

RECENT PROGRESS IN STUDYING THE BOUNDEDNESS  
OF CLASSICAL OPERATORS OF REAL ANALYSIS  
IN GENERAL MORREY-TYPE SPACES. I

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**Abstract.** The survey is aimed at providing detailed information about recent results in the problem of the boundedness in general Morrey-type spaces of various important operators of real analysis, namely of the maximal operator, fractional maximal operator, Riesz potential, singular integral operator, Hardy operator. The main focus is on the results which contain, for a certain range of the numerical parameters, necessary and sufficient conditions on the functional parameters characterizing general Morrey-type spaces, ensuring the boundedness of the aforementioned operators from one general Morrey-type space to another one. The major part of the survey is dedicated to the results obtained by the author jointly with his co-authores A. Gogatishvili, M.L. Goldman, H.V. Guliyev, V.S. Guliyev, P. Jain, R. Mustafaev, E.D. Nursultanov, R. Oinarov, A. Serbetci, T.V. Tararykova. Part I of the survey contains discussion of the definition and basic properties of the local and global general Morrey-type spaces, of embedding theorems, and of the boundedness properties of the maximal operator. Part II of the survey will contain discussion of boundedness properties of the fractional maximal operator, Riesz potential, singular integral operator, commutators of singular integral operator, Hardy operator. It will also contain discussion of interpolation theorems, of methods of proofs and of open problems.

## 1 Introduction

The theory of the boundedness of classical operators of real analysis, such as maximal operator, fractional maximal operator, Riesz potential, singular integral operator etc, from one weighted Lebesgue space to another one is by now well studied. For the overwhelming majority of the values of the numerical parameters necessary and sufficient conditions on the weight functions ensuring boundedness have been found.

These results have good applications in real analysis and in the theory of partial differential equations. In these areas, alongside with weighted Lebesgue spaces, general

Morrey-type spaces also play an important role.<sup>1</sup> However, until recently there were no results, containing necessary and sufficient conditions on the weight functions ensuring boundedness of the aforementioned operators from one general Morrey-type space to another one (apart from the cases in which this follows directly from the appropriate results for weighted Lebesgue spaces). The case of power-type weights was well studied, but for general Morrey-type spaces only sufficient conditions were known.

In the last several years necessary and sufficient conditions for the case of general Morrey-type spaces have been found, but for a comparatively restricted range of the numerical parameters.

In this area there are many open questions which may be of particular interest to experts in studying such problems for weighted Lebesgue spaces.

In this survey results on the boundedness of the aforementioned operators will be given, with emphasis on the results containing necessary and sufficient conditions for boundedness of these operators, and open problems will be discussed in detail.

Part I of the survey contains discussion of the definition and basic properties of the local and global general Morrey-type spaces, of embedding theorems, and of the boundedness properties of the maximal operator.

## 2 Morrey spaces

We shall use the following notation. For a Lebesgue measurable set  $G \subset \mathbb{R}^n$  and  $0 < p \leq \infty$ ,  $L_p(G)$  is the standard Lebesgue space of all functions  $f$  Lebesgue measurable on  $G$  for which

$$\|f\|_{L_p(G)} = \left( \int_G |f(y)|^p dy \right)^{\frac{1}{p}} < \infty$$

if  $0 < p < \infty$  and

$$\|f\|_{L_\infty(G)} = \operatorname{ess\,sup}_{x \in G} |f(x)| < \infty$$

if  $p = \infty$ . Also, for an open set  $G \subset \mathbb{R}^n$ ,  $L_p^{\operatorname{loc}}(G)$  is the set of all functions  $f$  such that  $f \in L_p(K)$  for any compact  $K \subset G$ . If  $G = \mathbb{R}^n$  then, for brevity, we write  $L_p$  for  $L_p(\mathbb{R}^n)$  and  $L_p^{\operatorname{loc}}$  for  $L_p^{\operatorname{loc}}(\mathbb{R}^n)$ .

The same convention refers to the case of weak Lebesgue spaces  $WL_p(G)$ , the space of all functions  $f$  Lebesgue measurable on  $G$  for which

$$\|f\|_{WL_p(G)} = \sup_{0 < t \leq |G|} t^{\frac{1}{p}} f^*(t) < \infty.$$

Here  $|G|$  is the Lebesgue measure of  $G$ , and  $f^*$  denotes the non-increasing rearrangement of  $f$ :

$$f^*(t) = \inf\{\tau : \lambda_f(\tau) \leq t\}, \quad t > 0,$$

where  $\lambda_f(\tau) = |\{x \in G : |f(x)| > \tau\}|$ ,  $\tau > 0$  is the distribution function of the function  $f$ .

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<sup>1</sup> Some of such applications are discussed in detail in the survey papers by V.S. Guliyev [21], P.G. Lemarié-Rieusset [26], M.A. Ragusa [34], and W. Sickel [36] published in this issue of the Eurasian Mathematical Journal.

Morrey spaces  $M_p^\lambda$ , named after C. Morrey, were introduced by him in 1938 in [30] and defined as follows: For  $\lambda \in \mathbb{R}$ ,  $0 < p \leq \infty$ ,  $f \in M_p^\lambda$  if  $f \in L_p^{\text{loc}}$  and

$$\|f\|_{M_p^\lambda} \equiv \|f\|_{M_p^\lambda(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\lambda} \|f\|_{L_p(B(x,r))} < \infty,$$

where  $B(x, r)$  is the open ball in  $\mathbb{R}^n$  centered at the point  $x \in \mathbb{R}^n$  of radius  $r > 0$ .

Here the notation is slightly altered compared with the original definition in [30], namely we write  $r^{-\lambda}$  rather than  $r^{-\frac{\lambda}{p}}$  for the reasons which will be clarified in Section 3. Also in [30]  $p \in [1, \infty]$ , but there is no problem in extending the range of this parameter to  $(0, \infty]$ .

In other words  $f \in M_p^\lambda$  if  $f \in L_p^{\text{loc}}(\mathbb{R}^n)$  and there exists  $c > 0$  (depending on  $f$ ) such that for all  $x \in \mathbb{R}^n$  and for all  $r > 0$

$$\|f\|_{L_p(B(x,r))} \leq c r^\lambda.$$

The minimal value of  $c$  in this inequality is  $\|f\|_{M_p^\lambda}$ .

If  $\lambda = 0$ , then

$$M_p^0 = L_p.$$

If  $\lambda = \frac{n}{p}$ , then

$$M_p^{\frac{n}{p}} = L_\infty.$$

If  $\lambda > \frac{n}{p}$  or  $\lambda < 0$ , then

$$M_p^\lambda = \Theta,$$

where  $\Theta \equiv \Theta(\mathbb{R}^n)$  is the set of all functions equivalent to 0 on  $\mathbb{R}^n$ .

So the admissible range of the parameters is

$$0 < p \leq \infty \quad \text{and} \quad 0 \leq \lambda \leq \frac{n}{p}. \tag{2.1}$$

(If  $p = \infty$  then the inequality for  $\lambda$  holds only if  $\lambda = 0$  and  $M_\infty^0 = L_\infty$ .)

Under these assumptions, which will always be assumed in the sequel, the space  $M_p^\lambda$  is a Banach space for  $1 \leq p \leq \infty$  and a quasi-Banach space for  $0 < p < 1$ .

Also the space  $M_p^\lambda$  does not coincide with a Lebesgue space, if and only if

$$0 < p < \infty \quad \text{and} \quad 0 < \lambda < \frac{n}{p}. \tag{2.2}$$

Furthermore,

$$L_\infty \cap L_p \subset M_p^\lambda.$$

If  $f \in L_p$ , then  $f \in M_p^\lambda$  if and only if  $\sup_{x \in \mathbb{R}^n, 0 < r \leq 1} r^{-\lambda} \|f\|_{L_p(B(x,r))} < \infty$ , hence under this assumption only local properties of  $f$  are of importance.

**Example 1.** If  $\alpha \in \mathbb{R}$  and conditions (2.2) are satisfied, then

$$|x|^\alpha \in M_p^\lambda \iff \alpha = \lambda - \frac{n}{p},$$

$$|x|^\alpha \chi_{B(0,1)}(x) \in M_p^\lambda \iff \alpha \geq \lambda - \frac{n}{p},$$

and

$$|x|^\alpha \chi_{\mathbb{R}^n \setminus B(0,1)}(x) \in M_p^\lambda \iff \alpha \leq \lambda - \frac{n}{p}.$$

Here  $\chi_G$  is the characteristic function of the set  $G \subset \mathbb{R}^n$  and  ${}^cG$  is the complement of the set  $G$ .

**Example 2.** If  $\alpha, \beta \in \mathbb{R}$  and conditions (2.2) are satisfied, then

$$f_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha & \text{if } |x| \leq 1, \\ |x|^\beta & \text{if } |x| > 1 \end{cases} \in M_p^\lambda$$

if and only if

$$\beta \leq \alpha \quad \text{and} \quad \frac{n}{p} + \beta \leq \lambda \leq \frac{n}{p} + \alpha.$$

Sometimes it is more useful to consider the local variant of Morrey spaces, namely the space of functions  $f \in L_p^{\text{loc}}(\mathbb{R}^n)$  which are such that

$$\sup_{r>0} r^{-\lambda} \|f\|_{L_p(B(x,r))} < \infty.$$

for a fixed <sup>2</sup>  $x \in \mathbb{R}^n$ , in which case the behaviour of the expression  $\|f\|_{L_p(B(x,r))}$  is important only in a neighbourhood of the point  $x$ , in contrast to the case of standard (global) Morrey space  $M_p^\lambda$  when the uniform in  $x \in \mathbb{R}^n$  behaviour of the expressions  $\|f\|_{L_p(B(x,r))}$  is assumed.

