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COMPACT–ANALYTICAL PROPERTIES OF VARIATIONAL FUNCTIONAL IN SOBOLEV SPACES $W^{1,p}$

I.V. Orlov

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Abstract. In the work, conditions of well–definiteness, compact continuity, compact differentiability and multiple compact differentiability of the Euler–Lagrange one–dimensional variational functional in Sobolev–Bochner spaces $W^{1,p}([a;b],F)$ are obtained in terms of belonging of the integrand to the corresponding Weierstrass pseudopolynomial classes.

1 Introduction. Preliminaries

The remarkable I.V. Skrypnik theorem [15] states that a variational functional

$$\Phi(y) = \int_a^b f(x, y, y')dx, \quad (y(\cdot) \in W^{1,2}[a;b]) \tag{1.1}$$

is twice strongly differentiable only if the integrand $f(x, y, \cdot)$ is purely quadratic. Skrypnik's result initiates the problem on the natural smoothness related to variational functionals in Sobolev spaces $W^{1,p}[a;b]$. Solving it will enable us, in particular, to avoid applying the classical direct methods (see, e.g., [4]–[6], [17]) in extremal problems.

The main idea, based on this observation and developed by the author jointly with E. Bozhonok, first in the case of $W^{1,2}[a;b]$ (see [2], [7]–[11]) is to study compact–analytical functional characteristics considered on each subspace of $W^{1,p}[a;b]$ generated by an appropriate compact set. It turned out that variational functional (1.1) has the required characteristics in Sobolev spaces (e.g., K–continuity, K–differentiability, K–extremum, etc.), if some natural conditions on the integrand are satisfied.

The topological background of the well–definiteness of K–characteristics (see [10], [12]) is the possibility to present any Banach space as the inductive limit of its subspaces spanned by absolutely convex compacts (K–scale). K–characteristic of a functional is introduced as analogous local characteristic on each space in K–scale.

For instance, the functional Φ is K–continuous (K–differentiable, twice K–differentiable, etc.) in $W^{1,p}[a;b]$, if, for any absolutely convex compact $C \subset W^{1,p}[a;b]$, the restriction of Φ to $span\,(C)$ is continuous (Fréchet differentiable, twice Fréchet differentiable, etc.) with respect to the norm $\|\cdot\|_C$, generated by C. In addition, K–derivatives (multiple K–derivatives), being linear (multilinear) operators, are continuous in the usual sense.

The paper consists of four main sections. In Section 2 we introduce the class of K–pseudopolynomials $K_p(z)$ of order p, $1 \leqslant p < \infty$, and prove that belonging of the integrand f of the principal variational functional to this class guarantees well–definiteness of the functional in the corresponding Sobolev–Bochner space $W^{1,p}([a;b],F)$ (Theorem 2.1).

In Section 3 we introduce the class of Weierstrass pseudopolynomials $WK_p(z)$ of order p and prove K–continuity of the variational functional in $W^{1,p}([a;b],F)$ under the condition $f \in WK_p(z)$ (Theorem 3.1).

In Section 4 we introduce a more narrow class of the Weierstrass pseudopolynomials $W^1K_p(z)$ and prove K–differentiability of the variational functional in $W^{1,p}([a;b],F)$ under the condition $f \in W^1K_p(z)$ (Theorem 4.1).

Finally, in Section 5 we introduce the general Weierstrass classes $W^nK_p(z)$ and prove n–multiple K–differentiability of the variational functional under the condition $f \in W^nK_p(z)$. Moreover, the formula for the K–variation of order n is presented via the coefficients of the K–pseudopolynomial representation of f. A series of the examples is considered.

2 Pseudopolynomiality. Well–definiteness conditions for variational functionals in Sobolev spaces

As is well known [4], well–definiteness of the variational functionals in Sobolev spaces $W^{1,p}$

$$\Phi(y) = \int_a^b f(x,y,y')dx, \quad (y(\cdot) \in W^{1,p}[a;b],\ 1 \leqslant p < \infty) \tag{2.1}$$

is closely connected to an estimates of the integrand f via $|y'|^p$. However, the classical sufficient well–definiteness condition

$$|f(x,y,z)| \leqslant \alpha + \beta \cdot |z|^p \quad (\beta > 0)$$

essentially restricts the class of admissible integrands. We introduce a much wider class of K–pseudopolynomial in z integrands of order p for which functional (2.1) is well defined as well. Below, Z_k^* is the space of k–linear symmetric continuous real forms acting in a Banach space Z, $k \in \mathbb{N}_0$, $Z_0^* = \mathbb{R}$.

Definition 1. Let X, Y, Z be real Banach spaces; $D_x \subset X$, $D_y \subset Y$, $D_z \subset Z$ be open domains; $f : D_x \times D_y \times D_z \to \mathbb{R}$; $p \in \mathbb{N}_0$. We say that f is *a K–pseudopolynomial of order p*, if f admits a representation of the form:

$$f(x,y,z) = \sum_{k=0}^{p} R_k(x,y,z) \cdot (z)^k, \tag{2.2}$$

where the coefficients $R_k : D_x \times D_y \times D_z \to Z_k^*$ $(k = \overline{0,p})$ are Borel mappings satisfying the condition of *the dominating in x, y mixed boundedness* (see general definition of *the dominating mixed smoothness* in [14], [16]):

for any compacts $C_x \subset D_x$, $C_y \subset D_y$, the coefficients R_k are bounded on $C_x \times C_y \times D_z$, independently of the choice of $z \in D_z$.

In this case we write $f \in K_p(z)$.

Example 2. In case of $Z = \mathbb{R}^m$, $R_k(x, y, z) \cdot (z)^k$ are homogeneous K–pseudopolynomials in $z = (z_1, \dots z_m)$ of order k and representation (2.2) takes the form

$$f(x,y,z) = \sum_{k=0}^{p} \left(\sum_{k_1+\ldots+k_m=k} r_k(x,y,z) \cdot z_1^{k_1} z_2^{k_2} \cdot \ldots \cdot z_m^{k_m} \right) =$$

$$= \sum_{k_1+\ldots+k_m \leqslant p} a_{k_1\ldots k_m}(x,y,z) \cdot z_1^{k_1} z_2^{k_2} \cdot \ldots \cdot z_m^{k_m} \ .$$

Let us check that K–pseudopolynomiality of the integrand f guarantees well–definiteness of the principal variational functional in the corresponding Sobolev–Bochner space $W^{1,p}([a;b], F)$, where F is an arbitrary real Banach space. In what follows, $\|y\|_{W^{1,p}}$ is norm in $W^{1,p}([a;b], F)$, $\|R_k(x, y, y')\|$ is norm in Z_k^*.

Theorem 2.1. *If an integrand* $f : [a;b] \times F \times F \to \mathbb{R}$ *is in the class* $K_p(z)$, $p \in \mathbb{N}$, *then the variational functional*

$$\Phi(y) = \int_a^b f(x,y,y')dx, \quad (y(\cdot) \in W^{1,p}([a;b], F)) \tag{2.3}$$

is well defined in the space $W^{1,p}([a;b], F)$. *Moreover, for each compact* $C \subset W^{1,p}([a;b], F)$ *the following power estimate:*

$$|\Phi(y)| \leqslant \alpha_C + \beta_C \cdot (\|y\|_{W^{1,p}})^p \qquad (y \in C) \tag{2.4}$$

holds.

Proof. Let us fix $y(\cdot) \in W^{1,p}([a;b], F)$ and denote $C_y = y([a;b])$, a compact in F. Then, according to Definition 1, there exist such constants $M_k < \infty$, $(k = \overline{0,p})$ that

$$|R_0(x,y,y')| \leqslant M_0, \qquad \|R_k(x,y,y')\| \leqslant M_k \quad (x \in [a;b], \ \ k = \overline{1,p}). \tag{2.5}$$

Using the K–pseudopolynomial representation (2.2) for f leads to

$$\Phi(y) = \sum_{k=0}^{p} \int_a^b R_k(x,y,y') \cdot (y')^k dx \ . \tag{2.6}$$

From (2.6), taking into account the well–known properties of multilinear continuous forms [3] and the Hölder–Minkowski inequality [13], it follows that
a) for $k = 0$:

$$\left|\int_a^b R_0(x,y,y')dx\right| \leqslant \int_a^b |R_0(x,y,y')|dx \leqslant M_0 \cdot (b-a)\,; \tag{2.7}$$

b) for $1 \leqslant k \leqslant p$:

$$\left|\int_a^b R_k(x,y,y')\cdot(y')^k dx\right| \leqslant \int_a^b \|R_k(x,y,y')\|\cdot\|y'\|^k dx \leqslant M_k \cdot \int_a^b \|y'\|^k dx$$

$$\leqslant M_k \left(\int_a^b \left(\|y'\|^k\right)^{\frac{p}{k}} dx\right)^{\frac{k}{p}} \cdot \left(\int_a^b dx\right)^{\frac{p-k}{p}} \leqslant M_k(b-a)^{\frac{p-k}{p}} \cdot \left(\|y\|_{W^{1,p}}\right)^k\,. \tag{2.8}$$

From (2.5)–(2.8) it follows that

$$|\Phi(y)| = \left|\sum_{k=0}^{p}\int_a^b R_k(x,y,y')\cdot(y')^k dx\right|$$

$$\leqslant \sum_{k=0}^{p}\int_a^b \|R_k(x,y,y')\|\cdot\|y'\|^k dx \leqslant \sum_{k=0}^{p} M_k\cdot(b-a)^{\frac{p-k}{p}}\cdot\left(\|y\|_{W^{1,p}}\right)^k < \infty\,. \tag{2.9}$$

Thus, $|\Phi(y)| < \infty$, i.e. functional (2.3) is well defined everywhere on $W^{1,p}([a;b],F)$. Let us obtain now estimate (2.4), whose coefficients depends only on the choice of a compact $C \subset W^{1,p}([a;b],F)$. As C is a compact, the set

$$\widetilde{C} = \{y(x)\ \big|\ a \leqslant x \leqslant b,\ \ y(\cdot) \in C\} = \bigcup_{y\in C} C_y$$

is also compact in F. Since the coefficients $R_k(x,y,z)$ of representation (2.6) are bounded locally compactly in x, y and globally in z, the estimates of type (2.5) are also satisfied on the set $[a;b]\times\widetilde{C}\times F$:

$$|R_0(x,y,z)| \leqslant \widetilde{M_0}, \qquad \|R_k(x,y,z)\| \leqslant \widetilde{M_k} \quad (x\in[a;b],\ \ k=\overline{1,p}). \tag{2.10}$$

Using estimates (2.10), with the constants $\widetilde{M_k}$ depending only on the choice of a compact C, and estimate (2.9) leads to the inequality

$$|\Phi(y)| \leqslant A_C^0 + A_C^1\cdot\|y\|_{W^{1,p}} + \ldots + A_C^p\cdot\left(\|y\|_{W^{1,p}}\right)^p\ , \tag{2.11}$$

with the coefficients A_C^0, A_C^1, … A_C^p depending only on the choice of C too.

