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# ORDER-SHARP ESTIMATES FOR HARDY-TYPE OPERATORS ON THE CONES OF FUNCTIONS WITH PROPERTIES OF MONOTONICITY

**M.L. Goldman**

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**Abstract.** Two-sided estimates are established for two types of generalized Hardy operators on the cones of functions in weighted Lebesgue spaces with some properties of monotonicity.

We prove the results announced in [7], and present some other equivalent forms for the criterion of boundedness. Some other equivalent descriptions, particular cases, and results in the case of degenerate measures will be given in our next paper.

## 1 Notation and formulation of results

**1.1.** Let $\beta$ and $\gamma$ be nonnegative Borel measures on $R_+ = (0,\infty)\,; p,q \in R_+$, $\Omega$ be a certain cone of nonnegative Borel measurable functions on $R_+$, and $A$ be a positive operator. We investigate the finiteness of the quantity

$$H_\Omega(A) = \sup_{f\in\Omega}\left[\left(\int_{R_+}(Af)^q\,d\gamma\right)^{1/q}\left(\int_{R_+}f^p d\beta\right)^{-1/p}\right]. \tag{1.1}$$

Here, as $\Omega$ we consider the cones of functions that are monotone with respect to the prescribed positive continuous functions $k$ and $m$:

$$\Omega_k = \{f \geqslant 0 : f(\tau)/k(\tau)\downarrow\}; \qquad \Omega^m = \{f \geqslant 0 : f(\tau)/m(\tau)\uparrow\}. \tag{1.2}$$

As operator $A$, we consider the generalized Hardy operators $A_\mu$, and $B_\mu$ where $\mu$ is a nonnegative Borel measure on $R_+$;

$$(A_\mu f)(t) = \int_{(0,t]} f d\mu; \quad (B_\mu f)(t) = \int_{[t,\infty)} f d\mu. \tag{1.3}$$

**1.2.** First, we formulate the result for $H_{\Omega_k}(B_\mu)$. For this purpose we need the following notation:

$$\omega_p(t) = \left( \int\limits_{(0,t)} k^p d\beta \right)^{1/p}, \quad t>0; \quad \Psi(t,\tau) = \int\limits_{[t,\tau)} k d\mu, \quad t<\tau; \tag{1.4}$$

$$V_p(t) = \sup_{\tau\in(t,\infty)} \left[ \frac{\Psi(t,\tau)}{\omega_p(\tau)} \right], \quad p\in(0,1]; \tag{1.5}$$

$$V_p(t) = \left\{ \int\limits_{(t,\infty)} \Psi^{p'}(t,\tau) \left( -d\left[ \frac{1}{\omega_p^{p'}(\tau)} \right] \right) \right\}^{1/p'}; \quad p>1, \quad \frac{1}{p}+\frac{1}{p'}=1; \tag{1.6}$$

$$W_q(\tau) = \left( \int\limits_{(0,\tau)} d\gamma \right)^{1/q}; \quad \xi_\alpha(\tau) = \omega_p^{-1}(\alpha\omega_p(\tau)), \quad \tau\in R_+. \tag{1.7}$$

Here $\alpha\in(0,1)$ is fixed; $\omega_p^{-1}$ is the right-continuous inverse function for the (increasing) continuous function $\omega_p$. Obviously, $\xi_\alpha(\tau)<\tau$.

The criterion of finiteness of $H_{\Omega_k}(B_\mu)$ will be formulated by using the following quantities:

$$E_{pq} = \sup_{\tau\in R_+} \left[ \left( \int\limits_{[\xi_\alpha(\tau),\tau)} \Psi^q(t,\tau)\, d\gamma(t) \right)^{1/q} \frac{1}{\omega_p(\tau)} \right], \quad p\leqslant q; \tag{1.8}$$

$$E_{pq} = \left\{ \int\limits_{R_+} \left( \int\limits_{[\xi_\alpha(\tau),\tau)} \Psi^q(t,\tau)\, d\gamma(t) \right)^{s/q} \left( -d\left[ \frac{1}{\omega_p^s(\tau)} \right] \right) \right\}^{1/s}, \quad p>q, \tag{1.9}$$

$$F_{pq} = \sup_{t\in R_+} \left[ V_p(t)\, W_q(t) \right], \quad p\leqslant q, \tag{1.10}$$

$$F_{pq} = \left\{ \int\limits_{R_+} V_p^s(t)\, d\left[ W_q^s(t) \right] \right\}^{1/s}, \quad p>q, \tag{1.11}$$

where as always in this paper $s=pq/(p-q)$ for $p>q$. In addition, introduce the non-degeneracy condition for measure $\beta$ :

$$\beta\in N_p(k) \quad \Leftrightarrow \quad \int\limits_{(0,1)} k^p d\beta = 1, \quad \int\limits_{[1,\infty)} k^p d\beta = \infty. \tag{1.12}$$

**Theorem 1.1.** *Let $\beta\in N_p(k)$ and the functions $\omega_p$ and $W_q$ be positive and continuous on $R_+$, $\omega_p(+0)=0$. Then there exists $c_0=c_0(p,q,\alpha)\in[1,\infty)$ such that*

$$c_0^{-1}(E_{pq}+F_{pq}) \leqslant H_{\Omega_k}(B_\mu) \leqslant c_0(E_{pq}+F_{pq}). \tag{1.13}$$

Some slightly different equivalent forms for the criterion of the boundedness of the operator $B_\mu$ are presented below in Theorems 4.4 and 5.1.

