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LIMITING VALUES OF THE CAUCHY TYPE INTEGRAL IN A THREE-DIMENSIONAL HARMONIC ALGEBRA

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Abstract. We establish sufficient conditions for the existence of the limiting values of a certain analog of the Cauchy type integral taking values in a three-dimensional harmonic algebra with two-dimensional radical.

1 Introduction

Let Γ be a closed Jordan rectifiable curve in the complex plane $\mathbb{C}$. By D^+ and D^- we denote, respectively, the interior and the exterior domains bounded by the curve Γ.

N. Davydov [1] established sufficient conditions for the existence of limiting values of the Cauchy type integral

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{g(t)}{t-\xi}dt, \qquad \xi\in\mathbb{C}\setminus\Gamma, \tag{1.1}$$

on Γ from the domains D^+ and D^-. This result stimulated development of the theory of Cauchy type integral on curves which are not piecewise-smooth.

In particular, using the mentioned result of the paper [1], the following result was proved (see [3]): if the curve Γ satisfies the condition (see [13])

$$\theta(\varepsilon):=\sup_{\xi\in\Gamma}\ \theta_\xi(\varepsilon)=O(\varepsilon), \quad \varepsilon\to 0 \tag{1.2}$$

(here $\theta_\xi(\varepsilon):=m\,\{t\in\Gamma:|t-\xi|\leqslant\varepsilon\}$, where m denotes the linear Lebesgue measure on Γ), and the modulus of continuity

$$\omega_g(\varepsilon):=\sup_{t_1,t_2\in\Gamma,|t_1-t_2|\leqslant\varepsilon}|g(t_1)-g(t_2)|$$

of a function $g:\Gamma\to\mathbb{C}$ satisfies the Dini condition

$$\int\limits_0^1\frac{\omega_g(\eta)}{\eta}d\eta<\infty, \tag{1.3}$$

then integral (1.1) has the limiting values in every point of Γ from the domains D^+ and D^-.

Condition (1.2) means that the measure of a part of the curve Γ in every disk centered at a point of the curve is commensurable with the radius of the disk.

In this paper we consider a certain analogue of Cauchy type integral taking values in a three-dimensional harmonic algebra with two-dimensional radical and study the question about the existence of its limiting values on the boundary of the domain of definition.

2 A three-dimensional harmonic algebra with a two-dimensional radical

Let $\mathbb{A}_3$ be a three-dimensional commutative associative Banach algebra with unit 1 over the field of complex numbers $\mathbb{C}$. Let $\{1, \rho_1, \rho_2\}$ be a basis of algebra $\mathbb{A}_3$ with the multiplication table: $\rho_1\rho_2 = \rho_2^2 = 0,\ \rho_1^2 = \rho_2$.

$\mathbb{A}_3$ is a *harmonic* algebra, i. e. there exists a *harmonic* basis $\{e_1, e_2, e_3\} \subset \mathbb{A}_3$ satisfying the following conditions (see [5, 6, 8, 9, 10]):

$$e_1^2 + e_2^2 + e_3^2 = 0, \qquad e_j^2 \neq 0 \ \text{ for } j = 1, 2, 3. \tag{2.1}$$

P. Ketchum [5] discovered that every function $\Phi(\zeta)$ analytic with respect to the variable $\zeta := xe_1 + ye_2 + ze_3$ with real x, y, z satisfies the equalities

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi(\zeta) = \Phi''(\zeta)\,(e_1^2 + e_2^2 + e_3^2) = 0 \tag{2.2}$$

due to equality (2.1). I. Mel'nichenko [8] noticed that functions twice differentiable in the sense of Gateaux form the largest class of functions Φ satisfying equalities (2.2).

All harmonic bases in $\mathbb{A}_3$ are constructed by I. Mel'nichenko in [10].

Consider a harmonic basis

$$e_1 = 1, \qquad e_2 = i + \frac{1}{2}\, i\rho_2, \qquad e_3 = -\rho_1 - \frac{\sqrt{3}}{2}\, i\rho_2$$

in $\mathbb{A}_3$ and the linear span $E_3 := \{\zeta = x + ye_2 + ze_3 : x, y, z \in \mathbb{R}\}$ over the field of real numbers $\mathbb{R}$, that is generated by the vectors $1, e_2, e_3$. Associate with a domain $\Omega \subset \mathbb{R}^3$ the domain $\Omega_\zeta := \{\zeta = x + ye_2 + ze_3 : (x, y, z) \in \Omega\}$ in E_3.

