 10.2:

**Theorem 10.3.** *Let  $X$  be defined as the space of locally integrable functions  $f$  such that*

$$\sup_{x_0 \in \mathbb{R}^3} \sup_{0 < R < \infty} R^{-1/2} \|1_{B(x_0,R)} f\|_{L^{2,1}} < \infty \quad (10.7)$$

and let  $\tilde{X}$  be the closure of  $\mathcal{D}$  in  $X$ . Then

- a) Uniqueness holds in  $(\mathcal{C}([0, T^*], \tilde{X}))^3$ .
- b)  $\tilde{X}$  is not included in the multiplier space  $\dot{X}_1 = \mathcal{M}(\dot{H}^1 \mapsto L^2)$
- c) there is no fully adapted critical space  $E$  such that  $\tilde{X} \subset E$ .

## 11 Self-similarity

In 1934, Leray [50] proposed to study backward self-similar solutions to the Navier–Stokes equations in order to try to exhibit solutions that blow up in a finite time. Leray's self-similar solutions on  $(0, T^*) \times \mathbb{R}^3$  are given by  $\vec{u}(t, x) = \lambda(t)\vec{U}(\lambda(t)x)$  with  $\lambda(t) = \frac{1}{\sqrt{2a(T^*-t)}}$  for some positive  $a$ .

In 1996, Nečas, Ružička, and Šverák [60] proved that the only solution  $\vec{U} \in (L^3(\mathbb{R}^3))^3$  was  $\vec{U} = 0$ . Their result was extended by Tsai [70] to the case  $\vec{U} \in (\mathcal{C}_0(\mathbb{R}^3))^3$  or to the case of solutions  $\vec{u}$  with local energy estimates near the blow-up point:  $\sup_{T_0 < t < T^*} \int_{|x| < 1} |\vec{u}(t, x)|^2 dx < \infty$  and  $\int_{T_0}^{T^*} \int_{|x| < 1} |\vec{\nabla} \otimes \vec{u}(t, x)|^2 dx dt < \infty$ . Their proof was based on Hopf's strong maximum principle applied to  $\Pi = \frac{1}{2}|\vec{U}|^2 + P + a\vec{X} \cdot \vec{U}$ , where  $P$  is associated to the pressure  $p(t, x)$  by  $\vec{\nabla} p(t, x) = \lambda(t)^3(\vec{\nabla} P)(\lambda(t)x)$ , and where  $\vec{X}$  be the identical vector field on  $\mathbb{R}^3$ :  $\vec{X}(x) = (x_1, x_2, x_3)$ ; another key ingredient was the regularity criterion of Caffarelli, Kohn, and Nirenberg.

In contrast with backward self-similarity, it is easy to construct forward self-similar solutions. In his book [12], Cannone gave a very clear strategy for exhibiting self-similar solutions to the Navier–Stokes equations. We take a shift-invariant Banach space of distributions  $X$  whose norm is homogeneous with the good scaling for the Navier–Stokes equations ( $\|f(\lambda x)\|_X = \frac{1}{\lambda}\|f\|_X$ ). This implies that  $X \subset \dot{B}_\infty^{-1,\infty}$ , or equivalently that for all  $f \in X$  we have  $\sup_{0 < t} \sqrt{t} \|e^{t\Delta} f\|_\infty \leq C\|f\|_X$ . We assume that  $X$  contains nontrivial homogeneous distributions, and we try to arrive at a theorem of global existence and uniqueness for solutions of the Navier–Stokes equations. Such a theorem will provide us with self-similar solutions when the initial data is homogeneous (with a small norm).