**1.3.** Next, we present the corresponding results concerning $H_{\Omega^m}(A_\mu)$. To this end we denote

$$\bar{\omega}_p(t) = \left( \int\limits_{(t,\infty)} m^p d\beta \right)^{1/p}, \quad t > 0; \quad \Phi(\tau, t) = \int\limits_{(\tau,t]} m d\mu, \quad \tau < t; \tag{1.14}$$

$$V_p^{(0)}(t) = \sup_{\tau \in (0,t)} \left[ \Phi(\tau, t) \frac{1}{\bar{\omega}_p(\tau)} \right], \quad p \in (0, 1]; \tag{1.15}$$

$$V_p^{(0)}(t) = \left\{ \int\limits_{(0,t)} \Phi^{p'}(\tau, t) d\left[ \frac{1}{\bar{\omega}_p^{p'}(\tau)} \right] \right\}^{1/p'}, \quad p > 1; \tag{1.16}$$

$$\bar{W}_q(\tau) = \left( \int\limits_{(\tau,\infty)} d\gamma \right)^{1/q}; \quad \varsigma_\alpha(\tau) = \bar{\omega}_p^{-1}(\alpha \bar{\omega}_p(\tau)), \quad \tau \in R_+. \tag{1.17}$$

Here $\alpha \in (0, 1)$ is fixed; $\bar{\omega}_p^{-1}$ is the right-continuous inverse function for the (decreasing) continuous function $\bar{\omega}_p$. Obviously, $\tau < \varsigma_\alpha(\tau)$. We also introduce the following quantities:

$$E_{pq}^{(0)} = \sup_{\tau \in R_+} \left[ \left( \int\limits_{(\tau, \varsigma_\alpha(\tau)]} \Phi^q(\tau, t) \, d\gamma(t) \right)^{1/q} \frac{1}{\bar{\omega}_p(\tau)} \right], \quad p \leqslant q; \tag{1.18}$$

$$E_{pq}^{(0)} = \left\{ \int\limits_{R_+} \left( \int\limits_{(\tau, \varsigma_\alpha(\tau)]} \Phi^q(\tau, t) \, d\gamma(t) \right)^{s/q} d\left[ \frac{1}{\bar{\omega}_p^s(\tau)} \right] \right\}^{1/s}, \quad p > q. \tag{1.19}$$

$$F_{pq}^{(0)} = \sup_{t \in R_+} \left[ V_p^{(0)}(t) \bar{W}_q(t) \right], \quad p \leqslant q, \tag{1.20}$$

$$F_{pq}^{(0)} = \left\{ \int\limits_{R_+} \left( V_p^{(0)} \right)^s(t) \left( -d\left[ \bar{W}_q^s(t) \right] \right) \right\}^{1/s}, \quad p > q. \tag{1.21}$$

The non-degeneracy condition on measure $\beta$ has the following form:

$$\beta \in \bar{N}_p(m) \quad \Leftrightarrow \quad \int\limits_{(0,1]} m^p d\beta = \infty, \quad \int\limits_{(1,\infty)} m^p d\beta = 1. \tag{1.22}$$

**Theorem 1.2.** *Let $\beta \in \bar{N}_p(m)$ and functions $\bar{\omega}_p$ and $\bar{W}_q$ be positive and continuous on $R_+$, $\bar{\omega}_p(+\infty) = 0$.*

$$c_0^{-1} \left( E_{pq}^{(0)} + F_{pq}^{(0)} \right) \leqslant H_{\Omega^m}(A_\mu) \leqslant c_0 \left( E_{pq}^{(0)} + F_{pq}^{(0)} \right), \tag{1.23}$$

where $c_0$ is the same as in Theorem 1.1. Some slightly different equivalent form for the criterion of the boundedness is presented below in Theorem 5.2.

**Remark 1.3.** Let in Theorems 1.1 and 1.2 $p \leqslant q$. Then, we can change definitions (1.8) and (1.18). Namely, estimates (1.13) and (1.23) with a certain constants $c_0, c_1; \quad c_i = c_i(p,q) \in [1,\infty), \quad i = 0,1$, remain true if we replace $E_{pq}$ in (1.13) or $E_{pq}^{(0)}$ in (1.23), by

$$\dot{E}_{pq} = \sup_{\tau \in R_+} \left[ \left( \int\limits_{(0,\tau)} \Psi^q(t,\tau)\, d\gamma(t) \right)^{1/q} \frac{1}{\omega_p(\tau)} \right], \quad p \leqslant q; \tag{1.24}$$

by

$$\dot{E}_{pq}^{(0)} = \sup_{\tau \in R_+} \left[ \left( \int\limits_{(\tau,\infty)} \Phi^q(\tau,t)\, d\gamma(t) \right)^{1/q} \frac{1}{\bar{\omega}_p(\tau)} \right], \quad p \leqslant q, \tag{1.25}$$

respectively.

**Remark 1.4.** The results concerning $H_{\Omega_k}(A_\mu)$ and $H_{\Omega^m}(B_\mu)$ were obtained in our paper [5; Theorems 1.2 and 1.4], and in some other forms in [1, 3, and 4]. The detailed comparison for corresponding results in [5, 1, and 3] was made in [6].

**Remark 1.5.** It was found by A. Gogatishvili that for some $p > q$ the formulations given by Theorems 1.1 and 1.3 in [5] were not correct (personal communication). Here we establish the corrected versions of these results. They were formulated in [7]. Correction is realized by inserting in (1.9) the function $\xi_\alpha$ defined by (1.7), and by inserting in (1.19) the function $\varsigma_\alpha$ defined in (1.17). Also, we establish some other new equivalent variants for the result.