Also by  $WM_p^\lambda$  we denote the weak Morrey space, the space the space of all functions  $f \in WL_p^{\text{loc}}$  with finite quasi-norm

$$\|f\|_{WM_p^\lambda} \equiv \|f\|_{WM_p^\lambda(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r>0} r^{-\lambda} \|f\|_{WL_p(B(x,r))}.$$

### 3 Comparison of Morrey spaces and spaces of smooth functions

Consider the Nikol'skii space <sup>3</sup>  $H_p^\lambda \equiv H_p^\lambda(\mathbb{R}^n)$  of functions possessing ‘‘common smoothness of order  $\lambda$  measured in the  $L_p$  metrics’’. For  $\lambda > 0, 1 \leq p \leq \infty$  they are defined in the following way:  $f \in H_p^\lambda$  if  $f \in L_p$  and

$$\|f\|_{H_p^\lambda} = \|f\|_{L_p} + \sup_{h \in \mathbb{R}^n, h \neq 0} |h|^{-\lambda} \|\Delta_h^\sigma f\|_{L_p} < \infty,$$

<sup>2</sup> Usually it suffices to consider  $x = 0$ .

<sup>3</sup> Detailed exposition of properties of these spaces can be found in [32], [2].

where  $\Delta_h^\sigma f$  is the difference of  $f$  of order  $\sigma \in \mathbb{N}$  with step  $h$  and  $\sigma > \lambda$ . (For different  $\sigma > \lambda$  the definitions are equivalent.) One can prove that if  $0 < \lambda < \frac{n}{p}$ , then

$$H_p^\lambda \subset M_p^\lambda.$$

(For  $n = 1$  see [24], for  $n > 1$  [31], [32].)

Clearly the converse inclusion does not hold, because if  $f \in M_p^\lambda$ , then clearly  $fg \in M_p^\lambda$  for any bounded measurable function  $g$ , which is not true for the case of the spaces  $H_p^\lambda$ .

So,  $M_p^\lambda$  is not a space of functions possessing any kind of common smoothness of order  $\lambda$ , but the expressions  $\|f\|_{L_p(B(x,r))}$  behave like the ones for functions  $f$  possessing certain smoothness of order  $\lambda$ .

**Example 3.** Let  $\eta \in C_0^\infty(\mathbb{R}^n)$  be such that  $\eta(x) = 1$  if  $|x| \leq 1$ . Then

$$|x|^\alpha \eta(x) \in H_p^\lambda \iff |x|^\alpha \eta(x) \in M_p^\lambda \iff \alpha \geq \lambda - \frac{n}{p}.$$

**Remark 1.** It appears that, in many situations in real analysis and especially in applications to the theory of partial differential equations, of primary importance is the behaviour of the expressions  $\|f\|_{L_p(B(x,r))}$  rather than smoothness properties of  $f$ . In such cases the usage of Morrey spaces is natural and effective.

## 4 Morrey-type spaces

**Definition 1.** Let  $0 < p, \theta \leq \infty$  and let  $w$  be a non-negative Lebesgue measurable function on  $(0, \infty)$ . We denote by  $LM_{p\theta, w(\cdot)} \equiv LM_{p\theta, w(\cdot)}(\mathbb{R}^n)$  the local Morrey-type space, the space of all functions  $f$  Lebesgue measurable on  $\mathbb{R}^n$  with finite quasi-norm

$$\|f\|_{LM_{p\theta, w(\cdot)}} = \left\| w(r) \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0, \infty)},$$

and by  $WLM_{p\theta, w(\cdot)} \equiv WLM_{p\theta, w(\cdot)}(\mathbb{R}^n)$  the weak local Morrey-type space, the space of all functions  $f$  Lebesgue measurable on  $\mathbb{R}^n$  with finite quasi-norm

$$\|f\|_{WLM_{p\theta, w(\cdot)}} = \left\| w(r) \|f\|_{WL_p(B(0,r))} \right\|_{L_\theta(0, \infty)}.$$

Furthermore, we denote by  $GM_{p\theta, w(\cdot)} \equiv GM_{p\theta, w(\cdot)}(\mathbb{R}^n)$  the global Morrey-type space, the space of all functions  $f$  Lebesgue measurable on  $\mathbb{R}^n$  with finite quasi-norm

$$\|f\|_{GM_{p\theta, w(\cdot)}} = \sup_{x \in \mathbb{R}^n} \|f(x + \cdot)\|_{LM_{p\theta, w(\cdot)}} = \sup_{x \in \mathbb{R}^n} \left\| w(r) \|f\|_{L_p(B(x,r))} \right\|_{L_\theta(0, \infty)},$$

and by  $WGM_{p\theta, w(\cdot)} \equiv WGM_{p\theta, w(\cdot)}(\mathbb{R}^n)$  the weak global Morrey-type space, the space of all functions  $f$  Lebesgue measurable on  $\mathbb{R}^n$  with finite quasi-norm

$$\|f\|_{WGM_{p\theta, w(\cdot)}} = \sup_{x \in \mathbb{R}^n} \|f(x + \cdot)\|_{WLM_{p\theta, w(\cdot)}} = \sup_{x \in \mathbb{R}^n} \left\| w(r) \|f\|_{WL_p(B(x,r))} \right\|_{L_\theta(0, \infty)}.$$

**Remark 2.** The spaces  $LM_{p\theta,w(\cdot)}$ ,  $GM_{p\theta,w(\cdot)}$  are mostly aimed at describing the behaviour of  $\|f\|_{L_p(B(0,r))}$ ,  $\|f\|_{L_p(B(x,r))}$  respectively, for small  $r > 0$  in a very general setting.

Note that if  $w(r) \equiv 1$ , then  $LM_{p\infty,1} = GM_{p\infty,1} = L_p$ . Furthermore,

$$GM_{p\infty,r^{-\lambda}} \equiv M_p^\lambda, \quad 0 < p \leq \infty, \quad 0 \leq \lambda \leq \frac{n}{p}.$$

**Definition 2.** Let  $0 < p, \theta \leq \infty$ . We denote by  $\Omega_\theta$  the set of all functions  $w$  which are non-negative, Lebesgue measurable on  $(0, \infty)$ , not equivalent to 0, and such that for some  $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty.$$

Moreover, we denote by  $\Omega_{p\theta}$ , the set of all functions  $w$  which are non-negative, Lebesgue measurable on  $(0, \infty)$ , not equivalent to 0, and such that for all  $t > 0$

$$\|w(r)r^{n/p}\|_{L_\theta(0,t)} < \infty, \quad \|w(r)\|_{L_\theta(t,\infty)} < \infty,$$

or, which is equivalent,

$$\left\| w_2(r) \left( \frac{r}{t+r} \right)^{\frac{n}{p}} \right\|_{L_{\theta_2}(0,\infty)} < \infty$$

for all  $t > 0$ .

**Lemma 4.1.** ([9], [12]) *Let  $0 < p, \theta \leq \infty$  and let  $w$  be a non-negative Lebesgue measurable function on  $(0, \infty)$ , which is not equivalent to 0.*

*Then the space  $LM_{p\theta,w(\cdot)}$  is non-trivial, in the sense that  $LM_{p\theta,w(\cdot)} \neq \Theta$ , if and only if  $w \in \Omega_\theta$ , and the space  $GM_{p\theta,w(\cdot)}$  is non-trivial if and only if  $w \in \Omega_{p\theta}$ .*

*Moreover, if  $w \in \Omega_\theta$  and  $\tau = \inf\{s > 0 : \|w\|_{L_\theta(s,\infty)} < \infty\}$ , then the space  $LM_{p\theta,w(\cdot)}$  contains all functions  $f \in L_p$  such that  $f = 0$  on  $B(0,t)$  for some  $t > \tau$ . If  $w \in \Omega_{p\theta}$ , then*

$$L_p \cap L_\infty \subset GM_{p\theta,w(\cdot)}.$$

**Remark 3.** Keeping in mind this statement it will always be assumed that  $w \in \Omega_\theta$  for the case of local Morrey-type spaces and that  $w \in \Omega_{p\theta}$  for the case of global Morrey-type spaces.