Since for $1 \leqslant k \leqslant p-1$

$$\Big(\|y\|_{W^{1,p}}\Big)^k \leqslant 1 + \Big(\|y\|_{W^{1,p}}\Big)^p$$

inequality (2.11) implies estimate (2.4). □

So, K–pseudopolynomiality of the integrand of variational functional (2.3) in the Sobolev–Bochner space $W^{1,p}([a;b],F)$, $p\in\mathbb{N}$, guarantees, besides well–definiteness of the functional, a power estimate of order p with respect to the Sobolev norm $\|y\|_{W^{1,p}}$ on each compact from the given Sobolev–Bochner space.

3 Compact–continuity conditions for variational functionals in Sobolev spaces

Let us pass to conditions of K–continuity of variational functionals in Sobolev–Bochner spaces. To this end, let us introduce an appropriate smoothness class $WK_p(z)$ of the K–pseudopolynomial integrands of order p.

Definition 2. Let, under the notation of Definition 1, the integrand f be continuous and belong to the class $K_p(z)$, $p\in\mathbb{N}$. We say that f is a *Weierstrass pseudopolynomial of order* p, briefly $f\in WK_p(z)$, if the coefficients R_k in K–pseudopolynomial representation (2.2) can be chosen in such a way that for arbitrary compacts $C_x\subset D_x$, $C_y\subset D_y$, they are uniformly continuous and bounded on $C_x\times C_y\times D_z$ (independently on the choice of z). In this case, introduce also notation $R_k\in W_K(z)$.

Let us prove now an important lemma which will be serve as a base for proofs of the consequent theorems on K–continuity and K–differentiability of variational functionals in $W^{1,p}$.

Basic Lemma. *Let the mappings*

$$\varphi:[a;b]\times F\to\mathbb{R},\quad \psi:[a;b]\to\mathbb{R}\quad (F \textit{ is a Banach space})$$

satisfy the following conditions:

i) $\varphi(x,u)=o(\|u\|^k)$, $0\leqslant k\leqslant p$, *as* $u\to 0$ *uniformly in* $x\in[a;b]$*;*

ii) $\psi\in L_1([a;b],\mathbb{R})$*;*

iii) the mapping $\chi(h)=\varphi(\cdot,h)\cdot\psi$ *is a continuous mapping from a compact* $C_1\subset L_p([a;b],F)$ *to* $L_1([a;b],\mathbb{R})$.

Then

$$\int\limits_a^b \underbrace{\varphi(x,h(x))\cdot\psi(x)}_{\chi(h)(x)}\,dx=o\left((\|h\|_{L_p})^k\right)\quad as\ \ \|h\|_{L_p}\to 0,\ \ h\in C_1\,. \tag{3.1}$$

Proof. 1) In view of continuity of the mapping χ, the set $\chi(C_1)\subset L_1([a;b],\mathbb{R})$ is compact. Therefore it is possible to apply the strengthened property of absolute continuity of the Lebesgue integral on functional compact (see, e.g. [16]) to integral (3.1). Denote, to this end, for every $N>0$,

$$E_N=\{x\in[a;b]\,\big|\,|\psi(x)|\leqslant N\},\ \ e_N=\{x\in[a;b]\,\big|\,|\psi(x)|>N\}.$$

Since $\psi \in L_1$ then $meas(e_N) \to 0$ as $N \to \infty$. Hence, according to the abovementioned property, there exists such $N > 0$ that

$$\left| \int\limits_{e_N} \chi(h) dx \right| \leqslant \frac{1}{2} \int\limits_a^b |\chi(h)| \, dx \,. \tag{3.2}$$

Because $[a;b] = E_N \dot{\bigcup} e_N$, it easily follows from (3.2) that

$$\left| \int\limits_a^b \chi(h) dx \right| \leqslant \frac{1}{1 - \frac{1}{2}} \int\limits_{E_N} |\chi(h)| \, dx = 2 \int\limits_{E_N} |\chi(h)| \, dx \,. \tag{3.3}$$

2) Fix $h \in C_1$ and, for arbitrary $\delta > 0$, set

$$E_{N\delta} = \{x \in E_N \,\big|\, \|h(x)\| < \delta\}, \;\; e_{N\delta} = \{x \in E_N \,\big|\, \|h(x)\| \geqslant \delta\}.$$

Evidently, $E_N = E_{N\delta} \dot{\bigcup} e_{N\delta}$, besides,

$$(\|h\|_{L_p} < \delta^{\frac{p+1}{p}}) \Rightarrow (meas \, e_{N\delta} < \delta) \,. \tag{3.4}$$

Indeed, assuming the contrary, we obtain

$$(\|h\|_{L_p})^p = \int\limits_a^b \|h(x)\|^p dx \geqslant \int\limits_{e_{N\delta}} \|h(x)\|^p dx \geqslant \delta^p \cdot meas \, e_{N\delta} \geqslant \delta^{p+1}.$$

Take advantage, again, of the strengthened property of absolute continuity of the Lebesgue integral for the integral in the right-hand side of (3.3). Taking into account (3.4), there exists such $\delta_1 > 0$ that

$$\left(\|h\|_{L_p} < \delta^{\frac{p+1}{p}} \right) \Rightarrow \left(meas \, e_{N\delta} < \delta \right) \Rightarrow \left(\int\limits_{e_{N\delta}} |\chi(h)| \, dx \leqslant \frac{1}{2} \int\limits_{E_N} |\chi(h)| \, dx \right). \tag{3.5}$$

By analogy with the case from part 1) of the proof, it easily follows from (3.5) that

$$\int\limits_{E_N} |\chi(h)| \, dx \leqslant 2 \cdot \int\limits_{E_{N\delta}} |\chi(h)| \, dx \,. \tag{3.6}$$

3) Let us fix now $\varepsilon > 0$ and, using property i) of the mapping φ, find such $\delta(\varepsilon) > 0$ that

$$|\varphi(x, u)| \leqslant \varepsilon \cdot \|u\|^k$$

for all $u \in F$ satisfying $\|u\| < \delta(\varepsilon)$ and $x \in [a;b]$. In particular,

$$|\varphi(x, h(x))| \leqslant \varepsilon \cdot \|h(x)\|^k \tag{3.7}$$

for $x \in E_{N\delta}$ and $\delta < \delta(\varepsilon)$.

4) Finally, it follows from (3.3), (3.6) and (3.7), for $\delta < \delta_1(\varepsilon) = \min(\delta_1, \delta(\varepsilon))$, that

$$\left|\int_a^b \chi(h)dx\right| \leqslant 2\cdot\int_{E_N} |\chi(h)|\,dx \leqslant 4\cdot\int_{E_{N\delta}} |\chi(h)|\,dx \leqslant 4N\cdot\int_{E_{N\delta}} |\varphi(x,h(x))|\,dx$$

$$\leqslant 4N\varepsilon\cdot\int_{E_{N\delta}} \|h\|^k dx \leqslant 4N\varepsilon\cdot\int_a^b \|h\|^k dx\,,$$

whence, using the Hölder–Minkowski inequality, we obtain

$$\left|\int_a^b \varphi(x,h(x))\cdot\psi(x)dx\right| \leqslant \left[4N\cdot(b-a)^{\frac{p-k}{p}}\right]\cdot\varepsilon\cdot\left(\|h\|_{L_p}\right)^k$$

for $\|h\|_{L_p} < (\delta_1(\varepsilon))^{\frac{p+1}{p}}$, $h \in C_1$. The last estimate implies the conclusion of the Lemma. □

Theorem 3.1. *If the integrand of variational functional* (2.3) *belongs to the Weierstrass class* $WK_p(z)$, $p \in \mathbb{N}$, *then functional* (2.3) *is* K*–continuous everywhere in the space* $W^{1,p}([a;b],F)$.