## 2 Discrete analogue of Theorem 1.1 on the cone of decreasing functions

**2.1.** First, we establish a particular case of Theorem 1.1 corresponding to the function $k(t) \equiv 1$, i.e., to the cone of decreasing functions

$$\Omega_1 = \{ f \geqslant 0 : f(\tau) \downarrow \}. \tag{2.1}$$

Thus, we consider here $H_{\Omega_1}(B_\mu)$ (see (1.1)–(1.3)). We preserve the notation of Section 1 and set there $k(t) \equiv 1$. In this Section we prove a criterion of the finiteness for $H_{\Omega_1}(B_\mu)$ in the discrete form. For this purposes we need some additional notation related to the following discretisation procedure. We fix $a > 1$, and introduce

$$\lambda_n = \omega_p^{-1}(a^n), \quad n \in \mathbb{Z}. \tag{2.2}$$

For $\beta \in N_p(1)$ we have

$$0 < \omega_p \uparrow \text{ on } R_+, \quad \omega_p(+0) = 0, \quad \omega_p(+\infty) = \infty, \tag{2.3}$$

so that

$$0 < \lambda_n < \lambda_{n+1}, \quad n \in \mathbb{Z}; \quad \lambda_n \to 0\,(n \to -\infty)\,, \lambda_n \to \infty\,(n \to +\infty)\,. \tag{2.4}$$

Introduce the discrete analogues of $E_{pq}$ and $F_{pq}$:

$$\tilde{E}^0_{pq} = \left\{ \sum_{n\in\mathbb{Z}} \left[ a^{-n} \left( \int_{\Delta_n} \Psi^q\left(t, \lambda_{n+1}\right) d\gamma\left(t\right) \right)^{1/q} \right]^s \right\}^{1/s}, \tag{2.5}$$

where as always $s = pq/(p-q)$ for $p > q$; $s = \infty$ for $p \leqslant q$ (in this case we understand $\tilde{E}^0_{pq}$ as the supremum over $n \in \mathbb{Z}$ of the expression in square brackets); $\Delta_n = [\lambda_n, \lambda_{n+1})$.

$$\tilde{F}_{pq} = \sup_n \left\{ V_p\left(\lambda_n\right) W_q\left(\lambda_n\right) \right\}, \qquad p \leqslant q, \tag{2.6}$$

$$\tilde{F}_{pq} = \left\{ \sum_{n\in\mathbb{Z}} V_p^s\left(\lambda_n\right) \left[ W_q^s\left(\lambda_n\right) - W_q^s\left(\lambda_{n-1}\right) \right] \right\}^{1/s}, \quad p > q. \tag{2.7}$$

**Theorem 2.1.** *Let $\beta \in N_p\left(1\right)$ and the function $\omega_p$ be positive and continuous on $R_+$, $\omega_p\left(+0\right) = 0$. Then there exists $\tilde{c}_0 = \tilde{c}_0\left(p, q, a\right) \in [1, \infty)$ such that*

$$\tilde{c}_0^{-1} \left( \tilde{E}^0_{pq} + \tilde{F}_{pq} \right) \leqslant H_{\Omega_1}\left(B_\mu\right) \leqslant \tilde{c}_0 \left( \tilde{E}^0_{pq} + \tilde{F}_{pq} \right). \tag{2.8}$$

*Here, if $p, q \in [\delta, \infty)$ for a certain $\delta \in R_+$ then*

$$1 \leqslant \tilde{c}_0\left(p, q, a\right) \leqslant c\left(a, \delta\right) < \infty. \tag{2.9}$$

**2.2.** To prove Theorem 2.1 we will need the following two Propositions (similar assertions are frequently met in the works devoted to this subject; in particular they were formulated and proved in [5; Propositions 2.1 and 2.2]).

**Proposition 2.2.** *Under the hypotheses of Theorem 2.1 the following estimate holds for $f \in \Omega_1$:*

$$\left(1 - a^{-p}\right) \left( \sum_{n\in\mathbb{Z}} \left[a^n f\left(\lambda_n\right)\right]^p \right) \leqslant \int_{R_+} f^p d\beta \leqslant a^p \left( \sum_{n\in\mathbb{Z}} \left[a^n f\left(\lambda_n\right)\right]^p \right). \tag{2.10}$$

**Proposition 2.3.** *Let $\sigma \in (0, \infty]\,; W_n > 0, n \in \mathbb{Z}$,*

$$W := \inf_n \left( W_{n+1} W_n^{-1} \right) > 1.$$

*Then, for all $\beta_n \geqslant 0, n \in \mathbb{Z}$, the following inequalities hold*

$$\left\{ \sum_{n\in\mathbb{Z}} \left[ W_n \sum_{m\geqslant n} \beta_m \right]^\sigma \right\}^{1/\sigma} \leqslant c \left\{ \sum_{n\in\mathbb{Z}} \left[W_n \beta_n\right]^\sigma \right\}^{1/\sigma}, \tag{2.11}$$

$$\left\{\sum_{n\in\mathbb{Z}}\left[W_n^{-1}\sum_{m\leqslant n}\beta_m\right]^\sigma\right\}^{1/\sigma}\leqslant c\left\{\sum_{n\in\mathbb{Z}}\left[W_n^{-1}\beta_n\right]^\sigma\right\}^{1/\sigma}. \tag{2.12}$$

*Here,* $c=\left(1-W^{-1}\right)^{-1}$ *if* $\sigma\in[1,\infty]$, *and* $c=\left(1-W^{-\sigma}\right)^{-1/\sigma}$ *if* $\sigma\in(0,1]$.