The algebra $\mathbb{A}_3$ has the unique maximal ideal $\{\lambda_1\rho_1 + \lambda_2\rho_2 : \lambda_1, \lambda_2 \in \mathbb{C}\}$ which is also radical of $\mathbb{A}_3$. Thus, it is obvious that the straight line $\{ze_3 : z \in \mathbb{R}\}$ is contained in the radical of algebra $\mathbb{A}_3$.

$\mathbb{A}_3$ is a Banach algebra with the Euclidean norm

$$\|a\| := \sqrt{|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2}\,,$$

where $a = \xi_1 + \xi_2 e_2 + \xi_3 e_3$ and $\xi_1, \xi_2, \xi_3 \in \mathbb{C}$.

We say that a continuous function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is *monogenic* in a domain $\Omega_\zeta \subset E_3$ if Φ is differentiable in the sense of Gateaux in every point of Ω_ζ, i. e. if for every $\zeta \in \Omega_\zeta$ there exists $\Phi'(\zeta) \in \mathbb{A}_3$ such that

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right) \varepsilon^{-1} = h\Phi'(\zeta) \quad \forall h \in E_3.$$

For monogenic functions $\Phi : \Omega_\zeta \to \mathbb{A}_3$ we established basic properties analogous to the properties of analytic functions of the complex variable: the Cauchy integral theorem, the Cauchy integral formula, the Morera theorem, the Taylor expansion (see [14]).

3 On existence of limiting values of a hypercomplex analogue of the Cauchy type integral on the line of integration

In what follows, $t_1, t_2, x, y, z \in \mathbb{R}^3$ and the variables x, y, z with subscripts are real. For example, x_0 and y_0 are real, etc.

Let $\Gamma_\zeta := \{\tau = t_1 + t_2 e_2 : t_1 + it_2 \in \Gamma\}$ be the curve congruent to the curve $\Gamma \subset \mathbb{C}$. Consider the domain $\Pi_\zeta^\pm := \{\zeta = x + ye_2 + ze_3 : x + iy \in D^\pm, z \in \mathbb{R}\}$ in E_3. By Σ_ζ we denote the common boundary of domains Π_ζ^+ and Π_ζ^-.

Consider the integral

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_{\Gamma_\zeta} \varphi(\tau)(\tau - \zeta)^{-1} d\tau \tag{3.1}$$

with a continuous density $\varphi : \Gamma_\zeta \to \mathbb{R}$. Function (3.1) is monogenic in the domains Π_ζ^+ and Π_ζ^-, but integral (3.1) is not defined for $\zeta \in \Sigma_\zeta$.

For a function $\varphi : \Gamma_\zeta \to \mathbb{R}$ consider the modulus of continuity

$$\omega_\varphi(\varepsilon) := \sup_{\tau_1, \tau_2 \in \Gamma_\zeta, \|\tau_1 - \tau_2\| \leqslant \varepsilon} |\varphi(\tau_1) - \varphi(\tau_2)|,$$

and the singular integral

$$\int_{\Gamma_\zeta} \Big(\varphi(\tau) - \varphi(\zeta_0)\Big)(\tau - \zeta_0)^{-1} d\tau := \lim_{\varepsilon \to 0} \int_{\Gamma_\zeta \setminus \Gamma_\zeta^\varepsilon(\zeta_0)} \Big(\varphi(\tau) - \varphi(\zeta_0)\Big)(\tau - \zeta_0)^{-1} d\tau,$$

where $\zeta_0 \in \Gamma_\zeta$ and $\Gamma_\zeta^\varepsilon(\zeta_0) := \{\tau \in \Gamma_\zeta : \|\tau - \zeta_0\| \leqslant \varepsilon\}$.

By $\widehat{\Phi}^\pm(\zeta_0)$ we denote the limiting value of function (3.1) when ζ tends to $\zeta_0 \in \Gamma_\zeta$ along a curve $\gamma_\zeta \subset \Pi_\zeta^\pm$ for which there exists a constant $m < 1$ such that the inequality

$$|z| \leqslant m\|\zeta - \zeta_0\| \tag{3.2}$$

holds for all $\zeta = x + ye_2 + ze_3 \in \gamma_\zeta$.