**Remark 4.** Let  $LM_{p\theta,w(\cdot)}(x)$ , where  $x \in \mathbb{R}^n$ , denote the space of all functions Lebesgue measurable on  $\mathbb{R}^n$  with finite quasi-norm

$$\|f\|_{LM_{p\theta,w(\cdot)}(x)} = \left\| w(r) \|f\|_{L_p(B(x,r))} \right\|_{L_\theta(0,\infty)}.$$

Most of the operators  $A$  considered in the survey, though not all of them, possess the property

$$(A(f(\cdot + h)))(x) = (Af)(x + h), \quad x, h \in \mathbb{R}^n. \quad (4.1)$$

For such operators the boundedness from  $LM_{p_1\theta_1, w_1(\cdot)}$  to  $LM_{p_2\theta_2, w_2(\cdot)}$  implies the boundedness from  $LM_{p_1\theta_1, w_1(\cdot)}(x)$  to  $LM_{p_2\theta_2, w_2(\cdot)}(x)$  for any  $x \in \mathbb{R}^n$ . Moreover,

$$\begin{aligned} \|A\|_{LM_{p_1\theta_1, w_1(\cdot)}(x) \rightarrow LM_{p_2\theta_2, w_2(\cdot)}(x)} &= \|A\|_{LM_{p_1\theta_1, w_1(\cdot)}(0) \rightarrow LM_{p_2\theta_2, w_2(\cdot)}(0)} \\ &\equiv \|A\|_{LM_{p_1\theta_1, w_1(\cdot)} \rightarrow LM_{p_2\theta_2, w_2(\cdot)}}, \end{aligned}$$

because

$$\|f\|_{LM_{p\theta, w(\cdot)}(x)} = \|f(x + \cdot)\|_{LM_{p\theta, w(\cdot)}(0)} \equiv \|f(x + \cdot)\|_{LM_{p\theta, w(\cdot)}}.$$

Hence it also implies the boundedness from  $GM_{p_1\theta_1, w_1(\cdot)}$  to  $GM_{p_2\theta_2, w_2(\cdot)}$  and

$$\|A\|_{GM_{p_1\theta_1, w_1(\cdot)} \rightarrow GM_{p_2\theta_2, w_2(\cdot)}} \leq \|A\|_{LM_{p_1\theta_1, w_1(\cdot)} \rightarrow LM_{p_2\theta_2, w_2(\cdot)}}.$$

However, it should be kept in mind that necessary and sufficient conditions on  $w_1$  and  $w_2$  ensuring the boundedness of  $A$  from  $LM_{p_1\theta_1, w_1(\cdot)}$  to  $LM_{p_2\theta_2, w_2(\cdot)}$  imply, in general, only sufficient conditions for the boundedness of  $A$  from  $GM_{p_1\theta_1, w_1(\cdot)}$  to  $GM_{p_2\theta_2, w_2(\cdot)}$ , and the problem of finding necessary and sufficient conditions on  $w_1$  and  $w_2$  ensuring the boundedness of  $A$  from  $GM_{p_1\theta_1, w_1(\cdot)}$  to  $GM_{p_2\theta_2, w_2(\cdot)}$  requires further investigation.

**Remark 5.** Let  $0 < \|w\|_{L_\theta(t, \infty)} < \infty$  for all  $t > 0$  and  $x \in \mathbb{R}^n$ . The fact that  $f \in LM_{p\theta, w(\cdot)}(x)$  provides certain information about the behaviour of  $f(y)$  for all  $y \in \mathbb{R}^n$ . In particular,  $f \in L_p^{loc}$ . However, the most important is the information about the behaviour of  $f(y)$  in a neighbourhood of the point  $x$ . For example, if  $f \in L_p$ , then  $\|w(r)\|f\|_{L_p(B(x, r))}\|_{L_\theta(t, \infty)} < \infty$  for any  $t > 0$ , and the fact that  $f \in LM_{p\theta, w(\cdot)}(x) \cap L_p$  describes the behaviour of the quasi-norms  $\|f\|_{L_p(B(x, r))}$  for small  $r > 0$  and hence completely depends on the behaviour of  $f(y)$  in a neighbourhood of  $x$ . For this reason the term *local* Morrey-type space is used for the space  $LM_{p\theta, w(\cdot)}(x)$ . If  $f$  belongs to the *global* Morrey-type space  $GM_{p\theta, w(\cdot)}$ , then the situation is different: this implies the uniform in  $x \in \mathbb{R}^n$  behaviour of the quasi-norms  $\|f\|_{L_p(B(x, r))}$ .

**Remark 6.** Let  $x \in \mathbb{R}^n$  be fixed and an operator  $A$  be bounded from  $LM_{p_1\theta_1, w_1(\cdot)}(x)$  to  $LM_{p_2\theta_2, w_2(\cdot)}(x)$ . This provides information about the behaviour of the quasi-norms  $\|Af\|_{L_{p_2}(B(x, r))}$  for small  $r > 0$  for  $f \in LM_{p_1\theta_1, w_1(\cdot)}(x)$ , hence by using the information about behaviour of the quasi-norms  $\|f\|_{L_{p_1}(B(x, r))}$  for small  $r > 0$  and some global information about  $f$ . If  $A$  is bounded from  $GM_{p_1\theta_1, w_1(\cdot)}$  to  $GM_{p_2\theta_2, w_2(\cdot)}$ , and one is interested in the behaviour of the quasi-norms  $\|Af\|_{L_{p_2}(B(x, r))}$  for small  $r > 0$ , then this information is obtained by using stronger assumption  $f \in GM_{p_1\theta_1, w_1(\cdot)}$  which, in particular, contains the information about the behaviour of  $\|f\|_{L_{p_1}(B(y, r))}$  not only for  $y = x$  but for all  $y \in \mathbb{R}^n$ . (In this case one, of course, gets more information, namely the information about the behaviour of  $\|Af\|_{L_{p_2}(B(y, r))}$  for all  $y \in \mathbb{R}^n$ .)

In applications to partial differential equations this means that the usage of local Morrey-type spaces can better describe local behaviour of solutions  $u$  to partial differential equations: the behaviour of the quasi-norms  $\|u\|_{L_p(B(x, r))}$  may be derived using the fact that the right-hand side  $f$  belongs to the local Morrey-type space  $LM_{p\theta, w(\cdot)}(x)$ , hence by using only the properties of  $\|f\|_{L_p(B(x, r))}$  for sufficiently small  $r > 0$  and certain global information about  $f$ , rather than using the fact that the right-hand side  $f$  belongs to the global Morrey-type space  $GM_{p\theta, w(\cdot)}$ , hence assuming uniform in  $y$  behaviour of  $\|f\|_{L_p(B(y, r))}$ .

**Remark 7.** Both variants of the boundedness of an operator, from a local Morrey-type space to another local Morrey-type space and from a global Morrey-type space to another global Morrey-type space, are of interest. However, for the reasons explained above, of primary interest is the boundedness from a local Morrey-type space to another local Morrey-type space. Moreover, by Remark 4 for a certain class of operators this implies the boundedness from a global Morrey-type space to another global Morrey-type space, whilst the converse, in general, is not true.

We shall also use the notation  $LM_{p\theta}^\lambda$ ,  $GM_{p\theta}^\lambda$  respectively, for the particular case in which  $w(r) = r^{-\lambda - \frac{1}{\theta}}$ . In this case

$$\|f\|_{LM_{p\theta}^\lambda} = \left( \int_0^\infty \left( \frac{\|f\|_{L_p(B(0,r))}}{r^\lambda} \right)^\theta \frac{dr}{r} \right)^{\frac{1}{\theta}} < \infty.$$

By Lemma 4.1 the space  $LM_{p\theta}^\lambda$  is non-trivial if and only if  $\lambda > 0$  for  $\theta < \infty$  and  $\lambda \geq 0$  for  $\theta = \infty$ , and the space  $GM_{p\theta}^\lambda$  is non-trivial if and only if  $0 < \lambda < \frac{n}{p}$  for  $\theta < \infty$  and  $0 \leq \lambda \leq \frac{n}{p}$  for  $\theta = \infty$ .

Note that the expression for  $\|f\|_{LM_{p\theta}^\lambda}$  is very similar to the semi-norms  $\|f\|_{b_{p\theta}^\lambda}$  of the Nikol'skii – Besov spaces  $B_{p\theta}^\lambda$ . In the latter case  $\lambda > 0$ ,  $1 \leq p, \theta \leq \infty$  and  $\|f\|_{L_p(B(0,r))}$  should be replaced by the  $L_p$  modulus of continuity  $\omega^\sigma(f, r) = \sup_{|h| \leq r} \|\Delta_h^\sigma f\|_{L_p(\mathbb{R}^n)}$  with  $\sigma > \lambda$ . Recall that  $\|f\|_{B_{p\theta}^\lambda} = \|f\|_{L_p} + \|f\|_{b_{p\theta}^\lambda}$ . If  $\theta = \infty$  then  $B_{p\infty}^\lambda \equiv H_p^\lambda$ . There are several definitions, equivalent for these values of the parameters, of the spaces  $B_{p\theta}^\lambda$ . The definition mentioned above makes sense for a wider range of the parameters, namely for  $\lambda > 0, 0 < p, \theta \leq \infty$ . For this range of the parameters the equivalence of the quasi-norms  $\|\cdot\|_{B_{p\theta}^\lambda}$  for different  $\sigma > \lambda$  was proved in [13].

If  $\theta = p$  then

$$\|f\|_{LM_{pp}^\lambda} = (\lambda p)^{-\frac{1}{p}} \left( \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^{\lambda p}} dx \right)^{\frac{1}{p}}. \quad (4.2)$$

For  $n = 1, 1 \leq p, \theta < \infty, 0 < \lambda < \frac{1}{p}$  the inclusion

$$B_{p\theta}^\lambda \subset GM_{p\theta}^\lambda$$

was proved by Yu.V. Kuznetsov [25]. In the case  $p = \theta$  it follows by equality (4.2) and the estimate of the right-hand side of (4.2) via  $\|f\|_{b_{pp}^\lambda}$  for functions  $f \in B_{pp}^\lambda$ , proved by G.N. Yakovlev [39], [40]. (See also [18].) Further results in this direction can be found in [4].