**Corollary 2.4.** *Under the hypotheses of Proposition* 2.3 *the following inequalities hold for any* $j\in\mathbb{Z}$ :

$$\left\{\sum_{n\leqslant j}\left[W_n\sum_{m=n}^{j}\beta_m\right]^\sigma\right\}^{1/\sigma}\leqslant c\left\{\sum_{n\leqslant j}\left[W_n\beta_n\right]^\sigma\right\}^{1/\sigma}, \tag{2.13}$$

$$\left\{\sum_{n\geqslant j}\left[W_n^{-1}\sum_{m=j}^{n}\beta_m\right]^\sigma\right\}^{1/\sigma}\leqslant c\left\{\sum_{n\geqslant j}\left[W_n^{-1}\beta_n\right]^\sigma\right\}^{1/\sigma}. \tag{2.14}$$

**2.3. Proof of Theorem 2.1.**

**1.** Let us apply Proposition 2.2 to the denominator in $H_{\Omega_1}(B_\mu)$, see (1.1). Then

$$a^{-1}\tilde{H}_0\leqslant H_{\Omega_1}(B_\mu)\leqslant\left(1-a^{-p}\right)^{-1/p}\tilde{H}_0, \tag{2.15}$$

where

$$\tilde{H}_0=\sup_{f\in\Omega_1}\left[\left(\int\limits_{R_+}(B_\mu f)^q\,d\gamma\right)^{1/q}\left(\sum_{n\in\mathbb{Z}}\left[a^n f(\lambda_n)\right]^p\right)^{-1/p}\right]. \tag{2.16}$$

For $b=\{b_n\}$ we define

$$f_0(b;t)=\sum_{n\in\mathbb{Z}}b_n\chi_{\Delta_n}(t);\quad \Delta_n=[\lambda_n,\lambda_{n+1}),\quad n\in\mathbb{Z}. \tag{2.17}$$

The denominator in (2.16) is independent of the values of $f\in\Omega_1$ outside the points $\lambda_n,\quad n\in\mathbb{Z}$, therefore for a given set $\tilde{b}=\left\{\tilde{b}_n\right\}$ of values $\tilde{b}_n=f(\lambda_n), n\in\mathbb{Z}$ the upper bound is attained at the greatest function $\tilde{f}_0\in\Omega_1$ among those corresponding to this set, namely at the function $\tilde{f}_0(t)=f_0\left(\tilde{b};t\right)$ (see (2.17) with $\tilde{b}$ instead of $b$). Therefore,

$$\tilde{H}_0=\sup_{0\leqslant\tilde{b}_n\downarrow}\left[\left(\int\limits_{R_+}\left(B_\mu\tilde{f}_0\right)^q d\gamma\right)^{1/q}\left(\sum_{n\in\mathbb{Z}}\left[a^n\tilde{b}_n\right]^p\right)^{-1/p}\right]. \tag{2.18}$$

Now, we introduce

$$H_0=\sup_{0\leqslant b_n}\left[\left(\int\limits_{R_+}(B_\mu f_0)^q\,d\gamma\right)^{1/q}\left(\sum_{n\in\mathbb{Z}}\left[a^n b_n\right]^p\right)^{-1/p}\right]. \tag{2.19}$$

where $f_0(t) = f_0(b;t)$ (see (2.17)). Obviously $\tilde{H}_0 \leqslant H_0$. Let us prove the reverse inequality. For any sequence $b = \{b_n\}$, $b_n \geqslant 0$, we define $\tilde{b} = \left\{\tilde{b}_n\right\}$ with

$$\tilde{b}_n = \left(\sum_{m\geqslant n} b_m^p\right)^{1/p}, \text{ so that } \tilde{b}_n \geqslant b_n \geqslant 0, \quad \tilde{b}_n \downarrow, \tag{2.20}$$

and

$$\left(\sum_{n\in\mathbb{Z}} [a^n b_n]^p\right)^{1/p} = \left(1 - a^{-p}\right)^{1/p} \left(\sum_{n\in\mathbb{Z}} \left[a^n \tilde{b}_n\right]^p\right)^{1/p}. \tag{2.21}$$

Therefore, $f_0(\cdot) = f_0(b;\cdot) \leqslant f_0\left(\tilde{b};\cdot\right) = \tilde{f}_0(\cdot) \quad \Rightarrow \quad B_\mu f_0 \leqslant B_\mu \tilde{f}_0$, and

$$\left(\int\limits_{R_+} (B_\mu f_0)^q \, d\gamma\right)^{1/q} \left(\sum_{n\in\mathbb{Z}} [a^n b_n]^p\right)^{-1/p}$$
$$\leqslant \left(1 - a^{-p}\right)^{-1/p} \left(\int\limits_{R_+} \left(B_\mu \tilde{f}_0\right)^q d\gamma\right)^{1/q} \left(\sum_{n\in\mathbb{Z}} \left[a^n \tilde{b}_n\right]^p\right)^{-1/p} \leqslant \left(1 - a^{-p}\right)^{-1/p} \tilde{H}_0.$$

Consequently,

$$\tilde{H}_0 \leqslant H_0 \leqslant \left(1 - a^{-p}\right)^{-1/p} \tilde{H}_0. \tag{2.22}$$

Now, (2.15) and (2.22) imply

$$a^{-1}\left(1 - a^{-p}\right)^{1/p} H_0 \leqslant H_{\Omega_1}(B_\mu) \leqslant \left(1 - a^{-p}\right)^{-1/p} H_0. \tag{2.23}$$