We can say that inequality (3.2) means that the curve is not tangential to the surface Σ_ζ outside of the plane of curve Γ_ζ.

The following theorem presents sufficient conditions for the existence of the limiting values $\widehat{\Phi}^\pm(\zeta_0)$ in points $\zeta_0 \in \Gamma_\zeta$.

Theorem 3.1. *Let Γ be a closed Jordan rectifiable curve satisfying condition* (1.2) *and let the modulus of continuity of a function $\varphi : \Gamma_\zeta \to \mathbb{R}$ satisfy the condition of type* (1.3). *Then integral* (3.1) *has the limiting values $\widehat{\Phi}^{\pm}(\zeta_0)$ for all $\zeta_0 \in \Gamma_\zeta$ that are expressed by the formulas:*

$$\widehat{\Phi}^{+}(\zeta_0) = \frac{1}{2\pi i}\int\limits_{\Gamma_\zeta}(\varphi(\tau) - \varphi(\zeta_0))(\tau - \zeta_0)^{-1}d\tau + \varphi(\zeta_0)$$

$$\widehat{\Phi}^{-}(\zeta_0) = \frac{1}{2\pi i}\int\limits_{\Gamma_\zeta}(\varphi(\tau) - \varphi(\zeta_0))(\tau - \zeta_0)^{-1}d\tau.$$

Theorem 3.1 can be proved in a similar way as for the Sokhotski–Plemelj formulas in the complex plane (see, e. g., [2, 1, 3]).

Note that additional assumptions about the function φ are required for the existence of limiting values of function (3.1) from Π^{+}_{ζ} or Π^{-}_{ζ} on the boundary Σ_ζ. We are going to state and prove such results in the next section.

4 On existence of the limiting values of a hypercomplex analogue of the Cauchy type integral on the boundary of the domain of definition

Now we consider the question about the existence of limiting values $\Phi^{\pm}(\zeta_0)$ of Cauchy type integral (3.1) when $\zeta \in \Pi^{\pm}_{\zeta}$ tends to $\zeta_0 := x_0 + y_0e_2 + z_0e_3 \in \Sigma_\zeta$. For the function $\varphi : \Gamma_\zeta \to \mathbb{R}$ we define a function $g : \Gamma \to \mathbb{R}$ as $g(t) := \varphi(\tau)$, where $t = t_1 + it_2 \in \Gamma$, $\tau = t_1 + t_2e_2$.

Lemma 4.1. *Suppose that Γ is a closed Jordan rectifiable curve. Suppose also that a function $g : \Gamma \to \mathbb{R}$ and its contour derivative g' are absolutely continuous on Γ. Then for all $\zeta = x + ye_2 + ze_3 \in \Pi^{\pm}_{\zeta}$ the following equality is true:*

$$\frac{1}{2\pi i}\int\limits_{\Gamma_\zeta}\varphi(\tau)(\tau-\zeta)^{-1}d\tau = \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{g(t)}{t-\xi}dt - \rho_1\frac{z}{2\pi i}\int\limits_{\Gamma}\frac{g'(t)}{t-\xi}dt +$$

$$+\rho_2\frac{1}{2\pi i}\left(-\frac{\sqrt{3}}{2}iz\int\limits_{\Gamma}\frac{g'(t)}{t-\xi}dt + \frac{z^2}{2}\int\limits_{\Gamma}\frac{g''(t)}{t-\xi}dt + \frac{i}{2}\int\limits_{\Gamma}\frac{(y-t_2)g'(t)}{t-\xi}dt\right), \tag{4.1}$$

where $\xi := x + iy$ and $t_2 := \operatorname{Im} t$.

Proof. It follows by Lemma 1.1 [10] that

$$(\tau-\zeta)^{-1} = \frac{1}{t-\xi} - \frac{z}{(t-\xi)^2}\rho_1 + \left(\frac{i}{2}\frac{y-t_2-\sqrt{3}z}{(t-\xi)^2} + \frac{z^2}{(t-\xi)^3}\right)\rho_2 \tag{4.2}$$

for all $\zeta = x+ye_2+ze_3 \in \Pi^{\pm}_{\zeta}$ and $\tau = t_1+t_2e_2 \in \Gamma_\zeta$, where $\xi := x+iy$ and $t := t_1+it_2$.