Proof. 1) Let us fix $y(\cdot) \in W^{1,p}([a;b],F)$ and an arbitrary absolutely convex compact $C \subset W^{1,p}([a;b],F)$. Use a canonical representation of the integrand f (2.2), where the coefficients $R_k : T = [a;b]\times F\times F \to F_k^*$, by condition $f \in WK_p(z)$, are uniformly continuous and bounded *dominantly in* x, y (i.e., locally compactly in x, y, and globally in z). Note that, by virtue of compactness of the set C in the space $W^{1,p}([a;b],F)$, the set

$$K^y := \bigcup_{h\in C}(y+h)([a;b])$$

is compact in F, too. Hence, on the set $T^y := [a;b]\times K^y\times F$ all the coefficient R_k are uniformly continuous and bounded. This implies the following estimates:

$$\|R_k(x,y,z)\| \leqslant M_k < \infty \qquad \left((x,y,z)\in T^y;\quad k=\overline{0,p}\right)\,. \tag{3.8}$$

Now, substituting representation (2.2) in (2.3), we find the increment of the variational functional Φ at the point $y(\cdot)$ for $h \in C$:

$$\Phi(y+h)-\Phi(y) = \int_a^b f(x,y+h,y'+h')dx - \int_a^b f(x,y,y')dx =$$

$$= \int_a^b \left[\sum_{k=0}^p R_k(x,y+h,y'+h')\cdot(y'+h')^k\right]dx - \int_a^b\left[\sum_{k=0}^p R_k(x,y,y')\cdot(y')^k\right]dx$$

$$= \sum_{k=0}^p \int_a^b \overbrace{\left[R_k(x,y+h,y'+h')\cdot(y'+h')^k - R_k(x,y,y')\cdot(y')^k\right]}^{\Delta_k}dx\,. \tag{3.9}$$

Let us fix k and transform the expression Δ_k :

$$\Delta_k = R_k(x, y+h, y'+h') \cdot \left(\sum_{l=0}^{k} C_k^l (h')^{k-l} (y')^l \right) - R_k(x, y, y') \cdot (y')^k$$

$$= \sum_{l=0}^{k-1} \underbrace{C_k^l R_k(x, y+h, y'+h') \cdot (h')^{k-l} \cdot (y')^l}_{A_{kl}}$$

$$+ \underbrace{\overbrace{[R_k(x, y+h, y'+h') - R_k(x, y, y')]}^{\Delta R_k} \cdot (y')^k}_{B_k} . \tag{3.10}$$

2) First let us estimate the integrals of A_{kl} $(l = \overline{0, k-1})$. Since, in view of (3.8),

$$\|A_{kl}\| \leqslant M_k \cdot \|h'\|^{k-l} \cdot \|y'\|^l ,$$

then

$$\left| \sum_{l=0}^{k-1} \int_a^b A_{kl} dx \right| \leqslant \sum_{l=0}^{k-1} C_k^l \cdot M_k \cdot \int_a^b \|h'\|^{k-l} \cdot \|y'\|^l dx . \tag{3.11}$$

Applying to the integrals in the right-hand side of in (3.11) the Hölder–Minkowski inequality [13] with $p_1 = p/(k-l)$ leads to the inequality

$$\left| \sum_{l=0}^{k-1} \int_a^b A_{kl} dx \right| \leqslant \sum_{l=0}^{k-1} C_k^l \cdot M_k \cdot \left(\int_a^b \|h'\|^p dx \right)^{\frac{k-l}{p}} \cdot \left(\int_a^b \|y'\|^{\frac{pl}{p-k+l}} dx \right)^{\frac{p-k+l}{p}}$$

$$\leqslant \sum_{l=0}^{k-1} C_k^l \cdot M_k \cdot (N_{kl})^l \cdot (\|y\|_{W^{1,p}})^l \cdot (\|h\|_{W^{1,p}})^{k-l} , \tag{3.12}$$

where, in view of $pl/(p-k+l) \leqslant p$, N_{kl} are constants in the inequality:

$$\|y\|_{W^{1,\frac{pl}{p-k+l}}} \leqslant N_{kl} \cdot \|y\|_{W^{1,p}} .$$

Finally, from (3.12) it follows that

$$\left| \sum_{l=0}^{k-1} \int_a^b A_{kl} dx \right| \to 0 \quad \text{as} \quad \|h\|_{W^{1,p}} \to 0, \quad h \in C_\Delta . \tag{3.13}$$

3) Now, using Basic Lemma, we estimate the integral of B_k. Since

$$\left| \int_a^b B_k dx \right| \leqslant \int_a^b |\Delta R_k \cdot (y')^k| dx \leqslant \int_a^b \|\Delta R_k\| \cdot \|y'\|^k dx ,$$

then, in the framework of Basic Lemma, this enables us to set

$$\varphi(x,u) = \|R_k(x, y(x)+u_1, y'(x)+u_2) - R_k(x, y(x), y'(x))\| \quad (u=(u_1,u_2)\in F\times F),$$

$$\psi(x) = \|y'(x)\|^k .$$

Let us check fulfillment of the conditions of Basic Lemma on the set T^0. Let

$$J(h) = (h, h'); \quad J : W^{1,p}([a;b], F) \to L_p([a;b], F\times F).$$

Since J is an isometry between the spaces above, then $J(C)$ is compact in $L_p([a;b], F\times F)$.

i) By uniform continuity of R_k on the set T^y,

$$\varphi(x,u) = o(1) = o(\|u\|^0) \text{ as } u=(u_1,u_2)\to 0, \ (u_1,u_2)\in J(C),$$

uniformly along $x\in[a;b]$.

ii) The function $\psi = \|y'\|^k \in L_1([a;b],\mathbb{R})$, as $y'\in L_p$, $k\leqslant p$.

iii) The mapping

$$\chi(h_1) = \|R_k(\cdot, y+h, y'+h') - R_k(\cdot, y, y')\|\cdot\|y'\|^k \quad (h_1=(h,h')\in J(C))$$

is a continuous mapping from $J(C)$ to $L_1([a;b],\mathbb{R})$, in view of continuity and boundedness of R_k and summability of $\|y'\|^k$. Thus, Basic Lemma is applicable, hence

$$\left|\int\limits_a^b B_k dx\right| \leqslant \int\limits_a^b \varphi(x, J(h))\cdot\psi(x)dx = o(1) \quad \text{as} \quad \|J(h)\|_{L_p} = \|h\|_{W^{1,p}} \to 0,\ h\in C. \tag{3.14}$$

4) Finally, from identities (3.9)–(3.10) and estimates (3.13)–(3.14) we obtain

$$\Phi(y+h)-\Phi(y) = \sum_{k=0}^{p}\left(\sum_{l=0}^{k-1}\int\limits_a^b A_{kl}dx + \int\limits_a^b B_k dx\right) \to 0 \quad \text{as} \quad \|h\|_{W^{1,p}}\to 0, \quad h\in C\,. \tag{3.15}$$

Since C is compact in $W^{1,p}([a;b],F)$, the norm $\|\cdot\|_C$ majorizes the norm $\|\cdot\|_{W^{1,p}}$, therefore condition (3.15) is fulfilled all the more as $\|h\|_C\to 0$. By virtue of the choice of C, this implies K–continuity of variational functional (2.3) at any point $y(\cdot)\in W^{1,p}([a;b],F)$. □

Remark 1. Thus, K–continuity of variational functional (2.3) is guaranteed by belonging of the integrand to the class of Weierstrass pseudopolynomials. Note that, in fact, Theo-
rem 3.1 states a stronger assertion, the usual continuity of all restrictions of functional (2.3) to subspaces *span*(C) (C is absolutely convex compact in $W^{1,p}([a;b],F)$) with the induced topologies. However these subspaces, in the infinite-dimensional case, are not closed.

It makes more suitable using the K–continuity (and further K–differentiability).

4 Compact–differentiability conditions for variational functionals in Sobolev spaces

Now we pass from the initial Weierstrass class $WK_p(z)$ that was introduced in section 2 to other Weierstrass class $W^1K_p(z)$, and show that belonging of the integrand to $W^1K_p(z)$ guarantees K–differentiability of the corresponding variational functional.

Definition 3. Let, under the notation of Definition 1, a functional

$$f : D_x \times D_y \times D_z \to \mathbb{R} \quad (D_x \subset X,\ D_y \subset Y,\ D_z \subset Z)$$

be K–pseudopolynomial of order p in z : $f \in K_p(z)$. We say that f is *a Weierstrass pseudopolynomial of the class* $W^1K_p(z)$ if $f \in C_1$ and f admits a K–pseudopolynomial representation of form (2.2), whose coefficients R_k have first order jets

$$(R_k, \nabla_{yz}R_k) \in W_K(z)\,. \quad \left(\text{Here } \nabla_{yz}R_k = \left(\frac{\partial R_k}{\partial y}, \frac{\partial R_k}{\partial z}\right)\right).$$

In this case, we write $R_k \in W^1_K(z)$. More in detail: for any compacts $C_x \subset D_x$, $C_y \subset D_y$, the jets $(R_k, \nabla_{yz}R_k)$ are uniformly continuous and bounded on $C_x \times C_y \times D_z$.

Let us give a simple example.

Example 3. Let $f(z) = R_p(z) \cdot (z)^p$ $(z \in \mathbb{R}^m,\ p \in \mathbb{N})$, in addition $R_p \in C^1$ and

$$\lim_{\|z\|\to\infty} R_p(z) = \lim_{\|z\|\to\infty} R'_p(z) = 0\,.$$

Then R_p and R'_p are continuous mappings having zero limits in infinity, whence their uniform continuity and boundedness globally in z follow. Hence, $f \in W^1K_p(z)$. An evident generalization is:

$$f(x,y,z) = R_p(x,y,z) \cdot (z)^p \quad (z \in \mathbb{R}^m,\ p \in \mathbb{N}),$$

where $R_p \in C^1$ and

$$\lim_{\|z\|\to\infty} R_p(z) = \lim_{\|z\|\to\infty} \nabla_{yz}R_p(z) = 0$$

dominantly in x, y.

Let us prove now K–differentiability in space $W^{1,p}$ of the principal variational functional with the integrand in $W^1K_p(z)$. The proof is again based on Basic Lemma.

Theorem 4.1. *Let the integrand f of variational functional* (2.3) *in the space $W^{1,p}([a;b],F)$ belong to the Weierstrass class $W^1K_p(z)$, $p \in \mathbb{N}$. Then Euler–Lagrange functional* (2.3) *is K–differentiable everywhere in $W^{1,p}([a;b],F)$. In addition, there holds the classical first variation formula*

$$\Phi'_K(y)h = \int_a^b \left[\frac{\partial f}{\partial y}(x,y,y')h + \frac{\partial f}{\partial z}(x,y,y')h'\right] dx\,. \tag{4.1}$$

By using pseudopolynomial representation (2.2) *of the integrand* f, *equality* (4.1) *takes the form*

$$\Phi'_K(y)h = \sum_{k=0}^{p} \int_a^b \left[\nabla_{yz} R_k(x,y,y') \cdot (h,h') \cdot (y')^k + R_k(x,y,y') \cdot (h') \cdot k(y')^{k-1}\right] dx\,. \tag{4.2}$$

Proof. 1) Fix $y(\cdot) \in W^{1,p}([a;b],F)$ and an arbitrary absolutely convex compact $C \subset W^{1,p}([a;b],F)$ in the given space. Note that the coefficients R_k, according to condition $f \in W^1K_p(z)$, have first order jets $(R_k, \nabla_{yz}R_k)$ which are uniformly continuous and bounded in T locally compactly in x, y and globally in z (i.e., dominantly in x, y).