We write  $B(\vec{f}, \vec{g}) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} (\vec{f} \otimes \vec{g}) ds$ , and we check on which Banach space  $\mathcal{E}$  based on  $X$  and containing the tendencies  $e^{t\Delta} \vec{u}_0$  for  $\vec{u}_0 \in X^d$  the bilinear transform  $B$  is continuous. The first space we can try is  $\mathcal{E} = L^\infty((0, \infty), X^d)$ . Of course, the bilinear product  $uv$  should be defined for elements of  $X$ ; hence, we should assume that  $X$  is embedded in  $L^2_{loc}$ ; we then find that  $X \subset \dot{M}^{2,3}$ . There are many instances of spaces  $X$  which can be treated this way and provide self-similar solutions:

\*) homogeneous Besov spaces  $\dot{B}_p^{s,\infty}$  where  $p < 3$  and  $s = 3/p - 1$  (Cannone [12], Chemin [14], Furioli, Lemarié-Rieusset and Terraneo [26])

\*) the Lorentz space  $L^{3,\infty}$  (Meyer [56])

\*) the space  $\dot{B}_{PM}^{2,\infty} = \{f/|\xi|^2 \hat{f} \in L^\infty\}$  used by Le Jan and Sznitman [40].

We may use the smoothing effect of  $e^{t\Delta}$  and start from a more singular initial value. Cannone [12] and Planchon [64] have shown that it is possible to take  $\vec{u}_0 \in \dot{B}_p^{s,\infty}$  where  $p \in (3, \infty)$  and  $s = 3/p - 1$ ; then  $\mathcal{E} = \{\vec{f} / \sup_{0 < t} t^{1/2-3/2p} \|\vec{f}(t, \cdot)\|_p < \infty\}$  is a good choice.

This latter example can even be generalized by replacing  $L^p$  by a Morrey–Campanato space (Kozono and Yamazaki [39]). As a matter of fact, the first instance of self-similar solutions was constructed with help of Morrey–Campanato spaces (Giga and Miyakawa [30]).

Thus, we are interested in homogeneous initial values. In particular, we should know when a distribution is homogeneous. We begin by a simple remark: if  $T$  is a distribution on  $\mathbb{R}^3$ , then the following assertions are equivalent:

(A)  $T$  is homogeneous with degree  $-1$ :

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^3) \quad \forall \lambda > 0 \quad \langle T(x) | \lambda^3 \varphi(\lambda x) \rangle = \lambda \langle T(x) | \varphi(x) \rangle \quad (11.1)$$

(B) There exists  $\omega \in \mathcal{D}'(S^2)$  such that  $T(x) = \omega(\sigma) r^{-1}$  :

$$\langle T(x) | \varphi(x) \rangle_{\mathcal{D}'(\mathbb{R}^3), \mathcal{D}(\mathbb{R}^3)} = \langle \omega(\sigma) | \int_0^\infty \varphi(r\sigma) r dr \rangle_{\mathcal{D}'(S^2), \mathcal{D}(S^2)} \quad (11.2)$$

The distribution  $\omega$  is then unique. We shall write  $\omega = T|_{S^2}$ .

We then have the following trace theorems: let  $T(x) = \omega(\sigma) r^{-1}$ , then:

i) (Besov spaces [12]) : for  $p \in [1, +\infty]$ ,  $T \in \dot{B}_p^{3/p-1,\infty}(\mathbb{R}^3) \Leftrightarrow \omega \in B_p^{3/p-1,p}(S^2)$ .

ii) (Lorentz space [3]) :  $T \in L^{3,\infty}(\mathbb{R}^3) \Leftrightarrow \omega \in L^3(S^2)$

iii) (Morrey spaces [45]) : if  $2 \leq p < 3$ ,  $T \in \dot{M}^{p,3}(\mathbb{R}^3) \Leftrightarrow \omega \in L^p(S^2)$

iv) (Morrey spaces [45]) : if  $1 \leq p < 2$ ,  $T \in \dot{M}^{p,3}(\mathbb{R}^3) \Leftrightarrow \omega \in M^{p,2}(S^2)$

Whereas small homogeneous initial value provide us with self-similar solutions, the problem remains open for large initial value. The case of a large initial value in  $\dot{M}^{2,3}$  has been discussed in [46] and [41]. Using the energy method in  $L^2_{uloc}$  and a scaling argument, one may exhibit global suitable solutions (see Theorem 6). But, due to the possible lack of uniqueness, we don't know whether we may find large self-similar solutions. If a large self-similar suitable solution exist, it is smooth for positive times, due to the Caffareli, Kohn and Nirenberg regularity criterion [31], [41].