Taking into account equality (4.2) and the relation

$$d\tau = dt + \frac{i}{2} dt_2 \rho_2 \,, \tag{4.3}$$

we represent integral (3.1) in the form

$$\Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{g(t)}{t-\xi} dt - \frac{1}{2\pi i} z\rho_1 \int\limits_{\Gamma} \frac{g(t)}{(t-\xi)^2} dt +$$

$$+ \frac{1}{2\pi i} \Bigg(-\frac{\sqrt{3}}{2} iz \int\limits_{\Gamma} \frac{g(t)}{(t-\xi)^2} dt + z^2 \int\limits_{\Gamma} \frac{g(t)}{(t-\xi)^3} dt \Bigg) \rho_2 +$$

$$+ \frac{1}{2\pi i} \frac{i}{2} \Bigg(\int\limits_{\Gamma} \frac{g(t)}{t-\xi} dt_2 + \int\limits_{\Gamma} \frac{(y-t_2) g(t)}{(t-\xi)^2} dt \Bigg) \rho_2 . \tag{4.4}$$

Inasmuch as the function g is absolutely continuous on Γ, the formula of integration by parts for the Stieltjes integral (see, e. g., [12, p. 27]) is applicable to the last integral included in equality (4.4). In such a way we obtain

$$\int\limits_{\Gamma} \frac{g(t)}{t-\xi} dt_2 + \int\limits_{\Gamma} \frac{(y-t_2) g(t)}{(t-\xi)^2} dt = \int\limits_{\Gamma} \frac{g(t) \frac{dt_2}{dt}}{t-\xi} dt +$$

$$+ \int\limits_{\Gamma} \frac{-g(t) \frac{dt_2}{dt} + g'(t)(y-t_2)}{t-\xi} dt = \int\limits_{\Gamma} \frac{g'(t)(y-t_2)}{t-\xi} dt. \tag{4.5}$$

Further, substituting expression (4.5) in equality (4.4), we obtain

$$\Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{g(t)}{t-\xi} dt - \frac{z}{2\pi i} \rho_1 \int\limits_{\Gamma} \frac{g(t)}{(t-\xi)^2} dt +$$

$$+ \frac{1}{2\pi i} \Bigg(-\frac{\sqrt{3}}{2} iz \int\limits_{\Gamma} \frac{g(t)}{(t-\xi)^2} dt + z^2 \int\limits_{\Gamma} \frac{g(t)}{(t-\xi)^3} dt + \frac{i}{2} \int\limits_{\Gamma} \frac{g'(t)(y-t_2)}{t-\xi} dt \Bigg) \rho_2 . \tag{4.6}$$

To complete the proof, in equality (4.6) it is necessary to integrate the second and the third integrals by parts and to integrate the fourth integral by parts twice. □

Theorem 4.1. *Suppose that Γ is a closed Jordan rectifiable curve satisfying condition* (1.2). *Suppose also that a function $g : \Gamma \to \mathbb{R}$ and its contour derivative g' are absolutely continuous on Γ and, moreover, the modules of continuity of the functions g, g' and g'' satisfy the conditions of type* (1.3).*Then integral* (3.1) *has the limiting values $\Phi^+(\zeta_0)$ and $\Phi^-(\zeta_0)$ for all $\zeta_0 := x_0 + y_0 e_2 + z_0 e_3 \in \Sigma_\zeta$ that are expressed by the formulas:*

$$\Phi^+(\zeta_0) = \widetilde{\varphi}(\zeta_0) + \frac{1}{2\pi i} \int\limits_{\Gamma'_\zeta} \Big(\widetilde{\varphi}(\tau) - \widetilde{\varphi}(\zeta_0) \Big) (\tau - \zeta_0)^{-1} d\tau, \tag{4.7}$$

$$\Phi^-(\zeta_0) = \frac{1}{2\pi i}\int\limits_{\Gamma'_\zeta}\Big(\widetilde{\varphi}(\tau)-\widetilde{\varphi}(\zeta_0)\Big)(\tau-\zeta_0)^{-1}d\tau, \tag{4.8}$$

where $\Gamma'_\zeta := \{\tau = t_1+t_2e_2+z_0e_3 : t_1+t_2e_2 \in \Gamma_\zeta\}$ *and*

$$\widetilde{\varphi}(t_1+t_2e_2+z_0e_3) := g(t)-z_0g'(t)\rho_1+\left(\frac{z_0^2}{2}g''(t)-\frac{\sqrt{3}}{2}iz_0g'(t)\right)\rho_2 \tag{4.9}$$

and $t := t_1+it_2$.