The spaces  $LM_{p\theta}^\lambda, GM_{p\theta}^\lambda$  and  $b_{p\theta}^\lambda$  behave similarly with respect to dilations  $\tau_\varepsilon$  ( $(\tau_\varepsilon f)(x) = f(\varepsilon x), x \in \mathbb{R}^n$ ):

$$\|\tau_\varepsilon f\|_{LM_{p\theta}^\lambda} = \varepsilon^{\lambda - \frac{n}{p}} \|f\|_{LM_{p\theta}^\lambda}, \quad \|\tau_\varepsilon f\|_{GM_{p\theta}^\lambda} = \varepsilon^{\lambda - \frac{n}{p}} \|f\|_{GM_{p\theta}^\lambda}$$

and

$$\|\tau_\varepsilon f\|_{b_{p\theta}^\lambda} = \varepsilon^{\lambda - \frac{n}{p}} \|f\|_{b_{p\theta}^\lambda}.$$



Moreover,

$$\|\tau_\varepsilon f\|_{B_{p\theta}^\lambda} \sim \varepsilon^{\lambda - \frac{n}{p}} \|f\|_{B_{p\theta}^\lambda} \quad \text{as } \varepsilon \rightarrow +\infty.$$

Also, for natural  $\lambda$ ,

$$\|\tau_\varepsilon f\|_{W_p^\lambda} \sim \varepsilon^{\lambda - \frac{n}{p}} \|f\|_{W_p^\lambda} \quad \text{as } \varepsilon \rightarrow +\infty,$$

where  $W_p^\lambda$  is the Sobolev space.

This implies that in the terminology, used in particular in [3], p. 32, the differential dimension of the spaces  $LM_{p\theta}^\lambda$ ,  $GM_{p\theta}^\lambda$ ,  $b_{p\theta}^\lambda$ ,  $B_{p\theta}^\lambda$ , and, for natural  $\lambda$ ,  $W_p^\lambda$  coincide and are equal to  $\lambda - \frac{n}{p}$ .

Assume that  $0 < \|w\|_{L_\theta(t, \infty)} < \infty$  for all  $t \in (0, \infty)$  and let

$$v(t) = \|w\|_{L_\theta(t, \infty)}^{-1}, \quad 0 < t < \infty.$$

Then

$$\|f\|_{LM_{p\theta, w(\cdot)}} = \theta^{\frac{1}{\theta}} \|f\|_{LM_{p\theta}^{v(\cdot)}},$$

where

$$\|f\|_{LM_{p\theta}^{v(\cdot)}} = \left( \int_0^\infty \left( \frac{\|f\|_{L_p(B(0, r))}}{v(r)} \right)^\theta \frac{dv(r)}{v(r)} \right)^{\frac{1}{\theta}}, \quad (4.3)$$

if  $\theta < \infty$ , and

$$\|f\|_{LM_{p\infty}^{v(\cdot)}} = \sup_{r>0} \frac{\|f\|_{L_p(B(0, r))}}{v(r)}, \quad (4.4)$$

if  $\theta = \infty$ .

The definition of the quasi-norm  $\|f\|_{LM_{p\theta}^{v(\cdot)}}$ , the definition of the space  $LM_{p\theta}^{v(\cdot)}$  respectively, may be used for any positive non-decreasing function on  $(0, \infty)$  not equivalent to a constant. (In this case the integral in (4.3) is the Stiltjes-Lebesgue integral.<sup>4</sup>) If  $v$  is also locally absolutely continuous on  $(0, \infty)$ , the case in which we are mostly interested in, equality (4) holds with the function  $w$  defined by

$$w(r) = \left( \theta v'(r) v(r)^{-\theta-1} \right)^{\frac{1}{\theta}}, \quad r \in (0, \infty),$$

if  $\theta < \infty$  and by

$$w(r) = v(r)^{-1}, \quad r \in (0, \infty),$$

if  $\theta = \infty$ .

---

<sup>4</sup> If  $\{r_k\}_{k=-\infty}^\infty$  is an increasing sequence of positive numbers satisfying  $\lim_{k \rightarrow -\infty} r_k = 0$ ,  $\lim_{k \rightarrow \infty} r_k = \infty$ ,  $\{v_k\}_{k=-\infty}^\infty$  is a sequence of positive numbers, and  $v(r) = \sum_{k=-\infty}^\infty v_k \chi_{(r_k, \infty)}$ , then

$$\|f\|_{LM_{p\theta}^{v(\cdot)}} = \left( \sum_{k=-\infty}^\infty \left( \frac{\|f\|_{L_p(B(0, r_k))}}{v_k} \right)^\theta \right)^{\frac{1}{\theta}}.$$

We shall also use the notation

$$GM_{p\infty,w(\cdot)} \equiv M_{p,w(\cdot)}, \quad WGM_{p\infty,w(\cdot)} \equiv WM_{p,w(\cdot)}.$$

The spaces  $M_{p,w(\cdot)}$ ,  $WM_{p,w(\cdot)}$  respectively, are the most straightforward generalizations of the Morrey spaces  $M_p^\lambda \equiv M_{p,r-\lambda}$ , the weak Morrey spaces  $WM_p^\lambda \equiv WM_{p,r-\lambda}$  respectively.

## 5 Embeddings

Let  $A, B$  be some sets and  $\varphi, \psi$  be non-negative functions defined on  $A \times B$ . (It may happen that  $\varphi(\alpha, \beta) = +\infty$  or  $\psi(\alpha, \beta) = +\infty$  for some  $\alpha \in A, \beta \in B$ .) We say that  $\varphi$  is *dominated* by  $\psi$  (or  $\psi$  *dominates*  $\varphi$ ) on  $A \times B$  *uniformly in*  $\alpha \in A$  and write

$$\varphi(\alpha, \beta) \lesssim \psi(\alpha, \beta) \quad \text{uniformly in } \alpha \in A,$$

if for each  $\beta \in B$  there exists  $c(\beta) > 0$  such that

$$\varphi(\alpha, \beta) \leq c(\beta) \psi(\alpha, \beta)$$

for all  $\alpha \in A$ . We also say that  $\varphi$  is *equivalent* to  $\psi$  on  $A \times B$  *uniformly in*  $\alpha \in A$  and write

$$\varphi(\alpha, \beta) \approx \psi(\alpha, \beta) \quad \text{uniformly in } \alpha \in A,$$

if  $\varphi$  and  $\psi$  dominate each other on  $A \times B$  uniformly in  $\alpha \in A$ .

**Lemma 5.1. ([11])** *Let  $0 < p, \theta \leq \infty$  and  $w_1, w_2 \in \Omega_\theta$ . Then<sup>5</sup> for each  $0 < p \leq \infty$*

$$LM_{p\theta,w_1(\cdot)} \subset LM_{p\theta,w_2(\cdot)} \iff \|w_2\|_{L_\theta(t,\infty)} \lesssim \|w_1\|_{L_\theta(t,\infty)} \quad \text{uniformly in } t \in (0, \infty)$$

and

$$LM_{p\theta,w_1(\cdot)} = LM_{p\theta,w_2(\cdot)} \iff \|w_1\|_{L_\theta(t,\infty)} \approx \|w_2\|_{L_\theta(t,\infty)} \quad \text{uniformly in } t \in (0, \infty).$$

For a measurable set  $\Omega \subset \mathbb{R}^n$  and a function  $v$  non-negative and measurable on  $\Omega$ , let  $L_{p,v(\cdot)}(\Omega)$  be the weighted  $L_p$ -space of all functions  $f$  measurable on  $\Omega$  for which

$$\|f\|_{L_{p,v(\cdot)}(\Omega)} = \|vf\|_{L_p(\Omega)} < \infty.$$

Moreover, let  $L_{p,v(\cdot)} \equiv L_{p,v(\cdot)}(\mathbb{R}^n)$  and  $\|f\|_{L_{p,v(\cdot)}} \equiv \|f\|_{L_{p,v(\cdot)}(\mathbb{R}^n)}$ .

Recall that  $L_{p,v_1} \subset L_{p,v_2}$  if and only if, for some  $c > 0$ ,  $v_2(x) \leq cv_1(x)$  for almost all  $x \in \mathbb{R}^n$ . In the case of local Morrey-type spaces the condition  $\|w_2\|_{L_\theta(t,\infty)} \lesssim \|w_1\|_{L_\theta(t,\infty)}$

<sup>5</sup> By the above convention the right-hand side of this equivalence means that, given  $0 < p \leq \infty$ , for each  $0 < \theta \leq \infty$  and  $w_1, w_2 \in \Omega_\theta$  there exists  $c > 0$  such that

$$\|w_2\|_{L_\theta(t,\infty)} \leq c\|w_1\|_{L_\theta(t,\infty)}$$

for all  $t \in (0, \infty)$ . (In this case  $A = (0, \infty)$ ,  $B = \{\theta, w_1, w_2 : 0 < p, \theta \leq \infty; w_1, w_2 \in \Omega_\theta\}$ . So, for a fixed  $0 < p \leq \infty$ ,  $c$  may depend on  $0 < \theta \leq \infty$  and  $w_1, w_2 \in \Omega_\theta$ , but is independent of  $t \in (0, \infty)$ .) However, for the whole range of the parameter  $p$ ,  $c$  may depend also on  $p$

uniformly on  $(0, \infty)$  arises because the definition of these spaces contains the function  $\|f\|_{L_p(B(0,r))}$  which is non-decreasing. The statements of Lemma 5.1 follow by the appropriate results for non-decreasing functions contained in [37], [38].