**2.** Next, our aim is to estimate $H_0$ (2.19). To this end we note that

$$(B_\mu f_0)(t) = \int\limits_{[t,\infty)} f_0 d\mu = h_1(t) + h_2(t), \tag{2.24}$$

where

$$0 \leqslant h_1(t) = b_n \int\limits_{[t,\lambda_{n+1})} d\mu = b_n \Psi(t, \lambda_{n+1}), \quad t \in \Delta_n, n \in \mathbb{Z};$$

$$0 \leqslant h_2(t) = \sum_{m\geqslant n} b_{m+1} \int\limits_{\Delta_{m+1}} d\mu = \sum_{m\geqslant n} b_{m+1} \Psi(\lambda_{m+1}, \lambda_{m+2}), \quad t \in \Delta_n, n \in \mathbb{Z}.$$

Now, notice that

$$\left(\int\limits_{R_+} h_1^q d\gamma\right)^{1/q} = \left(\sum_{n\in\mathbb{Z}} b_n^q \int\limits_{\Delta_n} \Psi^q(t, \lambda_{n+1}) \, d\gamma\right)^{1/q}, \tag{2.25}$$

$$\left(\int\limits_{R_+} h_2^q d\gamma\right)^{1/q} = \left(\sum_{n\in\mathbb{Z}}\left(\sum_{m\geqslant n} b_{m+1}\Psi\left(\lambda_{m+1},\lambda_{m+2}\right)\right)^q w_n^q\right)^{1/q}, \tag{2.26}$$

where

$$w_n = \left(\int\limits_{\Delta_n} d\gamma\right)^{1/q} = \left[W_q^q\left(\lambda_{n+1}\right) - W_q^q\left(\lambda_n\right)\right]^{1/q}. \tag{2.27}$$

By (2.24) and by triangle inequality in $L_q$ it follows that

$$\max_{j=1,2}\left(\int\limits_{R_+} h_j^q d\gamma\right)^{1/q} \leqslant \left(\int\limits_{R_+}\left(B_\mu f_0\right)^q d\gamma\right)^{1/q} \leqslant \bar{c}_q \sum_{j=1}^{2}\left(\int\limits_{R_+} h_j^q d\gamma\right)^{1/q},$$

where $\bar{c}_q = 1, \quad q \in [1,\infty); \quad \bar{c}_q = 2^{1/q-1}, \quad q \in (0,1)$.

We insert here (2.25) and (2.26) and obtain

$$\max\left\{H_1, H_2\right\} \leqslant H_0 \leqslant \bar{c}_q\left(H_1 + H_2\right), \tag{2.28}$$

where

$$H_j = \sup_{b_n\geqslant 0}\left\{I_j\left[\{b_n\}\right]\left(\sum_{n\in\mathbb{Z}}\left[a^n b_n\right]^p\right)^{-1/p}\right\}, \quad j = 1,2; \tag{2.29}$$

$$I_1\left[\{b_n\}\right] = \left(\sum_{n\in\mathbb{Z}} b_n^q \int\limits_{\Delta_n} \Psi^q\left(t,\lambda_{n+1}\right) d\gamma\right)^{1/q}, \tag{2.30}$$

$$I_2\left[\{b_n\}\right] = \left(\sum_{n\in\mathbb{Z}} w_n^q\left(\sum_{m\geqslant n} b_{m+1}\Psi\left(\lambda_{m+1},\lambda_{m+2}\right)\right)^q\right)^{1/q}. \tag{2.31}$$

First, we calculate $H_1$. By Jensen's inequality (when $p \leqslant q$) or Hölder's inequality (when $p > q$), combined with the assertion about their sharpness on the set of all nonnegative sequences, we have

$$H_1 = \left\{\sum_{n\in\mathbb{Z}}\left[a^{-n}\left(\int\limits_{\Delta_n}\Psi^q\left(t,\lambda_{n+1}\right)d\gamma(t)\right)^{1/q}\right]^s\right\}^{1/s} = \tilde{E}_{pq}^0. \tag{2.32}$$

Next, consider $H_2$ (2.29):

$$H_2 = \sup_{b_n\geqslant 0}\left\{\left(\sum_{n\in\mathbb{Z}} w_n^q\left(\sum_{m\geqslant n} b_{m+1}\Psi\left(\lambda_{m+1},\lambda_{m+2}\right)\right)^q\right)^{1/q}\left(\sum_{n\in\mathbb{Z}}\left[a^{n+1}b_{n+1}\right]^p\right)^{-1/p}\right\}.$$

We denote

$$\alpha_m = a^{m+1}b_{m+1}, \quad \varphi_m = a^{-(m+1)}\Psi\left(\lambda_{m+1},\lambda_{m+2}\right), \quad m \in \mathbb{Z}, \tag{2.33}$$

and obtain

$$H_2 = \sup_{\alpha_n \geqslant 0} \left\{ \left( \sum_{n\in\mathbb{Z}} w_n^q \left( \sum_{m\geqslant n} \alpha_m \varphi_m \right)^q \right)^{1/q} \left( \sum_{n\in\mathbb{Z}} \alpha_n^p \right)^{-1/p} \right\}.$$

We apply now the discrete generalized Hardy inequality (see [8; Theorem 2.1, Remark 2.5]. According to it we have

$$c_1^{-1}(p,q)\, F \leqslant H_2 \leqslant c_1(p,q)\, F, \quad c_1(p,q) \in [1,\infty), \tag{2.34}$$

where

$$F = \sup_n \left[\Phi_n W_n\right], \quad p \leqslant q; \tag{2.35}$$

$$F = \left\{ \sum_{n\in\mathbb{Z}} \Phi_n^s \left[W_n^s - W_{n-1}^s\right] \right\}^{1/s}, \quad p > q. \tag{2.36}$$