Proof. Passing to the limit in equality (4.1) with $\Pi^+_\zeta \ni \zeta \to \zeta_0$ and seeing that $\xi \to \xi_0 := x_0+iy_0$, we obtain

$$\Phi^+(\zeta_0) = g(\xi_0)-z_0\rho_1g'(\xi_0)+\left(\frac{z_0^2}{2}g''(\xi_0)-\frac{\sqrt{3}}{2}iz_0g'(\xi_0)\right)\rho_2+$$

$$+\frac{1}{2\pi i}\left[\int\limits_{\Gamma}\frac{g(t)-g(\xi_0)}{t-\xi_0}dt - z_0\rho_1\int\limits_{\Gamma}\frac{g'(t)-g'(\xi_0)}{t-\xi_0}dt+\right.$$

$$\left.+\rho_2\left(\frac{z_0^2}{2}\int\limits_{\Gamma}\frac{g''(t)-g''(\xi_0)}{t-\xi}dt-\frac{\sqrt{3}}{2}iz_0\int\limits_{\Gamma}\frac{g'(t)-g'(\xi_0)}{t-\xi_0}dt\right)\right]+$$

$$+\frac{1}{2\pi i}\frac{i}{2}\rho_2\int\limits_{\Gamma}\frac{(y_0-t_2)g'(t)}{t-\xi_0}dt. \tag{4.10}$$

Let us to prove the following equality:

$$\int\limits_{\Gamma}\frac{(y_0-t_2)g'(t)}{t-\xi_0}dt = \int\limits_{\Gamma}\frac{g(t)-g(\xi_0)}{t-\xi_0}dt_2+\int\limits_{\Gamma}\frac{\big(g(t)-g(\xi_0)\big)(y_0-t_2)}{(t-\xi_0)^2}dt. \tag{4.11}$$

For $\varepsilon > 0$ we consider the point $\xi'_\varepsilon \in \{t\in\Gamma : |t-\xi_0| = \varepsilon\}$ that is the first of the points going after ξ_0, and we consider the point $\xi''_\varepsilon \in \{t\in\Gamma : |t-\xi_0| = \varepsilon\}$ that is the last of the points going before ξ_0 with the given orientation of Γ. We consider also the arc $\Gamma^\varepsilon \subset \Gamma$ with the beginning point ξ'_ε and the end point ξ''_ε.

Now, we have

$$J := \int\limits_{\Gamma}\frac{g(t)-g(\xi_0)}{t-\xi_0}dt_2+\int\limits_{\Gamma}\frac{\big(g(t)-g(\xi_0)\big)(y_0-t_2)}{(t-\xi_0)^2}dt =$$

$$=\lim_{\varepsilon\to 0}\left(\int\limits_{\Gamma\setminus\Gamma_\varepsilon(\zeta_0)}\frac{g(t)-g(\xi_0)}{t-\xi_0}dt_2+\int\limits_{\Gamma\setminus\Gamma_\varepsilon(\zeta_0)}\frac{\big(g(t)-g(\xi_0)\big)(y_0-t_2)}{(t-\xi_0)^2}dt\right)=$$

$$=\lim_{\varepsilon\to 0}\left(\left(\int\limits_{\Gamma^\varepsilon}-\int\limits_{\Gamma^\varepsilon\cap\Gamma_\varepsilon(\zeta_0)}\right)\frac{g(t)-g(\xi_0)}{t-\xi_0}dt_2+\right.$$

$$+\Biggl(\int\limits_{\Gamma^\varepsilon}-\int\limits_{\Gamma^\varepsilon\cap\Gamma_\varepsilon(\zeta_0)}\Biggr)\frac{\bigl(g(t)-g(\xi_0)\bigr)(y_0-t_2)}{(t-\xi_0)^2}dt\Biggr). \tag{4.12}$$