Let  $0 < p, \theta \leq \infty$  and  $w \in \Omega_\theta$ . If  $p \leq \theta$ , then

$$L_{p,W(\cdot)} \subset LM_{p\theta,w(\cdot)}$$

and

$$\|f\|_{LM_{p\theta,w(\cdot)}} \leq \|f\|_{L_{p,W(\cdot)}}, \quad (5.1)$$

where for all  $x \in \mathbb{R}^n$

$$W(x) = \|w\|_{L_\theta(|x|,\infty)}.$$

If  $\theta \leq p$ , then

$$L_{p,W(\cdot)} \subset LM_{p\theta,w(\cdot)}$$

and

$$\|f\|_{L_{p,W(\cdot)}} \leq \|f\|_{LM_{p\theta,w(\cdot)}}. \quad (5.2)$$

Inequalities (5.1) and (5.2) are corollaries of the following inequality:

$$\left\| \|F(x, y)\|_{L_{p,x}(\mathbb{R}^n)} \right\|_{L_{q,y}(\mathbb{R}^m)} \leq \left\| \|F(x, y)\|_{L_{q,y}(\mathbb{R}^m)} \right\|_{L_{p,x}(\mathbb{R}^n)} \quad (5.3)$$

for functions  $F$  Lebesgue measurable on  $\mathbb{R}^{n+m}$ , where  $0 < p \leq q \leq \infty$ .

In particular, for  $0 < p \leq \infty$

$$LM_{pp,w(\cdot)} = L_{p,V(\cdot)},$$

and

$$\|f\|_{LM_{pp,w(\cdot)}} = \|f\|_{L_{p,V(\cdot)}}, \quad (5.4)$$

where for all  $x \in \mathbb{R}^n$

$$V(x) = \|w\|_{L_p(|x|,\infty)}.$$

Moreover, given a function  $v$  non-negative and measurable on  $\mathbb{R}^n$ , the equality

$$LM_{pp,w(\cdot)} = L_{p,v(\cdot)}$$

holds if and only if for certain  $c_1, c_2 > 0$

$$c_1 V(x) \leq v(x) \leq c_2 V(x)$$

for almost all  $x \in \mathbb{R}^n$ .

**Remark 8.** Inequality (5.1) implies that  $\|f\|_{LM_{p\theta,w(\cdot)}} \leq \|w\|_{L_p(0,\infty)} \|f\|_{L_p}$  for  $0 < p \leq \theta \leq \infty$ , because  $W(x) \leq \|w\|_{L_\theta(0,\infty)}$  for all  $x \in \mathbb{R}^n$ . However, this inequality holds for all  $0 < p, \theta \leq \infty$ , because clearly

$$\|f\|_{LM_{p\theta,w(\cdot)}} = \left\| \|w(r)\| \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)} \leq \|w\|_{L_\theta(0,\infty)} \|f\|_{L_p}.$$

Hence  $L_p \subset LM_{p\theta,w(\cdot)}$  if  $w \in L_\theta(0, \infty)$  and  $\|I\|_{L_p \rightarrow LM_{p\theta,w(\cdot)}} \leq \|w\|_{L_\theta(0,\infty)}$ , where  $I$  is the corresponding embedding operator.

On the other hand

$$\|I\|_{L_p \rightarrow LM_{p\theta, w(\cdot)}} \geq \sup_{\varrho > 0} \frac{\|w(r)\|_{\mathcal{X}_{B(0, \varrho)}} \|L_p(B(0, r))\|_{L_\theta(\varrho, \infty)}}{\|\mathcal{X}_{B(0, \varrho)}\|_{L_p(\mathbb{R}^n)}} = \|w\|_{L_\theta(0, \infty)}. \quad (5.5)$$

So the embedding

$$L_p \subset LM_{p\theta, w(\cdot)} \quad (5.6)$$

holds if and only if  $\|w\|_{L_\theta(0, \infty)} < \infty$  and

$$\|I\|_{L_p \rightarrow LM_{p\theta, w(\cdot)}} = \|w\|_{L_\theta(0, \infty)}. \quad (5.7)$$

Next we discuss conditions ensuring the validity of the embedding

$$L_{p_1} \subset LM_{p_2\theta, w(\cdot)} \quad (5.8)$$

where  $0 < p_1, p_2, \theta \leq \infty$ ,  $w \in \Omega_\theta$ , and  $p_1 \neq p_2$ .

Assume that  $p_1 < p_2$ . Since  $w \in \Omega_\theta$  there exists  $r_0 > 0$  such that  $\|w\|_{L_\theta(r_0, \infty)} > 0$ . We can find  $f \in L_{p_1}$  such that  $f \notin L_{p_2}(B(0, r_0))$ . Then  $f \notin LM_{p_2\theta, w(\cdot)}$ , because

$$\|f\|_{LM_{p_2\theta, w(\cdot)}} \geq \|f\|_{L_{p_2}(B(0, r_0))} \|w\|_{L_\theta(r_0, \infty)} = \infty.$$

Thus, embedding (5.8) cannot hold.

If  $0 < p_2 < p_1$ , then by Hölder's inequality it immediately follows that

$$\begin{aligned} \|I\|_{L_{p_1} \rightarrow LM_{p_2\theta, w(\cdot)}} &= \sup_{\substack{f \in L_{p_1} \\ f \not\approx 0}} \frac{\|w(r)\|_{L_{p_2}(B(0, r))} \|L_\theta(0, \infty)\|}{\|f\|_{L_{p_1}}} \\ &\leq v_n^{\frac{1}{p_2} - \frac{1}{p_1}} \left\| r^{n(\frac{1}{p_2} - \frac{1}{p_1})} w(r) \right\|_{L_\theta(0, \infty)} \end{aligned}$$

uniformly in  $w \in \Omega_\theta$ . Hence the condition

$$\left\| r^{n(\frac{1}{p_2} - \frac{1}{p_1})} w(r) \right\|_{L_\theta(0, \infty)} < \infty \quad (5.9)$$

is sufficient for the validity of embedding (5.8).

However, in spite of the fact that Hölder's inequality is sharp, it appears that this simple sufficient condition is also necessary if and only if  $\theta = \infty$ . If  $\theta < \infty$  it is not necessary (though is rather close to being necessary). In this case necessary and sufficient conditions are more sophisticated.

**Theorem 5.1.** ([7]) *Let  $0 < p_2 \leq p_1 \leq \infty$ ,  $0 < \theta \leq \infty$ , and  $w \in \Omega_\theta$ .*

1. *If  $p_2 = p_1$ ,  $0 < \theta \leq \infty$  or  $0 < p_2 < p_1$ ,  $\theta = \infty$ , then*

$$\|I\|_{L_{p_1} \rightarrow LM_{p_2\theta, w(\cdot)}} \approx \left\| r^{n(\frac{1}{p_2} - \frac{1}{p_1})} w(r) \right\|_{L_\theta(0, \infty)} \quad (5.10)$$

*uniformly in  $w \in \Omega_\theta$ .*

2. *If  $0 < p_2 < p_1$  and  $\theta < \infty$ , then*

$$\|I\|_{L_{p_1} \rightarrow LM_{p_2\theta, w(\cdot)}} \approx \|t^{n(\frac{1}{p_2} - \frac{1}{p_1}) - \frac{1}{s}} w(r)\|_{L_\theta(t, \infty)} \|L_s(0, \infty)\| \quad (5.11)$$

$$\approx \left\| t^{n(\frac{1}{p_2} - \frac{1}{p_1}) - \frac{1}{s}} \left\| \left( \frac{r}{r+t} \right)^{\frac{n}{p_2}} w(r) \right\|_{L_\theta(0,\infty)} \right\|_{L_s(0,\infty)} \quad (5.12)$$

uniformly in  $w \in \Omega_\theta$ , where

$$s = \begin{cases} \frac{p_1 \theta}{p_1 - \theta} & \text{if } \theta < p_1, \\ \infty & \text{if } \theta \geq p_1. \end{cases} \quad (5.13)$$

(Here the semi-norm  $\|\cdot\|_{L_\theta(0,\infty)}$  is taken in the variable  $r$  and the semi-norm  $\|\cdot\|_{L_s(0,\infty)}$  in the variable  $t$ .)

**Remark 9.** Since  $\theta \leq s$  by inequality (5.3)

$$\begin{aligned} & \left\| t^{n(\frac{1}{p_2} - \frac{1}{p_1}) - \frac{1}{s}} \left\| \left( \frac{r}{r+t} \right)^{\frac{n}{p_2}} w(r) \right\|_{L_{\theta,r}(0,\infty)} \right\|_{L_{s,t}(0,\infty)} \\ & \leq \left\| \left\| t^{n(\frac{1}{p_2} - \frac{1}{p_1}) - \frac{1}{s}} \left( \frac{r}{r+t} \right)^{\frac{n}{p_2}} w(r) \right\|_{L_{s,t}(0,\infty)} \right\|_{L_{\theta,r}(0,\infty)} \\ & = \left\| \xi^{n(\frac{1}{p_2} - \frac{1}{p_1}) - \frac{1}{s}} (1 + \xi)^{-\frac{n}{p_2}} \right\|_{L_s(0,\infty)} \cdot \left\| r^{n(\frac{1}{p_2} - \frac{1}{p_1})} w(r) \right\|_{L_\theta(0,\infty)}, \end{aligned}$$

which conforms with sufficient condition (5.9).

## 6 Maximal operator

Let  $f \in L_1^{loc}$ . The Hardy-Littlewood maximal operator  $M$  is defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)| dy.$$

The boundedness of  $M$  in Morrey spaces was investigated by F. Chiarenza and M. Frasca.

**Theorem 6.1. ([14])** For any  $1 < p \leq \infty$  and  $0 \leq \lambda \leq \frac{n}{p}$  the operator  $M$  is bounded from  $M_p^\lambda$  to  $M_p^\lambda$ .

For any  $0 \leq \lambda \leq n$  the operator  $M$  is bounded from  $M_1^\lambda$  to  $WM_1^\lambda$ .

Sufficient conditions for the boundedness of  $M$  from  $M_{p,w_1(\cdot)}$  to  $M_{p,w_2(\cdot)}$  were obtained by T. Mizuhara, E. Nakai, and V.S. Guliyev.