As it was noticed in a similar situation in the proof of Theorem 3.1 (part 1), the numerial set K^y is compact, by virtue of compactness of C. Hence, all the jets $(R_k, \nabla_{yz}R_k)$ are uniformly continuous and bounded on the set T^y. From here, in particular, the estimates

$$\|R_k(x,y,z)\| \leqslant M_k < \infty, \quad \|\nabla_{yz}R_k(x,y,z)\cdot(h,h')\| \leqslant M_{k1} < \infty$$
$$\big((x,y,z) \in T^y,\ h \in C,\ k = \overline{0,p}\big) \tag{4.3}$$

follow. Next we use equalities (3.9)–(3.10) from the proof of Theorem 3.1. Let us transform the expression Δ_k, taking into account that

$$R_k(x,y+u_1,z+u_2) - R_k(x,y,z) = \nabla_{yz}R_k(x,y,z)\cdot(u_1,u_2) + r_k(x,y,z;u_1,u_2)\cdot(u_1,u_2), \tag{4.4}$$

where

$$\|r_k(x,y,z;u_1,u_2)\| \to 0 \ \text{ as } \ \|(u_1,u_2)\| \to 0\,.$$

Substituting (4.4) in (3.10) and separating the last term of the sum in (3.10) (for $k \geqslant 2$) we obtain

$$\begin{aligned}\Delta_k = \sum_{l=0}^{k-2} &\underbrace{C_k^l \cdot R_k(x,y+h,y'+h') \cdot (h')^{k-l} \cdot (y')^l}_{A_{kl}} \\ &+ \underbrace{k \cdot [R_k(x,y+h,y'+h') - R_k(x,y,y')] \cdot (h') \cdot (y')^{k-1}}_{B_k} \\ &+ \underbrace{\nabla_{yz}R_k(x,y,y')\cdot(h,h')\cdot(y')^k}_{C_k} + \underbrace{r_k(x,y,y';h,h')\cdot(h,h')\cdot(y')^k}_{D_k} \\ &+ \underbrace{k \cdot R_k(x,y,y') \cdot (h') \cdot (y')^{k-1}}_{E_k}\ .\end{aligned} \tag{4.5}$$

Let us estimate the integrals of every summand in expression (4.5).

2) Using estimate (3.1) for the integrals of A_{kl}, which was obtained in the proof of Theorem 3.1

$$\left|\sum_{l=0}^{k-2}\int_a^b A_{kl}dx\right| \leqslant \sum_{l=0}^{k-2} C_k^l \cdot M_{k0}\cdot \int_a^b \|h'\|^{k-l}\cdot \|y'\|^l dx \tag{4.6}$$

and applying to the integrals in the right-hand side of (4.6) the Hölder–Minkowski inequality with $p_{kl} = p/(k-l)$ leads to

$$\left|\sum_{l=0}^{k-2}\int_a^b A_{kl}dx\right| \leqslant \sum_{l=0}^{k-2} C_k^l \cdot M_{k0}\cdot \left(\int_a^b \|h'\|^p dx\right)^{\frac{k-l}{p}} \cdot \left(\int_a^b \|y'\|^{\frac{lp}{p-k+l}} dx\right)^{\frac{p-k+l}{p}}$$

$$\leqslant \sum_{l=0}^{k-2} C_k^l\cdot M_{k0}\cdot(\|h\|_{W^{1,p}})^{k-l}\cdot(\|y\|_{W^{1,\frac{lp}{p-k+l}}})^l \leqslant \sum_{l=0}^{k-2} C_k^l\cdot M_{k0}\cdot(N_{kl}^p)^l\cdot(\|y\|_{W^{1,p}})^l\cdot(\|h\|_{W^{1,p}})^{k-l}, \tag{4.7}$$

where N_{kl}^p are the constants in the inequality

$$\|y\|_{W^{1,\frac{lp}{p-k+l}}} \leqslant N_{kl}^p \cdot \|y\|_{W^{1,p}} . \tag{4.8}$$

We took into account that $\frac{lp}{p-k+l} \leqslant p$ since $k-l \geqslant 2$, it follows immediately from (4.7) that

$$\left|\sum_{l=0}^{k-2}\int_a^b A_{kl}dx\right| = o(\|h\|_{W^{1,p}}) \quad \text{as} \quad \|h\|_{W^{1,p}} \to 0, \quad h \in C_\Delta . \tag{4.9}$$

3) Now, let us estimate the integral of B_k in (4.5) by using Basic Lemma. First of all,

$$\left|\int_a^b B_k dx\right| \leqslant k\cdot \int_a^b \underbrace{\|R_k(x,y+h,y'+h') - R_k(x,y,y')\|}_{\Delta R_k}\cdot\|h'\|\cdot\|y'\|^{k-1}dx .$$

This enables us, in the framework of Basic Lemma, to set

$$\varphi(x,u) = \|R_k(x,y(x)+u_1,y'(x)+u_2) - R_k(x,y(x),y'(x))\|\cdot\|u_2\| \quad (u=(u_1,u_2)\in F\times F),$$

$$\psi(x) = k\cdot\|y'(x)\|^{k-1} .$$

Let us check fulfillment of the conditions of Basic Lemma.

i) By virtue of uniform continuity of R_k on the set T^y,

$$\varphi(x,u) = o(\|u_2\|) = o(\|(u_1,u_2)\|) \text{ as } \|u\| = \|(u_1,u_2)\| \to 0$$

uniformly in $x \in [a;b]$, $u \in J(C)$.

ii) The function $\psi = k\cdot\|y'\|^{k-1} \in L_1([a;b],\mathbb{R})$ in view of $y' \in L_p$ and $k-1 \leqslant p$.

iii) The mapping

$$\chi(h_1) = \|R_k(\cdot,y+h,y'+h') - R_k(\cdot,y,y')\|\cdot\|h'\|\cdot k\|y'\|^{k-1} \quad (h_1 = (h,h') = Jh)$$

is a continuous mapping from $J(C)$ to $L_1([a;b],\mathbb{R})$, in view of continuity and boundedness of R_k, continuity of the mapping $h_1=(h,h')\mapsto\|h'\|$ and summability of the product $\|h'\|\cdot\|y'\|^{k-1}$. Thus, Basic Lemma is applicable, whence

$$\left|\int\limits_a^b B_k dx\right| \leqslant \int\limits_a^b \varphi(x,h_1)\cdot\psi(x)dx$$

$$= \int\limits_a^b \|\Delta R_k\|\cdot\|h'\|\cdot\|y'\|^{k-1}dx = o(\|(h,h')\|_{L_p}) = o(\|h\|_{W^{1,p}}) \tag{4.10}$$

as $\|h\|_{W^{1,p}}\to 0,\ h\in C$.

4) Next, let us estimate the integral of C_k in (4.5) with the help of the Hölder–Minkowski inequality. Using estimate (4.4) leads to

$$\left|\int\limits_a^b C_k dx\right| \leqslant \int\limits_a^b |\nabla_{yz}R_k(x,y,y')\cdot(h,h')|\cdot\|y'\|^k dx \leqslant M_{k1}\cdot\int\limits_a^b \|y'\|^k dx$$

$$\leqslant M_{k1}\cdot(b-a)^{\frac{p-k}{p}}\cdot(\|y'\|_{L_p})^k \leqslant \left[M_{k1}\cdot(b-a)^{\frac{p-k}{p}}\right]\cdot(\|y\|_{W^{1,p}})^k < \infty.$$

So, $\int\limits_a^b C_k dx$ is a bounded linear functional in h on the subspace $span(C)$ with respect to the norm $\|\cdot\|_C$. By virtue of arbitrariness of the choice of $C\subset W^{1,p}([a;b],F)$, this implies K–continuity of the functional and the last property, in view of linearity of the functional (see [8]) is equivalent to its usual continuity on the space $W^{1,p}([a;b],F)$.

5) Next, let us estimate the integral of D_k in (4.5) with the help of Basic Lemma. First of all,

$$\left|\int\limits_a^b D_k dx\right| \leqslant \int\limits_a^b |r_k(x,y,y';h,h')\cdot(h,h')|\cdot\|y'\|^k dx.$$

Note also that, in view of continuous differentiability of R_k and compactness of C,

$$|r_k(x,y,y';h,h')\cdot(h,h')| = o(\|(h,h')\|) \tag{4.11}$$

uniformly in $x\in[a;b]$. This enables us, in the framework of Basic Lemma, to set

$$\varphi(x,u) = |r_k(x,y,y';u_1,u_2)\cdot(u_1,u_2)| \quad (u=(u_1,u_2)\in F\times F),$$

$$\psi(x) = \|y'(x)\|^k .$$

Let us check fulfillment of the conditions of Basic Lemma.

i) From estimate (4.11) it follows immediately that

$$\varphi(x,u) = o(\|u\|) \text{ as } \|u\|\to 0, \text{ uniformly in } x\in[a;b]$$

ii) The function $\psi=\|y'\|^k\in L_1([a;b],\mathbb{R})$ in view of $y'\in L_p$ and $k\leqslant p$.