## 12 Euler equations

We now pay a few words to the resolution of Euler equations (1.12). In [47], we solved equations (1.12) in an abstract space  $A^{1+\sigma}$ .  $A^{1+\sigma}$  belongs to a scale of Banach spaces

$A^s$  (where  $s > 0$  stands for a regularity index) which satisfies the following hypotheses:

◇ **Hypothesis (H1) : integrability**

$A^s \subset L^1_{\text{loc}}(\mathbb{R}^3)$  (continuous embedding)

◇ **Hypothesis (H2) : monotony**

For  $s_1 < s_2$ ,  $A^{s_2} \subset A^{s_1}$

◇ **Hypothesis (H3) : regularity**

$f \in A^{1+s} \Leftrightarrow f \in A^s$  and  $\vec{\nabla} f \in A^s$  (with equivalence of the norms  $\|f\|_{A^{s+1}}$  and  $\|f\|_{A^s} + \|\vec{\nabla} f\|_{A^s}$ )

◇ **Hypothesis (H4) : stability**

If a sequence  $(f_n)_{n \in \mathbb{N}}$  is bounded in  $A^s$  and converges in  $\mathcal{D}'(\mathbb{R}^3)$  then the limit belongs to  $A^s$  and we have  $\|\lim_{n \rightarrow +\infty} f_n\|_{A^s} \leq C_s \liminf_{n \rightarrow +\infty} \|f_n\|_{A^s}$ .

◇ **Hypothesis (H5) : invariance**

The map  $(f, g) \in \mathcal{D} \times A^s \mapsto f * g$  extends to a bounded bilinear operator from  $L^1 \times A^s$  to  $A^s$ .

◇ **Hypothesis (H6) : interpolation**

If  $T$  is a linear operator which is bounded from  $A^{s_1}$  to  $A^{s_1}$  and from  $A^{s_2}$  to  $A^{s_2}$  then it is bounded from  $A^s$  to  $A^s$  for every  $s \in [s_1, s_2]$  and  $\|T\|_{\mathcal{L}(A^s, A^s)} \leq C(s, s_1, s_2) \max(\|T\|_{\mathcal{L}(A^{s_1}, A^{s_1})}, \|T\|_{\mathcal{L}(A^{s_2}, A^{s_2})})$ .

◇ **Hypothesis (H7) : transport by Lipschitz flows**

Let  $\vec{u} \in L^1((0, T), \mathbf{Lip})$  be a divergence-free vector field and let  $f_0 \in A^s$  for some  $s \in (0, 1)$ . Then the solution  $f \in \mathcal{C}([0, T], L^1_{\text{loc}})$  of the transport equation

$$\begin{cases} \partial_t f + \vec{u} \cdot \vec{\nabla} f = 0 \\ f|_{t=0} = f_0 \end{cases} \quad (12.1)$$

satisfies  $\sup_{0 \leq t \leq T} \|f(t, \cdot)\|_{A^s} \leq C_s e^{C_s \int_0^T \|\vec{u}\|_{\mathbf{Lip}} dt} \|f_0\|_{A^s}$ .