In equality (4.12) the integrals along the set $\Gamma^\varepsilon\cap\Gamma_\varepsilon(\zeta_0)$ tend to 0 with $\varepsilon\to 0$ because the functions $(y_0-t_2)/(t-\xi_0)$ and $(g(t)-g(\xi_0))/(t-\xi_0)$ are bounded. For this reason

$$J=\lim_{\varepsilon\to 0}\Biggl(\int\limits_{\Gamma^\varepsilon}\frac{g(t)-g(\xi_0)}{t-\xi_0}dt_2+\int\limits_{\Gamma^\varepsilon}\frac{\bigl(g(t)-g(\xi_0)\bigr)(y_0-t_2)}{(t-\xi_0)^2}dt\Biggr). \tag{4.13}$$

Further, integrating the second integral in equality (4.13) by parts, we obtain

$$\int\limits_{\Gamma^\varepsilon}\frac{g(t)-g(\xi_0)}{t-\xi_0}dt_2+\int\limits_{\Gamma^\varepsilon}\frac{\bigl(g(t)-g(\xi_0)\bigr)(y_0-t_2)}{(t-\xi_0)^2}dt=$$

$$=\int\limits_{\Gamma\setminus\Gamma_\varepsilon}\frac{\bigl(g(t)-g(\xi_0)\bigr)\frac{dt_2}{dt}}{t-\xi_0}dt-\frac{\bigl(g(t)-g(\xi_0)\bigr)(y_0-t_2)}{t-\xi_0}\Biggr|_{t=\xi'_\varepsilon}^{t=\xi''_\varepsilon}+$$

$$+\int\limits_{\Gamma\setminus\Gamma_\varepsilon}\frac{g'(t)(y_0-t_2)-\bigl(g(t)-g(\xi_0)\bigr)\frac{dt_2}{dt}}{t-\xi_0}dt=$$

$$=-\frac{\bigl(g(t)-g(\xi_0)\bigr)(y_0-t_2)}{t-\xi_0}\Biggr|_{t=\xi'_\varepsilon}^{t=\xi''_\varepsilon}+\int\limits_{\Gamma\setminus\Gamma_\varepsilon}\frac{g'(t)(y_0-t_2)}{t-\xi_0}dt. \tag{4.14}$$

Now, passing to the limit in equality (4.13) when $\varepsilon\to 0$ and taking into account equality (4.14), we obtain equality (4.11).

Thus, it follows from equalities (4.9) and (4.11) that we can rewrite equality (4.10) in the form

$$\Phi^+(\zeta_0)=\widetilde{\varphi}(\zeta_0)+\frac{1}{2\pi i}\Biggl[\int\limits_{\Gamma}\frac{g(t)-g(\xi_0)}{t-\xi_0}dt-z_0\rho_1\int\limits_{\Gamma}\frac{g'(t)-g'(\xi_0)}{t-\xi_0}dt+$$

$$+\rho_2\Biggl(\frac{z_0^2}{2}\int\limits_{\Gamma}\frac{g''(t)-g''(\xi_0)}{t-\xi}dt-\frac{\sqrt{3}}{2}iz_0\int\limits_{\Gamma}\frac{g'(t)-g'(\xi_0)}{t-\xi_0}dt+$$

$$+\frac{i}{2}\int\limits_{\Gamma}\frac{g(t)-g(\xi_0)}{t-\xi_0}dt_2+\frac{i}{2}\int\limits_{\Gamma}\frac{\bigl(g(t)-g(\xi_0)\bigr)(y_0-t_2)}{(t-\xi_0)^2}dt\Biggr)\Biggr]. \tag{4.15}$$

Finally, taking into account the equality

$$(\tau-\zeta_0)^{-1}d\tau=\frac{dt}{t-\xi_0}+\frac{i}{2}\rho_2\Biggl(\frac{dt_2}{t-\xi_0}+\frac{y_0-t_2}{(t-\xi_0)^2}dt\Biggr)\qquad\forall\,\tau\in\Gamma'_\zeta\setminus\{\zeta_0\} \tag{4.16}$$

that follows from relations (4.2) and (4.3), it is easy to conclude that the right-hand sides of equalities (4.15) and (4.7) are equal.