Here,

$$W_n = \left( \sum_{m\leqslant n} w_m^q \right)^{1/q} = W_q(\lambda_{n+1}), \quad \Phi_n = \left( \sum_{m\geqslant n} \varphi_m^\sigma \right)^{1/\sigma}, \tag{2.37}$$

with

$$\sigma = \infty \text{ if } p \in (0,1]; \quad \sigma = p' \text{ if } p > 1. \tag{2.38}$$

Now, we introduce

$$B_n = \left( \sum_{m\geqslant n} \left[a^{-m} \Psi(\lambda_n, \lambda_{m+1})\right]^\sigma \right)^{1/\sigma}, \quad n \in Z, \tag{2.39}$$

and show that

$$\left(1 - a^{-1}\right) B_{n+1} \leqslant \Phi_n \leqslant B_{n+1}. \tag{2.40}$$

Indeed, we see by (2.37) and (2.33) that

$$\Phi_n \leqslant \left( \sum_{m\geqslant n} \left[a^{-(m+1)} \Psi(\lambda_{n+1}, \lambda_{m+2})\right]^\sigma \right)^{1/\sigma}$$
$$= \left( \sum_{m\geqslant n+1} \left[a^{-m} \Psi(\lambda_{n+1}, \lambda_{m+1})\right]^\sigma \right)^{1/\sigma} = B_{n+1}$$

However,

$$B_{n+1} = \left( \sum_{m\geqslant n+1} \left[a^{-m} \sum_{i=n+1}^{m} \beta_i \right]^\sigma \right)^{1/\sigma}, \quad \beta_i = \Psi(\lambda_i, \lambda_{i+1}),$$

and we can apply inequality (2.14) with the appropriate notation:

$$W_m = a^m, \quad W = a, \sigma \in [1,\infty], \quad c = \left(1 - a^{-1}\right)^{-1}.$$

Then,

$$B_{n+1} \leqslant c\left(\sum_{m\geqslant n+1}\left[a^{-m}\beta_m\right]^{\sigma}\right)^{1/\sigma} = c\left(\sum_{m\geqslant n}\left[a^{-(m+1)}\beta_{m+1}\right]^{\sigma}\right)^{1/\sigma}$$
$$= c\left(\sum_{m\geqslant n}\left[a^{-(m+1)}\Psi\left(\lambda_{m+1},\lambda_{m+2}\right)\right]^{\sigma}\right)^{1/\sigma} = c\left(\sum_{m\geqslant n}\varphi_m^{\sigma}\right)^{1/\sigma} = c\Phi_n,$$

and (2.40) follows. Now, let us derive the estimate

$$a^{-1}\left(1-a^{-\sigma}\right)^{1/\sigma}B_n \leqslant V_p\left(\lambda_n\right) \leqslant B_n, \quad n\in\mathbb{Z}. \tag{2.41}$$

For $p\in(0,1]$ we have (see (1.5) and (2.39) with $\sigma=\infty$)

$$V_p\left(\lambda_n\right) = \sup_{m\geqslant n}\left\{\sup_{\tau\in(\lambda_m,\lambda_{m+1}]}\left[\Psi\left(\lambda_n,\tau\right)\frac{1}{\omega_p\left(\tau\right)}\right]\right\}$$
$$\leqslant \sup_{m\geqslant n}\left\{\Psi\left(\lambda_n,\lambda_{m+1}\right)\sup_{\tau\in(\lambda_m,\lambda_{m+1}]}\frac{1}{\omega_p\left(\tau\right)}\right\} = \sup_{m\geqslant n}\left\{\Psi\left(\lambda_n,\lambda_{m+1}\right)a^{-m}\right\} = B_n;$$

$$V_p\left(\lambda_n\right) \geqslant \sup_{m\geqslant n+1}\left\{\Psi\left(\lambda_n,\lambda_m\right)\sup_{\tau\in\Delta_m}\frac{1}{\omega_p\left(\tau\right)}\right\} = \sup_{m\geqslant n+1}\left\{\Psi\left(\lambda_n,\lambda_m\right)a^{-m}\right\}$$
$$= \sup_{m\geqslant n}\left\{\Psi\left(\lambda_n,\lambda_{m+1}\right)a^{-(m+1)}\right\} = a^{-1}B_n. \tag{2.42}$$

Thus, (2.41) holds with $\sigma=\infty$. Now, let $p>1, \sigma=p'$. We have, taking into account (1.6), that

$$V_p^{\sigma}\left(\lambda_n\right) = \sum_{m\geqslant n}\int_{(\lambda_m,\lambda_{m+1}]}\Psi^{\sigma}\left(\lambda_n,\tau\right)\left(-d\left[\frac{1}{\omega_p^{\sigma}\left(\tau\right)}\right]\right)$$
$$\leqslant \sum_{m\geqslant n}\Psi^{\sigma}\left(\lambda_n,\lambda_{m+1}\right)\int_{(\lambda_m,\infty)}\left(-d\left[\frac{1}{\omega_p^{\sigma}\left(\tau\right)}\right]\right) \leqslant \sum_{m\geqslant n}\Psi^{\sigma}\left(\lambda_n,\lambda_{m+1}\right)a^{-m\sigma} = B_n^{\sigma};$$