**Theorem 6.2. ([27], [29], [19])** Let  $1 \leq p < \infty$ . Moreover, let  $w_1 \in \Omega_{p,\infty}$ ,  $w_2 \in \Omega_{p,\infty}$  be positive functions satisfying the following condition:

$$\left\| w_1^{-1}(r) r^{-\frac{n}{p}-1} \right\|_{L_1(t,\infty)} \lesssim w_2^{-1}(t) t^{-\frac{n}{p}} \quad (6.1)$$

uniformly in  $t \in (0, \infty)$ .

Then for  $p > 1$   $M$  is bounded from  $M_{p,w_1(\cdot)}$  to  $M_{p,w_2(\cdot)}$  and for  $p_1 = 1$   $M$  is bounded from  $M_{1,w_1(\cdot)}$  to  $WM_{1,w_2(\cdot)}$ .

In the [27], [29] this statement was proved under the following additional assumptions: it was assumed that  $w_1 = w_2 = w$  and that  $w$  was a positive non-increasing function satisfying the pointwise doubling condition, namely that for some  $c > 0$

$$c^{-1}w(r) \leq w(t) \leq cw(r) \quad (6.2)$$

for all  $t, r > 0$  such that  $0 < r \leq t \leq 2r$ . In [19] it was proved without these additional assumptions. (See also [22], [23], [20].)

The known results on the boundedness of the maximal operator in general weighted Lebesgue spaces (see [35], [16], [15], [17]), inequalities (5.1), (5.2) and equality (5.4) imply the following statement for the case of Morrey-type spaces, including necessary and sufficient conditions for the boundedness of  $M$  from  $LM_{p_1 p_1, w_1(\cdot)}$  to  $LM_{p_2 p_2, w_2(\cdot)}$ .

**Theorem 6.3.** *Let  $1 < p_1 \leq p_2 < \infty$ ,  $0 < \theta_1, \theta_2 \leq \infty$ ,  $w_1 \in \Omega_{\theta_1}$ ,  $w_2 \in \Omega_{\theta_2}$ .*

*Let  $\theta_1 \leq p_1$  and  $p_2 \leq \theta_2$  and*

$$\sup_{R>0} R^{-n} \left\| t^{\frac{n-1}{p_1}} W_1(t)^{-1} \right\|_{L_{p_1'}(0,R)} \left\| t^{\frac{n-1}{p_2}} W_2(t) \right\|_{L_{p_2}(0,R)} < \infty. \quad (6.3)$$

*or equivalently*

$$\left\| M \left( \chi_B \widehat{W}_1^{\frac{p_1}{1-p_1}} \right) \right\|_{L_{p_2, \widehat{W}_2}(B)} \lesssim \left\| \widehat{W}_1^{\frac{1}{1-p_1}} \right\|_{L_{p_1}(B)}, \quad (6.4)$$

*uniformly in balls  $B \subset \mathbb{R}^n$ , where  $p_1' = \frac{p_1}{p_1-1}$ ,*

$$W_1(t) = \|w_1\|_{L_{\theta_1}(t, \infty)}, \quad W_2(t) = \|w_2\|_{L_{\theta_2}(t, \infty)}, \quad (6.5)$$

*for all  $t > 0$  and*

$$\widehat{W}_1(x) = \|w_1\|_{L_{\theta_1}(|x|, \infty)}, \quad \widehat{W}_2(x) = \|w_2\|_{L_{\theta_2}(|x|, \infty)}, \quad (6.6)$$

*for all  $x \in \mathbb{R}^n$ . Then  $M$  is bounded from  $LM_{p_1 \theta_1, w_1(\cdot)}$  to  $LM_{p_2 \theta_2, w_2(\cdot)}$  and from  $GM_{p_1 \theta_1, w_1(\cdot)}$  to  $GM_{p_2 \theta_2, w_2(\cdot)}$ . (In the latter case it is assumed that  $w_1 \in \Omega_{p_1 \theta_1}$ ,  $w_2 \in \Omega_{p_2 \theta_2}$ .)*

*If  $p_1 \leq \theta_1$  and  $p_2 \geq \theta_2$ , then condition (6.3), or equivalently (6.4), is necessary for the boundedness of  $M$  from  $LM_{p_1 \theta_1, w_1(\cdot)}$  to  $LM_{p_2 \theta_2, w_2(\cdot)}$ .*

*In particular, if  $\theta_1 = p_1$  and  $\theta_2 = p_2$ , then condition (6.3), or equivalently (6.4), is necessary and sufficient for the boundedness of  $M$  from  $LM_{p_1 p_1, w_1(\cdot)}$  to  $LM_{p_2 p_2, w_2(\cdot)}$ .*

If  $p_1 \neq \theta_1$  or  $p_2 \neq \theta_2$ , for the first time in the problem of boundedness of the maximal operator in general Morrey-type spaces, for a certain range of the numerical parameters necessary and sufficient conditions ensuring the boundedness were obtained by V.I. Burenkov and H.V. Guliyev [8], [9].

**Theorem 6.4.** ([8], [9], [6]) *If  $1 < p \leq \infty$ ,  $0 < \theta_1 \leq \theta_2 \leq \infty$ ,  $w_1 \in \Omega_{\theta_1}$ , and  $w_2 \in \Omega_{\theta_2}$ , then the condition*

$$\left\| w_2(r) \left( \frac{r}{t+r} \right)^{\frac{n}{p}} \right\|_{L_{\theta_2}(0, \infty)} \lesssim \|w_1\|_{L_{\theta_1}(t, \infty)} \quad (6.7)$$

uniformly in  $t \in (0, \infty)$  is necessary and sufficient for the boundedness of  $M$  from  $LM_{p\theta_1, w_1(\cdot)}$  to  $LM_{p\theta_2, w_2(\cdot)}$ . (The quasi-norm  $\|\cdot\|_{L_{\theta_2}(0, \infty)}$  is taken with respect to the variable  $r$ .)

Moreover,

$$\|M\|_{LM_{p\theta_1, w_1(\cdot)} \rightarrow LM_{p\theta_2, w_2(\cdot)}} \approx \sup_{0 < t < \infty} \|w_1\|_{L_{\theta_1}(t, \infty)}^{-1} \left\| w_2(r) \left( \frac{r}{t+r} \right)^{\frac{n}{p}} \right\|_{L_{\theta_2}(0, \infty)} \quad (6.8)$$

uniformly in  $w_1 \in \Omega_{\theta_1}$  and  $w_2 \in \Omega_{\theta_2}$ .

If  $p = 1$ , then condition (6.7) is necessary and sufficient for the boundedness of  $M$  from  $LM_{1\theta_1, w_1(\cdot)}(\mathbb{R}^n)$  to  $WLM_{1\theta_2, w_2(\cdot)}(\mathbb{R}^n)$  and

$$\|M\|_{LM_{1\theta_1, w_1(\cdot)}(\mathbb{R}^n) \rightarrow WLM_{1\theta_2, w_2(\cdot)}(\mathbb{R}^n)} \approx \sup_{0 < t < \infty} \|w_1\|_{L_{\theta_1}(t, \infty)}^{-1} \left\| w_2(r) \left( \frac{r}{t+r} \right)^n \right\|_{L_{\theta_2}(0, \infty)} \quad (6.9)$$

uniformly in  $w_1 \in \Omega_{\theta_1}$  and  $w_2 \in \Omega_{\theta_2}$ .

Keeping in mind that, in general, it may happen that  $\|w_1\|_{L_{\theta_1}(t, \infty)} = 0$  for a certain  $t > 0$  or that  $\|w_1\|_{L_{\theta_1}(t, \infty)} = +\infty$  for a certain  $t > 0$ , in (6.8) and (6.9) it is assumed that  $0^{-1} = +\infty$ ,  $(+\infty)^{-1} = 0$ , and  $0 \cdot (+\infty) = 0$ .

Inequality (6.7) implies that for the boundedness of  $M$  from  $LM_{p\theta_1, w_1(\cdot)}$  to  $LM_{p\theta_2, w_2(\cdot)}$  it is necessary that  $\|w_1\|_{L_{\theta_1}(0, \infty)} > 0$  for all  $t > 0$ . Otherwise by (6.7)  $w_2 \sim 0$  on  $(0, \infty)$  which contradicts the assumption  $w_2 \in \Omega_{\theta_2}$ . This also follows directly for any  $0 < \theta_1, \theta_2 \leq \infty$ . Indeed if  $\|w_1\|_{L_{\theta_1}(0, \infty)} = 0$  for a certain  $t > 0$ , then  $|x|\chi_{\mathfrak{c}_{B(0,t)}}(x) \in LM_{p\theta_1, w_1(\cdot)}$  but  $(Mf)(x) = +\infty$  for all  $x \in \mathbb{R}^n$ , hence  $Mf \notin LM_{p\theta_2, w_2(\cdot)}$  since  $w_2$  is not equivalent to 0 on  $(0, \infty)$ .

**Remark 10.** Condition (6.7) is equivalent to the two conditions

$$t^{-\frac{n}{p}} \left\| w_2(r) r^{\frac{n}{p}} \right\|_{L_{\theta_2}(0,t)} \lesssim \|w_1\|_{L_{\theta_1}(t, \infty)} \quad (6.10)$$

and

$$\|w_2(r)\|_{L_{\theta_2}(t, \infty)} \lesssim \|w_1\|_{L_{\theta_1}(t, \infty)} \quad (6.11)$$

uniformly in  $t \in (0, \infty)$ .