Thus, equality (4.7) is proved. Equality (4.8) is proved similarly. □

In the following theorem in comparison with Theorem 4.1, we make an additional assumption that the curve Γ is quasiconformal (see, e. g., [7]) but the number of assumptions about the function g is reduced.

Theorem 4.2. *Let Γ be a closed rectifiable quasiconformal curve satisfying condition* (1.2) *and the function $g : \Gamma \to \mathbb{R}$ be twice continuously differentiable on Γ and, moreover, the modulus of continuity of the function g'' satisfy the condition of type* (1.3). *Then integral* (3.1) *has the limiting values $\Phi^+(\zeta_0)$ and $\Phi^-(\zeta_0)$ for all $\zeta_0 := x_0 + y_0e_2 + z_0e_3 \in \Sigma_\zeta$ that are expressed by formulas (4.7) and (4.8).*

Proof. It follows by Lemma 4 [4] that the function g and its contour derivative g' are Lipschitz functions if the quasiconformal curve Γ satisfies condition (1.2) and the modulus of continuity of the function g'' satisfies a condition of type (1.3). Consequently, the functions g and g' are absolutely continuous on Γ and, moreover, the modules of continuity of the functions g and g' satisfy conditions of type (1.3).

Now, to complete the proof, it suffices to apply Theorem 4.1. □

Note that integral (3.1) has a discontinuous jump on the surface Σ_ζ in Theorems 4.1 and 4.2, namely

$$\Phi^+(\zeta_0) - \Phi^-(\zeta_0) = \widetilde{\varphi}(\zeta_0).$$

The results of this paper have been announced in the preprint [11].

References

[1] N.A. Davydov, *The continuity of an integral of Cauchy type in a closed domain.* Doklady Akad. Nauk SSSR. 64, no. 6 (1949), 759–762 (in Russian).

[2] F.D. Gakhov, *Boundary Value Problems.* Pergamon Press, Oxford, 1966.

[3] O.F. Gerus, *Finite-dimensional smoothness of Cauchy-type integrals.* Ukr. Math. J. 29, no. 5 (1977), 490–493.

[4] O.F. Gerus, *On the modulus of continuity of solid derivatives of a Cauchy-type integral.* Ukr. Math. J. 50, no. 4, 539–549 (1998).

[5] P.W. Ketchum, *Analytic functions of hypercomplex variables.* Trans. Amer. Math. Soc. 30, no. 4 (1928), 641–667.

[6] K.S. Kunz, *Application of an algebraic technique to the solution of Laplace's equation in three dimensions.* SIAM J. Appl. Math. 21, no. 3 (1971), 425–441.

[7] O. Lehto, K.I. Virtanen, *Quasikonforme Abbildungen.* Springer–Verlag, Berlin – New York. 32, no. 995, 1965.

[8] I.P. Mel'nichenko, *The representation of harmonic mappings by monogenic functions.* Ukr. Math. J. 27, no. 5 (1975), 499–505.

[9] I.P. Mel'nichenko, *Algebras of functionally invariant solutions of the three-dimensional Laplace equation.* Ukr. Math. J. 55, no. 9 (2003), 1551–1559.

[10] I.P. Mel'nichenko, S.A. Plaksa, *Commutative algebras and spatial potential fields.* Inst. Math. NAS Ukraine, Kiev, 2008 (in Russian).

[11] S.A. Plaksa, V.S. Shpakivskyi, *On limiting values of Cauchy type integral in a harmonic algebra with two-dimensional radical.* Preprint 2010.4 of Inst. Math. NAS Ukraine, Kiev, 2010 (in Russian).

[12] I.I. Privalov, *Boundary properties of analytic functions.* 2nd ed., GITTL, Moscow–Leningrad, 1950 (in Russian).

[13] V.V. Salaev, *Direct and inverse estimates for a singular Cauchy integral along a closed curve.* Mat. Zametki. 19, no. 3 (1976), 365–380.

[14] V.S. Shpakivskyi, S.A. Plaksa, *Integral theorems and a Cauchy formula in a commutative three-dimensional harmonic algebra.* Bulletin Soc. Sci. et Lettr. Łódź. 60, no. 2 (2010), 47–54.

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