◇ **Hypothesis (H8) : singular integrals**

Let  $T$  be a bounded linear operator from  $\mathcal{D}(\mathbb{R}^3)$  to  $\mathcal{D}'(\mathbb{R}^3)$  (with distribution kernel  $K(x, y) \in \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}^3)$ ) which satisfies the following conditions

- $T$  is bounded on  $L^2$  :  $\|T(f)\|_2 \leq C_0 \|f\|_2$
- outside from the diagonal  $x = y$ ,  $K$  is a continuous function such that

$$|K(x, y)| \leq C_0 \frac{1}{|x-y|^3(1+|x-y|)}$$

- outside from the diagonal,  $K$  satisfies  $|\vec{\nabla}_x K(x, y)| \leq C_0 |x-y|^{-4}$  and  $|\vec{\nabla}_y K(x, y)| \leq C_0 |x-y|^{-4}$

- $T(1) = T^*(1) = 0$  in  $BMO$

Then,  $T$  is bounded from  $A^s$  to  $A^s$  for all  $0 < s < 1$  and  $\|T\|_{\mathcal{L}(A^s, A^s)} \leq C_s C_0$

We further consider an hypothesis on some  $\sigma > 0$ :

◇ **Hypothesis (H9): pointwise products with  $A^\sigma$**

$A^\sigma \subset L^\infty$  (continuous embedding) and, for all  $s \in (0, \sigma]$ , the product  $(f, g) \mapsto fg$  is a bounded bilinear operator from  $A^\sigma \times A^s$  to  $A^s$ .

We then have the following theorem on the Cauchy problem for the Euler equations with initial data in  $A^{1+\sigma}$  :

**Theorem 12.1.** *Let  $A^s$  be a scale of spaces satisfying hypotheses (H1) to (H8) and let  $\sigma > 0$  satisfy hypothesis (H9). Let  $\vec{v}_0 \in A^{1+\sigma}$  be a divergence free vector field. Then*

there exists a positive  $T$  such that the Cauchy problem

$$\begin{cases} \partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} = \sum_{i=1}^3 [v_i, \mathbb{P} \partial_i] \vec{v} \\ \vec{v}|_{t=0} = \vec{v}_0 \\ \operatorname{div} \vec{v} = 0 \end{cases} \quad (12.2)$$

has a unique solution  $\vec{v} \in \mathcal{C}([0, T], A^\sigma)$  such that  $\sup_{0 \leq t \leq T} \|\vec{v}\|_{A^{\sigma+1}} < +\infty$ .

### Examples.

\*) Besov spaces [13]:  $A^{\sigma+1} = B_{p,q}^{1+\sigma}$  with  $1 \leq p \leq +\infty$ ,  $\sigma > 3/p$  and  $1 \leq q \leq +\infty$ , or with  $1 \leq p < +\infty$ ,  $\sigma = 3/p$  and  $q = 1$ . (The case of  $p = +\infty$ ,  $\sigma = 0$  and  $q = 1$  [63] is not covered by Theorem 12.1). [Work in the scale  $B_{p,q}^s$  for  $0 < s \leq 1 + \sigma$ ].

\*) Triebel–Lizorkin spaces [16], [47]:  $A^{\sigma+1} = F_{p,q}^{1+\sigma}$  with  $1 \leq p, q < +\infty$  and  $\sigma > 3/p$ . [Work in the scale  $F_{p,q}^s$  for  $0 < s \leq 1 + \sigma$ ].

\*) Besov-Lorentz spaces:  $A^{\sigma+1} = B_{L^{p,q,r}}^{\sigma+1}$  with  $1 < p < +\infty$ ,  $1 \leq q \leq +\infty$ ,  $\sigma > 3/p$  and  $1 \leq r \leq +\infty$  (or  $\sigma = 3/p$  and  $r = 1$ ). [Work in the scale  $B_{L^{p,q,r}}^s$  for  $0 < s \leq 1 + \sigma$ ]. The case  $r = +\infty$  was discussed in [67].

\*) Besov-Morrey spaces:  $A^{\sigma+1} = B_{\dot{M}^{p,q,r}}^{\sigma+1}$  with  $1 < p \leq q < +\infty$ ,  $\sigma > 3/q$  and  $1 \leq r \leq +\infty$  (or  $\sigma = 3/q$  and  $r = 1$ ). [Work in the scale  $B_{\dot{M}^{p,q,r}}^s$  for  $0 < s \leq 1 + \sigma$ ]. Some other spaces are discussed [47], such as Sobolev-Morrey spaces.

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