iii) The mapping

$$\chi(h_1) = |r_k(x,y,y';h,h') \cdot (h,h')| \cdot \|y'\|^k \quad (h_1 = Jh = (h,h') \in J(C)) \tag{4.12}$$

is a continuous mapping from $J(C)$ to $L_1([a;b],\mathbb{R})$ in view of continuity in $h_1 = (h,h')$, boundedness of the first multiple from the right in (4.12) and summability of the second multiple therein. Thus, Basic Lemma is applicable, whence

$$\left|\int_a^b D_k dx\right| \leqslant \int_a^b \varphi(x,h_1)\cdot\psi(x)dx = \int_a^b |r_k(x,y,y';h,h')\cdot(h,h')|\cdot\|y'\|^k dx$$

$$= o(\|(h,h')\|_{L_p}) = o(\|h\|_{W^{1,p}}) \text{ as } \|h\|_{W^{1,p}} \to 0,\ h \in C\,. \tag{4.13}$$

6) Finally, let us estimate the integral of E_k in (4.5) using the first estimate in (4.3) and the Hölder–Minkowski inequality. Taking into account that $\|h'\|\cdot\|y'\|^{k-1} \in L_1$, we obtain

$$\left|\int_a^b E_k dx\right| \leqslant k\cdot\int_a^b \|R_k(x,y,y')\|\cdot\|h'\|\cdot\|y'\|^{k-1}dx \leqslant kM_{k0}\cdot\int_a^b \|h'\|\cdot\|y'\|^{k-1}dx$$

$$\leqslant k\cdot M_{k0}\cdot\|h'\|_{L_p}\cdot(\|y'\|_{L_p})^{k-1} \leqslant \left(k\cdot M_{k0}\cdot(\|y\|_{W^{1,p}})^{k-1}\right)\cdot\|h\|_{W^{1,p}}\,.$$

Thus, $\int_a^b E_k dx$ is a bounded linear functional in h on the subspace $span(C)$ with respect to the norm $\|\cdot\|_{W^{1,p}}$ and all the more with respect to the norm $\|\cdot\|_C$. Continuity of the functional on the whole space $W^{1,p}([a;b],F)$ follows from here, by analogy with part 4) of the proof.

7) So, from the obtained estimates (4.9)–(4.10), (4.13) and the conclusions of parts 4)–6) of the proof it follows that

$$\int_a^b \Delta_k dx = \int_a^b \left[\nabla_{yz}R_k(x,y,y')\cdot(h,h')\cdot(y')^k + kR_k(x,y,y')\cdot(h')\cdot(y')^{k-1}\right]dx$$

$$+o(\|h\|_{W^{1,p}}), \tag{4.14}$$

where the integral functional in the right-hand side of (4.14) is continuous. Since, in view of compactness of C, the norm $\|\cdot\|_C$ majorizes the norm $\|\cdot\|_{W^{1,p}}$ in $span(C)$ then the second term in the right-hand side of (4.14) is $o(\|h\|_C)$.

Therefore, by summing up equalities (4.14) in $k = \overline{0,p}$ we arrive at K–differentiability of Φ and equality (4.2).

8) Finally, let us show that equality (4.2) can be transformed to standard form (4.1). Using K–pseudopolynomial representation (2.2) leads to

$$\frac{\partial f}{\partial y}(x,y,y')h = \frac{\partial R_0}{\partial y}(x,y,y')\cdot h + \sum_{k=1}^{p}\frac{\partial R_k}{\partial y}(x,y,y')\cdot h\cdot(y')^k\ ,$$

$$\frac{\partial f}{\partial z}(x,y,y')h' = \frac{\partial R_0}{\partial z}(x,y,y')\cdot h'$$

$$+\sum_{k=1}^{p}\Big[\frac{\partial R_k}{\partial z}(x,y,y')\cdot(h')\cdot(y')^k + R_k(x,y,y')\cdot(h')\cdot k(y')^{k-1}\Big].$$

From here it follows that

$$\nabla_{yz} f(x,y,y')(h,h') = \frac{\partial f}{\partial y}(x,y,y')\cdot h + \frac{\partial f}{\partial z}(x,y,y')\cdot h'$$

$$= \Big[\frac{\partial R_0}{\partial y}(x,y,y')\cdot h + \frac{\partial R_0}{\partial z}(x,y,y')\cdot h'\Big] + \sum_{k=1}^{p}\Big[\Big(\frac{\partial R_k}{\partial y}(x,y,y')\cdot(h)\cdot(y')^k$$

$$+\frac{\partial R_k}{\partial z}(x,y,y')\cdot(h')\cdot(y')^k\Big) + R_k(x,y,y')\cdot(h')\cdot k(y')^{k-1}\Big] = \nabla_{yz}R_0(x,y,y')\cdot(h,h')$$

$$+\sum_{k=1}^{p}\Big[\nabla_{yz}R_k(x,y,y')\cdot(h,h')\cdot(y')^k + R_k(x,y,y')\cdot(h')\cdot k(y')^{k-1}\Big]$$

$$= \sum_{k=0}^{p}\Big[\nabla_{yz}R_k(x,y,y')\cdot(h,h')\cdot(y')^k + R_k(x,y,y')\cdot(h')\cdot k(y')^{k-1}\Big],$$

and it is none other than the integrand in the right-hand side of (4.2). □

In the conclusion of this section we give examples of some classes of integrands which are enclosed by Theorem 4.1.

Example 4. Some types of integrands in the Weierstrass class $W^1K_p(z)$.

1) Let

$$f(x,y,z) = \sum_{k=0}^{p} R_k(x,y)\cdot(z)^k \quad (R_k \in C^1_{xy}).$$

Here independence R_k of z automatically implies that $R_k \in W^1_K(z)$, whence $f \in W^1K_p(z)$.

2) Let us generalize the preceding example. Let

$$f(x,y,z) = \sum_{k\in\mathcal{K}} R_k(x,y)\cdot(z)^k + \sum_{k'\in\mathcal{K}'} R_{k'}(x,y,z)\cdot(z)^{k'} \quad (\mathcal{K}\dot\cup\mathcal{K}' = \{\overline{0,p}\}),$$

where $R_k \in C^1_{xy}$ for $k\in\mathcal{K}$, $R_{k'} \in W^1_K(z)$ for $k'\in\mathcal{K}'$. Then $f \in W^1K_p(z)$ as well.

3) Let

$$f(x,y,z) = \sum_{k=0}^{p} \varphi_k(\underbrace{r_k(x,y,z)}_{t})\cdot(z)^k \quad (\varphi_k \in C^1_t,\ \ r_k \in W^1_K(z)).$$

Then, obviously, $R_k = \varphi_k(r_k) \in W_K(z)$ and

$$\nabla_{yz}R_k = \frac{d\varphi_k}{dt}\cdot\nabla_{yz}r_k \in W_K(z)\,,$$

whence it follows that $f \in W^1K_p(z)$.

5 General Weierstrass classes. Multiple K–differentiability conditions for variational functionals in Sobolev spaces

To pass to the high order K–derivatives of variational functionals we need a corresponding generalization of the Weierstrass classes.

Definition 4. Let, under the notation of Definition 1, a functional $f : D_x \times D_y \times D_z \to \mathbb{R}$ be K–pseudopolynomial of order p: $f \in K_p(z)$, $p \in \mathbb{N}$, where $f \in C^n(D_x \times D_y \times D_z)$, $n \in \mathbb{N}_0$. We say that f belongs to the Weierstrass class $W^n K_p(z)$, if there exists a K–pseudopolynomial representation (2.2) whose all coefficients R_k have n–th order jets in y, z

$$(R_k, \nabla_{yz} R_k, \ldots, \nabla^n_{yz} R_k) \quad (k = \overline{0, p}) \tag{5.1}$$

in the Weierstrass class $W_K(z)$. In this case, we write $R_k \in W^n_K(z)$.

In the other words, the coefficients $R_k(x, y, z)$ of representation 2.2 have dominating (in x, y) mixed smoothness of order n. More explicitly: for any compacts $C_x \subset D_x$, $C_y \subset D_y$, jets (5.1) are uniformly continuous and bounded on $C_x \times C_y \times D_z$.

Remark 2. Obviously,

$$W^0 K_p(z) = W K_p(z),\ W^n K_p(z) \subset W^{n-1} K_p(z);$$

$$W^0_K(z) = W_K(z),\ W^n_K(z) \subset W^{n-1}_K(z).$$

Let us give a simple example generalizing Example 3.

Example 5. Let $f(z) = R_p(z) \cdot (z)^p$ $(z \in \mathbb{R}^m,\ p \in \mathbb{N})$, in addition $R_p \in C^n$ and

$$\lim_{\|z\| \to \infty} R_p(z) = \lim_{\|z\| \to \infty} R'_p(z) = \ldots = \lim_{\|z\| \to \infty} R^{(n)}_p(z) = 0\,. \tag{5.2}$$

Then R_p, $R'_p, \ldots, R^{(n)}_p$ are continuous mappings with zero limits at infinity, whence their uniform continuity and boundedness globally in z follow. Hence, $f \in W^n K_p(z)$. In particular, any rapidly decreasing function $R_p \in \mathcal{S}(\mathbb{R}^m)$ satisfies conditions (5.2).

An evident generalization is:

$$f(x, y, z) = R_p(x, y, z) \cdot (z)^p \quad (z \in \mathbb{R}^m,\ p \in \mathbb{N}),$$

where $R_p \in C^n$ and

$$\lim_{\|z\| \to \infty} R_p(z) = \lim_{\|z\| \to \infty} \nabla_{yz} R_p(z) = \ldots = \lim_{\|z\| \to \infty} \nabla^n_{yz} R_p(z) = 0$$

dominantly in x, y.

Remark 3. It is easy to see that $f \in W^n K_p(z)$ implies that $(\partial^{s+t} f / \partial y^s \partial z^t) \in W^{n-s-t} K_p(z)$ with $s + t \leqslant n$ and

$$\frac{\partial^{s+t} f}{\partial y^s \partial z^t}(x, y, z) = \sum_{k=0}^{p} R^{s,t}_k(x, y, z) \cdot (z)^k\,, \tag{5.3}$$

where $R^{s,t}_k \in W^{n-s-t}_K(z)$.

Now, let us show that belonging of an integrand to the Weierstrass class $W^n K_p(z)$ guarantees n–multiple K–differentiability of the principal variational functional.