$$V_p^{\sigma}\left(\lambda_n\right) \geqslant \sum_{m\geqslant n+1}\Psi^{\sigma}\left(\lambda_n,\lambda_m\right)\int_{\Delta_m}\left(-d\left[\frac{1}{\omega_p^{\sigma}\left(\tau\right)}\right]\right)$$
$$= \sum_{m\geqslant n+1}\Psi^{\sigma}\left(\lambda_n,\lambda_m\right)\left[a^{-m\sigma}-a^{-(m+1)\sigma}\right] = \left(1-a^{-\sigma}\right)\sum_{m\geqslant n+1}\Psi^{\sigma}\left(\lambda_n,\lambda_m\right)a^{-m\sigma}$$
$$= a^{-\sigma}\left(1-a^{-\sigma}\right)\sum_{m\geqslant n+1}\Psi^{\sigma}\left(\lambda_n,\lambda_m\right)a^{-(m-1)\sigma} = a^{-\sigma}\left(1-a^{-\sigma}\right)B_n^{\sigma},$$

and (2.41) follows for $p>1$. Thus, (2.41) is established for all $p\in R_+$.

Estimates (2.41) and (2.40) imply

$$\left(1-a^{-1}\right) V_p\left(\lambda_{n+1}\right) \leqslant \Phi_n \leqslant a\left(1-a^{-\sigma}\right)^{-1/\sigma} V_p\left(\lambda_{n+1}\right), \quad n \in \mathbb{Z}. \tag{2.43}$$

We insert these estimates in (2.35) and in (2.36) and (2.37), obtain that, taking into account

$$\left(1-a^{-1}\right) G \leqslant F \leqslant a\left(1-a^{-\sigma}\right)^{-1/\sigma} G,$$

where

$$G=\sup_n\left\{V_p\left(\lambda_{n+1}\right) W_q\left(\lambda_{n+1}\right)\right\}=\tilde{F}_{pq}, \qquad p \leqslant q,$$

$$G=\left\{\sum_{n \in \mathbb{Z}} V_p^s\left(\lambda_{n+1}\right)\left[W_q^s\left(\lambda_{n+1}\right)-W_q^s\left(\lambda_n\right)\right]\right\}^{1/s}=\tilde{F}_{pq}, \quad p>q.$$

These assertions together with (2.34) yield

$$\tilde{c}_1^{-1}(p, q, a) \tilde{F}_{pq} \leqslant H_2 \leqslant \tilde{c}_1(p, q, a) \tilde{F}_{pq}. \tag{2.44}$$

Finally, by (2.23), (2.28), (2.32), and (2.44) we obtain estimate (2.8) which completes the proof of Theorem 2.1. □

**Remark 2.5.** By (2.42) it follows in particular, that for $p \in (0,1]$

$$V_p\left(\lambda_n\right) \geqslant a^{-(m+1)} \Psi\left(\lambda_n, \lambda_{m+1}\right), \quad m \geqslant n. \tag{2.45}$$

The same estimate remains true for $p>1$. Indeed, similarly to (2.43), we have for $m \geqslant n$

$$\begin{aligned}
V_p^\sigma\left(\lambda_n\right)= & \int_{(\lambda_n, \infty)} \Psi^\sigma\left(\lambda_n, \tau\right)\left(-d\left[\frac{1}{\omega_p^\sigma(\tau)}\right]\right) \\
& \geqslant \int_{[\lambda_{m+1}, \infty)} \Psi^\sigma\left(\lambda_n, \tau\right)\left(-d\left[\frac{1}{\omega_p^\sigma(\tau)}\right]\right) \\
& \geqslant \Psi^\sigma\left(\lambda_n, \lambda_{m+1}\right) \int_{[\lambda_{m+1}, \infty)}\left(-d\left[\frac{1}{\omega_p^\sigma(\tau)}\right]\right)=\Psi^\sigma\left(\lambda_n, \lambda_{m+1}\right) a^{-(m+1) \sigma}.
\end{aligned}$$

## 3 Some equivalent criteria of the finiteness of $H_{\Omega_1}\left(B_\mu\right)$ in the discrete form

In this Section we preserve the notations of Section 2.

**Proposition 3.1.** *Let the conditions of Theorem 2.1 be satisfied and let*

$$\tilde{E}_{pq}=\left\{\sum_{n \in \mathbb{Z}}\left[a^{-n}\left(\int_{\Delta_n} \Psi^q\left(t, \lambda_{n+2}\right) d \gamma(t)\right)^{1/q}\right]^s\right\}^{1/s}. \tag{3.1}$$

*Then*

$$c_0\left(\tilde{E}_{pq}+\tilde{F}_{pq}\right)\leqslant\left(\tilde{E}^0_{pq}+\tilde{F}_{pq}\right)\leqslant\left(\tilde{E}_{pq}+\tilde{F}_{pq}\right). \tag{3.2}$$

**Corollary 3.2.** *These assertions together with Theorem 2.1 imply following the two-sided estimate:*

$$\tilde{c}_1\left(\tilde{E}_{pq}+\tilde{F}_{pq}\right)\leqslant H_{\Omega_1}\left(B_\mu\right)\leqslant\tilde{c}_0\left(\tilde{E}_{pq}+\tilde{F}_{pq}\right) \tag{3.3}$$

**Remark 3.3.** Inequality (3.3) was proved directly in [5; Theorem 2.1].

To prove Proposition 3.1 we need the following lemma.