By Lemma 5.1 condition (6.11) is equivalent to the embedding  $LM_{p\theta_2, w_2(\cdot)} \subset LM_{p\theta_1, w_1(\cdot)}$ . Hence the necessity of condition (6.11) is obvious, because  $(Mf)(x) \geq |f(x)|$  for almost all  $x \in \mathbb{R}^n$  hence

$$\|I\|_{LM_{p\theta_1, w_1(\cdot)} \rightarrow LM_{p\theta_2, w_2(\cdot)}} \leq \|M\|_{LM_{p\theta_1, w_1(\cdot)} \rightarrow LM_{p\theta_2, w_2(\cdot)}}$$

**Remark 11.** In [8],[9] Theorem 6.4 is proved under the additional assumption  $\theta_1 \leq p_1$ , in [6] without this assumption by using a different method.

In the formulation of Theorem 6.4 there is a natural assumption  $w_2 \in \Omega_{\theta_2}$  (non-triviality of the space  $LM_{p\theta_2, w_2}$ ). However, inequality (6.11) and Definition 2 imply that the stronger assumption  $w_2 \in \Omega_{p\theta_2}$  is necessary for the boundedness of  $M$  from  $LM_{p\theta_1, w_1(\cdot)}$  to  $LM_{p\theta_2, w_2(\cdot)}$ . Moreover, if  $\theta_2 = \infty$  and  $\theta_1 < \infty$  it is also necessary that

$$\lim_{t \rightarrow \infty} \left\| w_2(r) \left( \frac{r}{t+r} \right)^{\frac{n}{p}} \right\|_{L_{\infty}(0, \infty)} = 0. \quad (6.12)$$

**Corollary 6.1.** *Let  $1 < p < \infty$ ,  $0 < \theta_1 \leq \theta_2 \leq \infty$ ,  $w_2 \in \Omega_{p\theta_2}$  and, if  $\theta_2 = \infty$  and  $\theta_1 < \infty$ , then also condition (6.12) be satisfied.*

*Then*

1)  *$M$  is bounded from  $LM_{p\theta_1, w_1^*(\cdot)}$  to  $LM_{p\theta_2, w_2(\cdot)}$ , where  $w_1^*$  is a non-increasing continuous function on  $(0, \infty)$  defined by*

$$\|w_1^*\|_{L_{\theta_1}(t, \infty)} = \left\| w_2(r) \left( \frac{r}{t+r} \right)^{n/p} \right\|_{L_{\theta_2}(0, \infty)}, \quad t \in (0, \infty). \quad (6.13)$$

2) *If  $w_1 \in \Omega_{\theta_1}$  and  $M$  is bounded from  $LM_{p\theta_1, w_1(\cdot)}$  to  $LM_{p\theta_2, w_2(\cdot)}$ , then*

$$LM_{p\theta_1, w_1} \subset LM_{p\theta_1, w_1^*}. \quad (6.14)$$

*(Hence  $LM_{p\theta_1, w_1^*(\cdot)}$  is the maximal among the spaces  $LM_{p\theta_1, w_1(\cdot)}$ ,  $w_1 \in \Omega_{\theta_1}$ , for which  $M$  is bounded from  $LM_{p\theta_1, w_1}$  to  $LM_{p\theta_2, w_2}$ .)*

Note that equality (6.13), under the assumptions (2) and (if  $\theta_2 = \infty$  and  $\theta_1 < \infty$ ) (6.12), defines a non-increasing continuous function  $w_1^*$  uniquely. In particular, if  $\theta_1 = \infty$ , then

$$w_1^*(t) = \left\| w_2(r) \left( \frac{r}{t+r} \right)^{n/p} \right\|_{L_{\theta_2}(0, \infty)}, \quad t \in (0, \infty).$$

In Theorem 6.4  $\theta_1 \leq \theta_2$ . In the case in which  $LM_{p\theta_1, w_1(\cdot)} = L_p$ , i. e.  $\theta_1 = \infty$ ,  $w_1 \equiv 1$ , necessary and sufficient conditions were obtained also for  $\theta_2 < \theta_1$ .

We start with the following simple observations aimed at clarifying necessary assumptions on  $0 < p_1, p_2, \theta \leq \infty$  for which for certain  $w \in \Omega_\theta$  the operator  $M$  can be bounded from  $L_{p_1}$  to  $LM_{p_2\theta, w(\cdot)}$ .

**Remark 12.** Let  $0 < \theta \leq \infty$ ,  $w \in \Omega_\theta$ , and  $0 < p_1 < p_2 \leq \infty$ . Then there exists  $r_0 > 0$  such that  $\|w\|_{L_\theta(r_0, \infty)} > 0$ . We can find  $f \in L_{p_1}$  such that  $f \notin L_{p_2}(B(0, r_0))$ . Then  $Mf \notin L_{p_2}(B(0, r_0))$  and therefore  $Mf \notin LM_{p_2\theta, w(\cdot)}$ , because

$$\|Mf\|_{LM_{p_2\theta, w(\cdot)}} \geq \|Mf\|_{L_{p_2}(B(0, r_0))} \|w\|_{L_\theta(r_0, \infty)}.$$

Thus, in the problem of boundedness of the maximal operator  $M : L_{p_1} \rightarrow LM_{p_2\theta, w}$  one should assume that  $p_2 \leq p_1$ .

**Remark 13.** Assume that  $0 < \theta \leq \infty$ ,  $w \in \Omega_\theta$ , and  $p_1 = p_2 = p > 1$ . Since  $Mf(x) \geq |f(x)|$  for almost all  $x \in \mathbb{R}^n$  by (5.5)

$$\|M\|_{L_p \rightarrow LM_{p\theta, w(\cdot)}} \geq \|I\|_{L_p \rightarrow LM_{p\theta, w(\cdot)}} \geq \|w\|_{L_\theta(0, \infty)}.$$

On the other hand, by applying the classical  $L_p$ -estimate for the maximal function, it follows that

$$\|M\|_{L_p \rightarrow LM_{p\theta, w(\cdot)}} = \sup_{\substack{f \in L_p \\ f \not\equiv 0}} \frac{\|w(r)\|_{L_\theta(B(0, r))} \|Mf\|_{L_p(B(0, r))}}{\|f\|_{L_p}} \Big\|_{L_\theta(0, \infty)}$$



$$\lesssim \sup_{\substack{f \in L_p \\ f \not\equiv 0}} \frac{\|w(r)\|f\|_{L_p} \|L_\theta(0,\infty)}{\|f\|_{L_p}} = \|w\|_{L_\theta(0,\infty)}$$

uniformly in  $w \in \Omega_\theta$ . Thus

$$\|M\|_{L_p \rightarrow LM_{p\theta,w(\cdot)}} \approx \|w\|_{L_\theta(0,\infty)} \quad (6.15)$$

uniformly in  $w \in \Omega_\theta$ .

For similar reasons by the equality  $\|\chi_{B(0,\varrho)}\|_{WL_1(B(0,\varrho))} = \|\chi_{B(0,\varrho)}\|_{L_1(B(0,\varrho))}$  and the boundedness of  $M$  from  $L_1$  to  $WL_1$  it follows that

$$\|M\|_{L_1 \rightarrow WLM_{1\theta,w(\cdot)}} \approx \|w\|_{L_\theta(0,\infty)} \quad (6.16)$$

uniformly in  $w \in \Omega_\theta$ .

Equivalences (6.15) and (6.16) also follow by equivalences (6.8) and (6.9) with  $w_1 \equiv 1, w_2 = w, \theta_1 = \infty, \theta_2 = \theta$ , because

$$\sup_{0 < t < \infty} \left\| w(r) \left( \frac{r}{t+r} \right)^{\frac{n}{p}} \right\|_{L_\theta(0,\infty)} = \|w\|_{L_\theta(0,\infty)}.$$

If  $p_1 = p_2 = 1$ , then  $\|M\|_{L_{p_1} \rightarrow LM_{1\theta,w(\cdot)}} = \infty$  for all  $0 < \theta \leq \infty$  and  $w \in \Omega_\theta$ . This follows if one considers test-functions  $\chi_{B(0,\varepsilon)}$  and passes to the limit as  $\varepsilon \rightarrow 0^+$ .

Summarizing, if one investigates the boundedness of  $M$  from  $L_{p_1}$  to  $LM_{p_2\theta,w(\cdot)}$ , then one should always assume that

$$0 < \theta \leq \infty, \quad 1 \leq p_1 \leq \infty, \quad 0 < p_2 \leq p_1 \text{ if } p_1 > 1, \quad 0 < p_2 < 1 \text{ if } p_1 = 1,$$

and  $w \in \Omega_\theta$ .

**Remark 14.** What happens if  $0 < p_2 < p_1$ ? If  $p_1 > 1$ , then by applying Hölder's inequality and the boundedness of  $M$  from  $L_{p_1}$  to  $L_{p_1}$  it immediately follows that

$$\|Mf\|_{L_{p_2}(B(0,r))} \leq (v_n r^n)^{\frac{1}{p_2} - \frac{1}{p_1}} \|Mf\|_{L_{p_1}} \lesssim r^{n(\frac{1}{p_2} - \frac{1}{p_1})} \|f\|_{L_{p_1}}$$

uniformly in  $r > 0$  and

$$\|M\|_{L_{p_1} \rightarrow LM_{p_2\theta,w(\cdot)}} = \sup_{\substack{f \in L_{p_1} \\ f \not\equiv 0}} \frac{\|w(r)\|Mf\|_{L_{p_2}(B(0,r))} \|L_\theta(0,\infty)}{\|f\|_{L_{p_1}}}$$

$$\lesssim \|r^{n(\frac{1}{p_2} - \frac{1}{p_1})} w(r)\|_{L_\theta(0,\infty)}$$

uniformly in  $w \in \Omega_\theta$ . Hence the condition (5.9) is sufficient for boundedness of the maximal operator  $M$  from  $L_{p_1}$  to  $LM_{p_2\theta,w(\cdot)}$ .