Theorem 5.1. *Let the integrand f of variational functional* (2.3) *belong to the Weierstrass class $W^n K_p(z)$, $(p \in \mathbb{N}, n \in \mathbb{N})$. Then Euler–Lagrange functional* (2.3) *is n times K–differentiable everywhere in $W^{1,p}([a;b], F)$. In addition, there holds the following n–th variation formula*

$$\Phi_K^{(n)}(y) \cdot (h_1, \dots, h_n) = \int_a^b \left[\sum_{l=0}^{n} \frac{\partial^n f}{\partial y^{n-l} \partial z^l}(x, y, y') \cdot \left(\sum_{i=(i_1,\dots,i_n):|i|=l\ (i_s=0,1)} (h_1^{(i_1)}, h_2^{(i_2)}, \dots, h_n^{(i_n)}) \right) \right] dx. \tag{5.4}$$

In particular, on the diagonal $h_1 = h_2 = \dots = h_n = h$ K—derivative $\Phi_K^{(n)}(y)$ takes the form

$$\Phi_K^{(n)}(y) \cdot (h)^n = \int_a^b \left[\sum_{l=0}^{n} C_n^l \cdot \frac{\partial^n f}{\partial y^{n-l} \partial z^l}(x, y, y') \cdot (h)^{n-l} \cdot (h')^l \right] dx. \tag{5.5}$$

Moreover, using K–pseudopolynomial representation (2.2) *leads to the formula*

$$\Phi_K^{(n)}(y) \cdot (h)^n = \sum_{k=0}^{p} \int_a^b \left[\sum_{l=0}^{n-1} C_{n-1}^l \cdot \Big(\nabla_{yz} R_k^{n-1-l,l}(x, y, y') \cdot (h, h') \cdot (y')^k \right.$$

$$\left. + R_k^{n-1-l,l}(x, y, y') \cdot (h') \cdot k(y')^{k-1} \Big) \cdot (h)^{n-1-l} \cdot (h')^l \right] dx\,. \tag{5.6}$$

Proof. Let us carry out the proof by induction. For $n = 1$ formula (5.6) reduces to the proved above formula for the first K–variation (4.1) together with its K–pseudopolynomial variant (4.2).

Suppose by the inductive hypothesis, that for a given n formula (5.6) holds, under the hypotheses of the theorem. Let us prove now analogous equality of $(n+1)$–th order, under the assumption $f \in W^{n+1} K_p(z)$. Call to mind that [3] a symmetric n–form is uniquely defined by its diagonal values (under preserving continuity).

1) Like in the proof of Theorem 4.1, let us fix $y(\cdot) \in W^{1,p}([a;b], F)$ and an arbitrary absolutely convex compact $C \subset W^{1,p}([a;b], F)$.

In this case, according to supposition $f \in W^{n+1} K_p(z)$ the $(n+1)$–th order jets in y, z of the coefficients $R_k : T = [a;b] \times F \times F \to F_k^*$,

$$(R_k, \nabla_{yz} R_k, \dots, \nabla_{yz}^{n+1} R_k) \qquad (k = \overline{0, p}) \tag{5.7}$$

are uniformly continuous and bounded on T locally compactly in x, y and globally in z.

As it was noticed already above (in the proofs of Theorems 3.1 and 4.1), in view of compactness of the set $(y + C)$ the numerical set K^y is compact as well. Hence, on the set $T^y = [a;b] \times K^y \times F$ all the jets (5.7) are bounded and uniformly continuous. From here, in particular, there follow the estimates

$$\left\| \tfrac{\partial^{s+t} R_k}{\partial y^s \partial z^t}(x,y,z) \cdot (h)^s (h')^t \right\| \leqslant M^k_{st} < \infty \quad \big((x,y,z) \in T^{y,\Delta} \\ h \in C_\Delta,\ s+t \leqslant n+1,\ k = \overline{0,p}\big)\,. \tag{5.8}$$

From estimates (5.8), Remark 3, and representation (5.3) there also follow the estimates

$$\left\| R_k^{n-l,l}(x,y,z) \cdot (h)^{n-l}(h')^l \right\| \leqslant M^{k0}_{n-l,l} < \infty; \tag{5.9}$$

$$\left\| \nabla_{yz} R_k^{n-l,l}(x,y,z) \cdot (h,h') \cdot (h)^{n-l}(h')^l \right\| \leqslant M^{k1}_{n-l,l} < \infty; \tag{5.10}$$

where $(x,y,z) \in T^y$, $h \in C$, $s+t \leqslant n+1$, $k = \overline{0,p}$.

2) Using the inductive hypothesis and representation (5.5), write the increment of the functional $\Phi_K^{(n)}(y)$ on a diagonal multivector $(h)^n$ in the form

$$[\Phi_K^{(n)}(y+h) - \Phi_K^{(n)}(y)] \cdot (h)^n$$

$$= \sum_{l=0}^{n} C_n^l \cdot \int_a^b \underbrace{\left[\frac{\partial^n f}{\partial y^{n-l} \partial z^l}(x, y+h, y'+h') - \frac{\partial^n f}{\partial y^{n-l} \partial z^l}(x,y,y') \right]}_{\Delta f^{n-l,l}} \cdot (h)^{n-l} \cdot (h')^l dx\,. \tag{5.11}$$

Let us transform the integrands in the right-hand side of (5.11), using representation (5.3) and expansion into principal and small parts

$$R_k^{n-l,l}(x, y+h, y'+h') - R_k^{n-l,l}(x,y,y') = \Delta R_k^{n-l,l}$$

$$= \nabla_{yz} R_k^{n-l,l}(x,y,y') \cdot (h,h') + \nabla_{yz} r_k^{n-l,l}(x,y,y';h,h') \cdot (h,h').$$

From here it follows that

$$\Delta f^{n-l,l} = \sum_{k=0}^{p} \Delta \left(R_k^{n-l,l}(x,y,y') \cdot (y')^k \right) \cdot (h)^{n-l} \cdot (h')^{n-l}$$

$$= \sum_{k=0}^{p} \Bigg[R_k^{n-l,l}(x, y+h, y'+h') \cdot \Big((y')^k$$

$$+ k(y')^{k-1} \cdot (h') + \sum_{m=0}^{k-2} C_k^m \cdot (y')^m \cdot (h')^{k-m} \Big) - R_k^{n-l,l}(x,y,y') \cdot (y')^k \Bigg] \cdot (h)^{n-l} \cdot (h')^l$$

$$= \sum_{k=0}^{p} \Bigg[\Big(R_k^{n-l,l}(x,y,y') + \nabla_{yz} R_k^{n-l,l}(x,y,y') \cdot (h,h') + \nabla_{yz} r_k^{n-l,l}(x,y,y';h,h') \cdot (h,h') \Big) \cdot$$

$$\cdot \Big((y')^k + k(y')^{k-1} \cdot (h') + \sum_{m=0}^{k-2} C_k^m \cdot (y')^m \cdot (h')^{k-m} \Big) - R_k^{n-l,l}(x,y,y') \cdot (y')^k \Bigg] \cdot (h)^{n-l} \cdot (h')^l$$

$$= \sum_{k=0}^{p} \Big[\underbrace{\nabla_{yz} R_k^{n-l,l}(x,y,y') \cdot (h,h') \cdot (h)^{n-l} \cdot (h')^l \cdot (y')^k}_{A_{kl}}$$

$$+ \underbrace{R_k^{n-l,l}(x,y,y') \cdot (h)^{n-l} \cdot (h')^{l+1} \cdot k(y')^{k-1}}_{B_{kl}} \Big] + \sum_{k=0}^{p} \Big[\underbrace{\Delta R_k^{n-l,l} \cdot (h)^{n-l} \cdot (h')^{l+1} \cdot k(y')^{k-1}}_{C_{kl}}$$

$$+ \underbrace{\nabla_{yz} r_k^{n-l,l}(x,y,y';h,h') \cdot (h,h') \cdot (h)^{n-l} \cdot (h')^l \cdot (y')^k}_{D_{kl}}$$

$$+ \sum_{m=0}^{k-2} \underbrace{C_k^m \cdot R_k^{n-l,l}(x,y+h,y'+h') \cdot (h)^{n-l} \cdot (h')^{l+k-m} \cdot (y')^m}_{E_{klm}} \Big] . \tag{5.12}$$

Here in the first square brackets from the right the principal terms of the decomposition, A_{kl} and B_{kl}, are collected, in the second square brackets from the right the small terms of the decomposition, C_{kl}, D_{kl} and D_{klm}, are collected. Let us estimate the integrals of each of the summands in (5.12).

4) First, let us estimate the integrals of A_{kl} and B_{kl}.

a) Note that the functional in h,

$$\int_a^b A_{kl} dx = \int_a^b \nabla_{yz} R_k^{n-l,l}(x,y,y') \cdot (h,h') \cdot (h)^{n-l} \cdot (h')^l \cdot (y')^k dx \ , \tag{5.13}$$

is homogeneous of order $(n+1)$. Using estimates (5.10) and the Hölder–Minkowski inequality (as $\|y'\|^k \in L_1$ for $k \leqslant p$) leads to

$$\left| \int_a^b A_{kl} dx \right| \leqslant \int_a^b \left\| \nabla_{yz} R_k^{n-l,l}(x,y,y') \cdot (h,h') \cdot (h)^{n-l} \cdot (h')^l \right\| \cdot \|y'\|^k dx$$

$$\leqslant M_{n-l,l}^{k1} \cdot \int_a^b \|y'\|^k dx \leqslant \left[M_{n-l,l}^{k1} \cdot (b-a)^{\frac{p-k}{p}} \right] \cdot (\|y\|_{W^{1,p}})^k \qquad \text{as} \quad h \in C_\Delta \, ,$$

whence boundedness of the functional on the subspace $span(C)$ with respect to the norm $\|\cdot\|_C$ follows. By arbitrariness of the choice of $C \subset W^{1,p}([a;b],F)$ this implies K–continuity of functional (5.13) and therefore, by virtue of its homogeneity ([10], [12]) usual continuity of the given functional on $W^{1,p}([a;b],F)$.

b) The homogeneous functional in h, of order $(n+1)$

$$\int_a^b B_{kl} dx = k \cdot \int_a^b R_k^{n-l,l}(x,y,y') \cdot (h)^{n-l} \cdot (h')^{l+1} \cdot k(y')^{k-1} dx$$

can be estimated in a similar way. Namely,

$$\left| \int_a^b B_k dx \right| \leqslant k \cdot \int_a^b \|R_k^{n-l,l}(x,y,y') \cdot (h)^{n-l} \cdot (h')^l\| \cdot \|h'\| \cdot \|y'\|^{k-1} dx$$

$$\leqslant k \cdot M^{k0}_{n-l,l} \cdot \int_a^b \|h'\| \cdot \|y'\|^{k-1} dx \leqslant k \cdot M^{k0}_{n-l,l} \cdot \|h'\|_{L_p} \cdot (\|y'\|_{L_{\frac{(k-1)p}{p-l}}})^{k-1}$$

$$\leqslant \left[k \cdot M^{k0}_{n-l,l} \cdot \|y\|_{W^{1,\frac{(k-1)p}{p-l}}} \right] \cdot \|h\|_{W^{1,p}} \ ,$$

whence continuity of the functional with respect to the norm $\|\cdot\|_{W^{1,p}}$, and moreover, continuity with respect to the norm $\|\cdot\|_C$ on the subspace $span(C)$ follows. Repeating of the argument of part a) implies usual continuity of the given functional in $W^{1,p}([a;b],F)$.

5) Let us begin estimating the integrals of the small terms C_{kl}, D_{kl} and E_{klm} of decomposition (5.12). Here we use Basic Lemma.

a) The estimate

$$\left| \int_a^b C_{kl} dx \right| \leqslant k \cdot \int_a^b \left(\|\Delta R^{n-l,l}_k(x,h,h') \cdot (h)^{n-l} \cdot (h')^l\| \cdot \|h'\| \right) \cdot \|y'\|^{k-1} dx$$

enables us in the framework of Basic Lemma to use the functions

$$\varphi(x,u) = \|\Delta R^{n-l,l}_k(x,u_1,u_2) \cdot (u_1)^{n-l} \cdot (u_2)^l)\| \cdot \|u_2\| \quad (u=(u_1,u_2) \in F \times F),$$

$$\psi(x) = k \cdot \|y'(x)\|^{k-1} .$$

Let us check fulfillment of conditions i)–iii) of Basic Lemma for φ and ψ.

i) In view of uniform continuity $R^{n-l,l}_k$ on T^y,

$$\|\Delta R^{n-l,l}_k(x,u)\| \to 0 \ \text{ as } \ \|u\| \to 0 \ \text{ uniformly in } \ x \in [a;b].$$

Therefore the estimate

$$|\varphi(x,u)| \leqslant \|\Delta R^{n-l,l}_k(x,u)\| \cdot \|u_1\|^{n-l} \cdot \|u_2\|^{l+1} \leqslant \|\Delta R^{n-l,l}_k(x,u)\| \cdot \|(u_1,u_2)\|^{n+1}$$

implies $\varphi(x,u) = o(\|u\|^{n+1})$ as $\|u\| \to 0$ uniformly in $x \in [a;b]$.

ii) The function $\psi = k \cdot \|y'\|^{k-1} \in L_1$ because $y' \in L_p$ and $(k-1) \leqslant p$.

iii) The mapping

$$\chi(h_1) = \varphi(\cdot,(h,h')) \cdot \psi$$

$$= \|(R^{n-l,l}_k(\cdot,y+h,y'+h') - R^{n-l,l}_k(\cdot,y,y')) \cdot (h)^{n-l} \cdot (h')^l\| \cdot (\|h'\| \cdot \|y'\|^{k-1}) \qquad (5.14)$$

is, obviously, continuous in $h \in J(C)$ mapping into $L_1([a;b],\mathbb{R})$, in view of continuity in h_1 and boundedness on $J(C)$ of the first multiple in the right-hand side of (5.14) and of summability of the second multiple, arising from h', $y' \in L_p$ and $1+(k-1)=k \leqslant p$.

Thus, Basic Lemma is applicable, whence for $h \in C$ it follows

$$\left| \int_a^b C_{kl} dx \right| \leqslant \int_a^b \varphi(x,(h,h')) \cdot \psi(x) dx = o\left((\|(h,h')\|_{L_p})^{n+1} \right) = o\left((\|h\|_{W^{1,p}})^{n+1} \right) \ . \tag{5.15}$$

b) The estimate

$$\left| \int_a^b D_{kl} dx \right| \leqslant \int_a^b \|\nabla_{yz} r_k^{n-l,l}(x,y,y';h,h') \cdot (h,h') \cdot (h)^{n-l} \cdot (h')^l\| \cdot \|y'\|^k dx$$

enables us in the framework of Basic Lemma to use the functions

$$\varphi(x,u) = \|\nabla_{yz} r_k^{n-l,l}(x,y,y';(u_1,u_2)) \cdot (u_1,u_2) \cdot (u_1)^{n-l} \cdot (u_2)^l\| \quad (u=(u_1,u_2) \in F \times F),$$

$$\psi(x) = \|y'(x)\|^k .$$

Let us check fulfillment of conditions i)–iii) of Basic Lemma for φ and ψ.

i) In view of uniform continuity $\nabla_{yz} r_k^{n-l,l}$ on T^y,

$$\|\nabla_{yz} r_k^{n-l,l}(x,y,y';(u_1,u_2))\| \to 0 \text{ as } \|u\| \to 0 \text{ uniformly in } x \in [a;b].$$

Therefore the estimate

$$|\varphi(x,u)| \leqslant \|\nabla_{yz} r_k^{n-l,l}(x,y,y';u)\| \cdot \|(u_1,u_2)\| \cdot \|u_1\|^{n-l} \cdot \|u_2\|^l$$

$$\leqslant \|\nabla_{yz} r_k^{n-l,l}(x,y,y';u)\| \cdot \|u\|^{n+1} = o(\|u\|^{n+1}) \text{ uniformly in } x \in [a;b]$$

shows fulfillment of condition i).

ii) The function $\psi = \|y'\|^k \in L_1$ because $y' \in L_p$ and $(k) \leqslant p$.

iii) The mapping

$$\chi(h_1) = \varphi(\cdot,(h,h')) \cdot \psi$$
$$= \|\nabla_{yz} r_k^{n-l,l}(x,y,y';h,h') \cdot (h,h') \cdot (h)^{n-l} \cdot (h')^l\| \cdot \|y'\|^k \tag{5.16}$$

is, obviously, continuous in $h_1 = (h,h') \in J(C)$ mapping into $L_1([a;b],\mathbb{R})$ in view of continuity in h_1 and boundedness on $J(C)$ of the first multiple in the right-hand side of (5.16) and of summability of the second multiple.

Thus, Basic Lemma is applicable, whence for $h \in C$ it follows

$$\left| \int_a^b D_{kl} dx \right| \leqslant \int_a^b \varphi(x,(h,h')) \cdot \psi(x) dx = o\left(\left(\|(h,h')\|_{L_p} \right)^{n+1} \right) = o\left((\|h\|_{W^{1,p}})^{n+1} \right) . \tag{5.17}$$

c) The estimate

$$\left| \int_a^b E_{klm} dx \right| \leqslant k \cdot \int_a^b \|R_k^{n-l,l}(x,y+h,y'+h') \cdot (h)^{n-l} \cdot (h')^l\| \cdot \|h'\|^{k-m} \cdot \|y'\|^m dx$$

enables us, in the framework of Basic Lemma, to use the functions

$$\varphi(x,u) = \|R_k^{n-l,l}(x,y+u_1,y'+u_2) \cdot (u_1)^{n-l} \cdot (u_2)^l\| \cdot \|u_2\|^{k-m} \quad (u=(u_1,u_2) \in F \times F),$$

$$\psi(x) = \|y'(x)\|^m .$$

Let us check fulfillment of conditions i)–iii) of Basic Lemma for φ and ψ.

i) Using estimate (5.9) for $R_k^{n-l,l}$ on T^y leads to

$$|\varphi(x,u)| \leqslant \|R_k^{n-l,l}(x,y+u_1,y'+u_2)\|\cdot\|u_1\|^{n-l}\cdot\|u_2\|^{l+k-m} \leqslant M_{n-l,l}^{k0}\cdot(\|u_1\|+\|u_2\|)^{n+(k-m)}$$

$$= O(\|u\|^{n+2}) = o(\|u\|^{n+1}) \text{ as } \|u\| \to 0 \text{ uniformly in } x \in [a;b].$$

ii) The function $\psi = k\cdot\|y'\|^{k-1} \in L_1$ because $y' \in L_p$ and $(k-1) \leqslant p$.

iii) The mapping

$$\chi(h_1) = \varphi(\cdot,(h,h'))\cdot\psi$$
$$= \|R_k^{n-l,l}(\cdot,y+h,y'+h')\cdot(h)^{n-l}\cdot(h')^l\|\cdot(\|h'\|^{k-m}\cdot\|y'\|^m) \qquad (5.18)$$

is , obviously, continuous in $h \in J(C)$ mapping into $L_1([a;b],\mathbb{R})$, in view of continuity in h_1 of the first multiple in the right-hand side of (5.18), arising from uniform continuity $R_k^{n-l,l}$ on T^y, and of summability of the second multiple, arising from h', $y' \in L_p$ and $(k-m)+m \leqslant p$.