If  $0 < p_2 < 1$ , then condition (5.9) with  $p_1 = 1$  is also sufficient for the boundedness of  $M$  from  $L_1$  to  $WLM_{L_{p_2\theta, w(\cdot)}}$ . This follows since by the boundedness of  $M$  from  $L_1$  to  $WL_1$

$$\begin{aligned} \|Mf\|_{L_{p_2}(B(0,r))} &= \|(Mf)\chi_{B(0,r)}\|_{L_{p_2}(B(0,r))} \leq \|((Mf)\chi_{B(0,r)})^*\|_{L_{p_2}(0,|B_r|)} \\ &\leq \left( \sup_{0 < t \leq |B_r|} t((Mf)\chi_{B(0,r)})^*(t) \right) \|t^{-1}\|_{L_{p_2}(0,|B_r|)} \\ &= (1 - p_2)^{-\frac{1}{p_2}} (v_n r^n)^{\frac{1}{p_2}-1} \|Mf\|_{WL_1(B(0,r))} \lesssim r^{n(\frac{1}{p_2}-1)} \|f\|_{L_1} \end{aligned}$$

uniformly in  $r > 0$ .

Denote by  $L_p^\downarrow$  the space of all functions  $f \in L_p$  of the form  $f(x) = g(|x|)$ ,  $x \in \mathbb{R}^n$ , where  $g$  is a non-negative non-increasing function on  $(0, \infty)$ .

**Theorem 6.5.** ([7]) *Let  $n \in \mathbb{N}$ ,  $0 < p_2 \leq p_1 \leq \infty$ ,  $0 < \theta \leq \infty$ , and  $w \in \Omega_\theta$ .*

1. *If  $p_1 > 1$ , then  $M$  is bounded from  $L_{p_1}$  to  $LM_{p_2\theta_2, w_2(\cdot)}$  if and only if*

$$L_{p_1}^\downarrow \subset LM_{p_2\theta_2, w_2(\cdot)},$$

and

$$\begin{aligned} \|M\|_{L_{p_1} \rightarrow LM_{p_2\theta_2, w_2(\cdot)}} &\approx \|M\|_{L_{p_1}^\downarrow \rightarrow LM_{p_2\theta_2, w_2(\cdot)}} \\ &\approx \|I\|_{L_{p_1}^\downarrow \rightarrow LM_{p_2\theta_2, w_2(\cdot)}} = \|I\|_{L_{p_1} \rightarrow LM_{p_2\theta_2, w_2(\cdot)}} \end{aligned}$$

uniformly in  $w \in \Omega_\theta$ , where  $I$  is the corresponding embedding operator.

2. *If  $p_1 = 1$ , then  $M$  is bounded from  $L_1$  to  $WLM_{1\theta, w(\cdot)}$  if and only if*

$$L_1^\downarrow \subset LM_{1\theta, w(\cdot)},$$

and

$$\begin{aligned} \|M\|_{L_1 \rightarrow WLM_{1\theta, w(\cdot)}} &\approx \|M\|_{L_1^\downarrow \rightarrow WLM_{1\theta, w(\cdot)}} \\ &\approx \|I\|_{L_1^\downarrow \rightarrow LM_{1\theta, w(\cdot)}} = \|I\|_{L_1 \rightarrow LM_{1\theta, w(\cdot)}} \end{aligned}$$

uniformly in  $w \in \Omega_\theta$ .

Theorems 5.1 and 6.5 imply the following statement.

**Theorem 6.6.** ([7]) *Let  $0 < p_2 \leq p_1 \leq \infty$ ,  $0 < \theta \leq \infty$ , and  $w \in \Omega_\theta$ .*

1. *If  $1 < p_2 = p_1$ ,  $0 < \theta \leq \infty$  or  $0 < p_2 < p_1$ ,  $p_1 > 1$ ,  $\theta = \infty$ , then*

$$\|M\|_{L_{p_1} \rightarrow LM_{p_2\theta, w(\cdot)}} \approx \left\| r^{n(\frac{1}{p_2} - \frac{1}{p_1})} w(r) \right\|_{L_\theta(0, \infty)}$$

uniformly in  $w \in \Omega_\theta$ .

In particular, if  $1 < p \leq \infty$ ,  $0 < \theta \leq \infty$ , then

$$\|M\|_{L_p \rightarrow LM_{p\theta, w(\cdot)}} \approx \|w(r)\|_{L_\theta(0, \infty)}$$

uniformly in  $w \in \Omega_\theta$ . Also for all  $0 < \theta \leq \infty$

$$\|M\|_{L_1 \rightarrow WLM_{1\theta, w(\cdot)}} \approx \|w(r)\|_{L_\theta(0, \infty)}$$

uniformly in  $w \in \Omega_\infty$ .

2. If  $0 < p_2 < p_1, p_1 > 1$ , and  $\theta < \infty$ , then

$$\|M\|_{L_{p_1} \rightarrow LM_{p_2\theta, w(\cdot)}} \approx \|t^{n(\frac{1}{p_2} - \frac{1}{p_1}) - \frac{1}{s}} \|w(r)\|_{L_\theta(t, \infty)}\|_{L_s(0, \infty)} \quad (6.17)$$

uniformly in  $w \in \Omega_\theta$ , where  $s$  is defined by equality (5.13). (Here the semi-norm  $\|\cdot\|_{L_\theta(0, \infty)}$  is taken in the variable  $r$  and the semi-norm  $\|\cdot\|_{L_s(0, \infty)}$  in the variable  $t$ .)

**Example 4.** Let  $n \in \mathbb{N}, 0 < p_2 \leq p_1 \leq \infty, p_1 > 1, 0 < \theta \leq \infty, \lambda_1, \lambda_2 \in \mathbb{R}$ , and

$$w(r) = \begin{cases} r^{-\lambda_1 - \frac{1}{\theta}} & \text{if } 0 < r \leq 1, \\ r^{-\lambda_2 - \frac{1}{\theta}} & \text{if } 1 \leq r < \infty. \end{cases}$$

Then  $w \in \Omega_\theta$  if and only if  $\lambda_2 > 0$  for  $\theta < \infty$  and  $\lambda_2 \geq 0$  for  $\theta = \infty$ .

Under this assumption  $M$  is bounded from  $L_{p_1}$  to  $LM_{p_2\theta_2, w_2(\cdot)}$  if and only if

1) for  $p_2 < p_1 \leq \theta \leq \infty$

$$\lambda_1 \leq n\left(\frac{1}{p_2} - \frac{1}{p_1}\right), \quad \lambda_2 \geq n\left(\frac{1}{p_2} - \frac{1}{p_1}\right),$$

2) for  $p_2 < p_1, \theta < p_1$

$$\lambda_1 < n\left(\frac{1}{p_2} - \frac{1}{p_1}\right), \quad \lambda_2 > n\left(\frac{1}{p_2} - \frac{1}{p_1}\right),$$

3) for  $p_2 = p_1$

$$\lambda_1 \leq 0 \text{ if } \theta = \infty, \quad \lambda_1 < 0 \text{ if } \theta < \infty$$

(if  $p_2 = p_1 = 1$ , this condition is necessary and sufficient for the boundedness of  $M$  from  $L_1$  to  $WLM_{1\theta, w(\cdot)}$ ).

**Example 5.** (Particular case of Example 4.) Let  $n \in \mathbb{N}, 0 < p_2 \leq p_1 \leq \infty, p_1 > 1, 0 < \theta \leq \infty, \lambda > 0$  for  $\theta < \infty$  and  $\lambda \geq 0$  for  $\theta = \infty$ .

Then  $M$  is bounded from  $L_{p_1}$  to  $LM_{p_2\theta}^\lambda \equiv LM_{p_2\theta, r^{-\lambda - \frac{1}{\theta}}}$  if and only if

$$p_1 \leq \theta \quad \text{and} \quad \lambda = n\left(\frac{1}{p_2} - \frac{1}{p_1}\right).$$

(The necessity of the above equality easily follows by the dilation argument.)

If  $p_1 = p_2 = p > 1$ , then  $M$  is bounded from  $L_p$  to  $LM_{p\theta}^\lambda$  only in the case  $\theta = \infty$  and  $\lambda = 0$ , in which  $LM_{p\theta}^0 = L_p$ . Similarly, if  $p_1 = p_2 = 1$ , then  $M$  is bounded from  $L_1$  to  $WLM_{1\theta}^\lambda$  only in this case.

**Example 6.** Let  $n \in \mathbb{N}, 0 < p_2 < p_1, p_1 > 1, 0 < \theta < p_1, \gamma \in \mathbb{R}$ , and

$$w(r) = r^{-n(\frac{1}{p_2} - \frac{1}{p_1}) + \frac{1}{\theta}} (1 + |\ln r|)^{-\gamma}.$$

Then  $M$  is bounded from  $L_{p_1}$  to  $LM_{p_2\theta_2, w_2(\cdot)}$  if and only if  $\gamma > \frac{1}{\theta} - \frac{1}{p_1}$ .

**Remark 15.** Examples 4 and 6 imply, in particular, that the right-hand side of equivalence (6.17) is not equivalent to the right-hand side of equivalence (6.6) for all  $0 < p_2 < p_1 \leq \infty, p_1 > 1$ , and  $0 < \theta < \infty$ .

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