Thus, Basic Lemma is applicable, whence for $h \in C$ it follows

$$\left|\int_a^b E_{klm}dx\right| \leqslant C_k^m\cdot\int_a^b \varphi(x,(h,h'))\cdot\psi(x)dx = o\left(\left(\|(h,h')\|_{L_p}\right)^{n+1}\right) = o\left((\|h\|_{W^{1,p}})^{n+1}\right). \qquad (5.19)$$

6) So, from decompositions (5.11)–(5.12), from the obtained above estimates (5.15), (5.17), (5.19), and from the results of part 4) of the proof it follows that

$$(\Phi_K^{(n)}(y+h)-\Phi_K^{(n)}(y))\cdot(h)^n = \sum_{l=0}^n C_n^l\cdot\int_a^b\Bigg(\sum_{k=0}^p\Big[\nabla_{yz}R_k^{n-l,l}(x,y,y')\cdot(h,h')\cdot(y')^k$$
$$+kR_k^{n-l,l}(x,y,y')\cdot(h')\cdot(y')^{k-1}\Big]\cdot(h)^{n-l}\cdot(h')^l\Bigg)dx + o\left((\|h\|_{W^{1,p}})^{n+1}\right), \qquad (5.20)$$

where the integral functional in the right-hand side of (5.20) is continuous. Since, in view of compactness of C, the norm $\|\cdot\|_C$ majorizes the norm $\|\cdot\|_{W^{1,p}}$ in *span*(C) then the small term in the right-hand side of (5.20) is $o\left((\|h\|_C)^{n+1}\right)$. Thus, we come to $(n+1)$–multiple K–differentiability Φ and to the equality

$$\Phi_K^{(n+1)}(y)\cdot(h)^{n+1} = \sum_{l=0}^n C_n^l\cdot\int_a^b\Bigg(\sum_{k=0}^p\Big[\nabla_{yz}R_k^{n-l,l}(x,y,y')\cdot(h,h')\cdot(y')^k$$
$$+k\cdot R_k^{n-l,l}(x,y,y')\cdot(h')\cdot(y')^{k-1}\Big]\cdot(h)^{n-l}\cdot(h')^l\Bigg)dx\,. \qquad (5.21)$$

7) Finally, let us check that equality (5.21) can be transformed to form (5.5) with n replaced by$n+1$.

a) First, let us prove the identity

$$\frac{\partial^{n+1} f}{\partial y^{n+1-l} \partial z^l}(x,y,y') \cdot h + \frac{\partial^{n+1} f}{\partial y^{n-l} \partial z^{l+1}}(x,y,y') \cdot h'$$

$$= \sum_{k=0}^{p} \Big(\nabla_{yz} R_k^{n-l,l}(x,y,y') \cdot (h,h') \cdot (y')^k + k \cdot R_k^{n-l,l}(x,y,y') \cdot (h') \cdot (y')^{k-1} \Big). \quad (5.22)$$

In the first place,

$$\frac{\partial^{n+1} f}{\partial y^{n+1-l} \partial z^l}(x,y,y') \cdot h = \frac{\partial}{\partial y} \left(\frac{\partial^n f}{\partial y^{n-l} \partial z^l} \right) (x,y,y') \cdot h$$

$$= \frac{\partial}{\partial y} \left(\sum_{k=0}^{p} R_k^{n-l,l}(x,y,z) \cdot (z)^k \right) \Big|_{z=y'} \cdot h = \sum_{k=0}^{p} \frac{\partial R_k^{n-l,l}}{\partial y}(x,y,y') \cdot (h) \cdot (y')^k . \quad (5.23)$$

Secondly, in analogous way,

$$\frac{\partial^{n+1} f}{\partial y^{n-l} \partial z^{l+1}}(x,y,y') \cdot h' = \frac{\partial}{\partial z} \left(\frac{\partial^n f}{\partial y^{n-l} \partial z^l} \right) (x,y,y') \cdot h'$$

$$= \frac{\partial}{\partial z} \left(\sum_{k=0}^{p} R_k^{n-l,l}(x,y,z) \cdot (z)^k \right) \Big|_{z=y'} \cdot h'$$

$$= \left(\sum_{k=0}^{p} \left[\frac{\partial R_k^{n-l,l}}{\partial y}(x,y,z) \cdot (z)^k + R_k^{n-l,l}(x,y,z) \cdot k \cdot (z)^{k-1} \right] \Big|_{z=y'} \right) \cdot h'$$

$$= \sum_{k=0}^{p} \left[\frac{\partial R_k^{n-l,l}}{\partial y}(x,y,y') \cdot (h') \cdot (y')^k + R_k^{n-l,l}(x,y,y') \cdot (h') \cdot k \cdot (y')^{k-1} \right]. \quad (5.24)$$

The term by term adding of equalities (5.23) and (5.24) results in (5.22).

b) Now, let us transform the right-hand side of (5.21), using equalities (5.22). It follow that

$$\Phi_K^{(n+1)}(y) \cdot (h)^{n+1} = \int_a^b \Bigg(\sum_{l=0}^{n} \Big[\frac{\partial^{n+1} f}{\partial y^{n+1-l} \partial z^l}(x,y,y') \cdot h$$

$$+ \frac{\partial^{n+1} f}{\partial y^{n-l} \partial z^{l+1}}(x,y,y') \cdot h' \Big] \cdot (h)^{n-l} \cdot (h')^l \Bigg) dx$$

$$= \sum_{l=0}^{n} \Bigg[\int_a^b C_n^l \cdot \frac{\partial^{n+1} f}{\partial y^{n+1-l} \partial z^l}(x,y,y') \cdot (h)^{n+1-l} \cdot (h')^l dx$$

$$+ \int_a^b C_n^l \cdot \frac{\partial^{n+1} f}{\partial y^{(n+1)-(l+1)} \partial z^{l+1}}(x,y,y') \cdot (h)^{(n+1)-(l+1)} \cdot (h')^{l+1} dx \Bigg]. \quad (5.25)$$

Finally, replacing $l+1$ by l in the last integrals of the right-hand side of (5.25) we get the equality

$$\Phi_K^{(n+1)}(y)\cdot(h)^{n+1}=\sum_{l=0}^{n+1}\overbrace{(C_n^l+C_n^{l-1})}^{C_{n+1}^l}\cdot\int\limits_a^b\frac{\partial^{n+1}f}{\partial y^{n+1-l}\partial z^l}(x,y,y')\cdot(h)^{n+1-l}\cdot(h')^l dx\,,$$

which is none other than the equality (5.5) with n replaced by $n+1$.

Thus, by induction, both formula (5.5) and its pseudopolynomial variant (5.6) are proved. □

Useful examples, can be obtained by considering particular cases of Theorem 5.1 for $n=2$.

Example 6. For $n=2$ formula (5.5) takes the form

$$\Phi_K''(y)(h)^2=\int\limits_a^b\Big[\frac{\partial^2 f}{\partial y^2}(x,y,y')\cdot(h)^2+2\frac{\partial^2 f}{\partial y\partial z}(x,y,y')\cdot(h)\cdot(h')+\frac{\partial^2 f}{\partial z^2}(x,y,y')\cdot(h')^2\Big]dx.$$

It is also possible to obtain a variant of formula (5.6) where the integrand is expressed explicitly via the coefficients of the initial K–pseudopolynomial representation of f.

Remark 4. Equality (5.6) can be transformed to the following form

$$\Phi_K^{(n)}(y)(h)^n=\sum_{k=0}^{p}\int\limits_a^b\Big[\sum_{l=0}^{n}C_n^l\cdot k(k-1)\cdots(k-l+1)\cdot\nabla_{yz}^{n-l}R_k(x,y,y')\cdot(h,h')^{n-l}\cdot(y')^{k-l}\Big]dx\,. \tag{5.26}$$

In particular, for $n=1$ (5.26) reduces to equality (4.2), for $n=2$ (5.26) reduces to the equality

$$\Phi_K''(y)(h)^2=\sum_{k=0}^{p}\int\limits_a^b\Big[\nabla_{yz}^2R_k(x,y,y')\cdot(h,h')^2\cdot(y')^k+2k\cdot\nabla_{yz}R_k(x,y,y')\cdot(h,h')\cdot(y')^{k-1}$$

$$+k(k-1)\cdot R_k(x,y,y')\cdot(y')^{k-2}\Big]dx\,.$$

It is clear that the number of non–zero summands in the sum under the integral sign in (5.26) does not exceed k. The readers are recommended to carry out the calculations independently, as an exercise.

In the conclusion of this section we give examples of some classes of integrands, which are enclosed by Theorem 5.1.

Example 7. Some types of integrands in the Weierstrass class $W^n K_p(z)$.

1) Let

$$f(x,y,z)=\sum_{k=0}^{p} R_k(x,y)\cdot(z)^k \quad (R_k\in C^n_{xy};\ k=\overline{0,p}).$$

Here independence $\nabla^m_{yz} R_k$ $(m=\overline{0,n})$ from z automatically leads to the conditions $R_k\in W^n_K(z)$, whence $f\in W^n K_p(z)$.

2) As an obvious generalization of the preceding example, let consider

$$f(x,y,z)=\sum_{k\in\mathcal{K}} R_k(x,y)\cdot(z)^k+\sum_{k'\in\mathcal{K}'} R_{k'}(x,y,z)\cdot(z)^{k'},$$

where $\mathcal{K}\dot{\cup}\mathcal{K}'=\{\overline{0,p}\}$, $R_k\in C^n_{xy}$ $(k\in\mathcal{K})$, $R_{k'}\in W^n_K(z)$ $(k'\in\mathcal{K}')$.

3) Let

$$f(x,y,z)=\sum_{k=0}^{p}\varphi_k(\underbrace{r_k(x,y,z)}_{t})\cdot(z)^k \quad (\varphi_k\in C^n_t,\ \ r_k\in W^n_K(z)).$$

Then $f\in W^n K_p(z)$.

4) Note one more evident example. Let $R_k\in C^n_{xyz}$ $(k=\overline{0,p})$ and R_k be periodic in z (with the periods not depending on x, y). Then $f\in W^n K_p(z)$ as well.

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Igor Vladimirovich Orlov
Faculty of Mathematics and Informatics
V. Vernadsky Taurida National University
4 Vernadsky Avenu,
95007 Simferopol, Ukraine
E-mail: igor_v_orlov@mail.ru

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