**Lemma 3.4.** *For $m\in N, l\in N_0, 0<q\leqslant s<\infty$, or $0<q<\infty, s=\infty$ consider the quantity*

$$A_{m,l}=\left\{\sum_{n\in\mathbb{Z}}\left[a^{-n}\left(\int\limits_{[\lambda_{n-l},\lambda_{n+1})}\Psi^q\left(t,\lambda_{n+m}\right)d\gamma\left(t\right)\right)^{1/q}\right]^s\right\}^{1/s}.$$

*Under notation of Subsection 2.1, the following estimates hold with $c=c\left(a,m,l,p,q\right)\in R_+$ :*

$$A_{1,0}\leqslant A_{m,l}\leqslant c\left(A_{1,0}+\tilde{F}_{pq}\right). \tag{3.4}$$

**Proof.** The left hand side inequality is evident. Let us prove the right–hand–side one.

**1.** We notice that

$$\Psi\left(\lambda_{n+1},\lambda_{n+m}\right)\leqslant a^{m+n}V_p\left(\lambda_{n+1}\right), m\geqslant 2. \tag{3.5}$$

(see (2.45)). First, let $s=\infty$. It corresponds to the case $p\leqslant q$, when $\tilde{F}_{pq}$ is determined by (2.6). In this case

$$A_{m,0}=\sup_n\left[a^{-n}\left(\int\limits_{\Delta_n}\Psi^q\left(t,\lambda_{n+m}\right)d\gamma\left(t\right)\right)^{1/q}\right]. \tag{3.6}$$

The equality

$$\Psi\left(t,\lambda_{n+m}\right)=\Psi\left(t,\lambda_{n+1}\right)+\Psi\left(\lambda_{n+1},\lambda_{n+m}\right), \tag{3.7}$$

together with the triangle inequality in $L_q$ imply

$$A_{m,0}\leqslant c\left(A_{1,0}+\tilde{A}_{m,0}\right), \tag{3.8}$$

with $c=\bar{c}_q$, where $\bar{c}_q=1, \quad q\in[1,\infty); \quad \bar{c}_q=2^{1/q-1}, \quad q\in(0,1)$. Here,

$$\tilde{A}_{m,0}=\sup_n\left[a^{-n}\Psi\left(\lambda_{n+1},\lambda_{n+m}\right)\left(\int\limits_{\Delta_n}d\gamma\left(t\right)\right)^{1/q}\right].$$

Now, we apply (3.5), (1.7), and (2.6) and obtain that

$$\tilde{A}_{m,0}\leqslant a^m\sup_n\left[V_p\left(\lambda_{n+1}\right)W_q\left(\lambda_{n+1}\right)\right]=a^m\tilde{F}_{pq}.$$

Thus, (3.4) follows in the case $l = 0, m \geqslant 2$.

Next let $l, m \in N$. Then,

$$A_{m,l} \leqslant \left( \sum_{j=0}^{l} a^{(j-l)q} A_{l+m-j,0}^q \right)^{1/q}. \tag{3.9}$$

Indeed,

$$\int\limits_{[\lambda_{n-l}, \lambda_{n+1})} \Psi^q(t, \lambda_{n+m})\, d\gamma = \sum_{j=0}^{l} \int\limits_{\Delta_{n-l+j}} \Psi^q(t, \lambda_{n+m})\, d\gamma, \tag{3.10}$$

and

$$A_{m,l}^q \leqslant \sum_{j=0}^{l} \sup_n \left[ a^{-nq} \left( \int\limits_{\Delta_{n-l+j}} \Psi^q(t, \lambda_{n+m})\, d\gamma \right) \right]$$
$$= \sum_{j=0}^{l} a^{(j-l)q} \sup_k \left[ a^{-kq} \left( \int\limits_{\Delta_k} \Psi^q(t, \lambda_{k+l-j+m})\, d\gamma \right) \right] = \sum_{j=0}^{l} a^{(j-l)q} A_{l+m-j,0}^q.$$

Now, to each term in (3.9) we apply the already proved variant of estimate (3.4) and obtain (3.4) in the general case.

**2.** Let now $0 < q \leqslant s < \infty$, and let $\tilde{F}_{pq}$ be determined by (2.7).

For $l = 0, m \geqslant 2$ we see by the expression for $A_{m,l}$ that

$$A_{m,0} = \left( \left\| \left\{ a^{-nq} \left( \int\limits_{\Delta_n} \Psi^q(t, \lambda_{n+m})\, d\gamma \right) \right\} \right\|_{l_{s/q}} \right)^{1/q}. \tag{3.11}$$

By (3.7), it follows that

$$\Psi^q(t, \lambda_{n+m}) \leqslant c_q \left[ \Psi^q(t, \lambda_{n+1}) + \Psi^q(\lambda_{n+1}, \lambda_{n+m}) \right],$$

where $c_q = 1, q \in (0,1]$; $c_q = 2^{q-1}, q > 1$. We insert this estimate in (3.11) and, taking into account the triangle inequality in $l_{s/q}$ (here $s \geqslant q$), we obtain that

$$A_{m,0} \leqslant c_q^{1/q} \left( A_{1,0}^q + \tilde{A}_{m,0}^q \right)^{1/q},$$

where now

$$\tilde{A}_{m,0} = \left( \left\| \left\{ a^{-nq} \Psi^q(\lambda_{n+1}, \lambda_{n+m}) \left( \int\limits_{\Delta_n} d\gamma \right) \right\} \right\|_{l_{s/q}} \right)^{1/q}$$
$$= \left\{ \sum_{n \in \mathbb{Z}} a^{-ns} \Psi^s(\lambda_{n+1}, \lambda_{n+m}) \left[ W_q^q(\lambda_{n+1}) - W_q^q(\lambda_n) \right]^{s/q} \right\}^{1/s}. \tag{3.12}$$

Thus, to prove (3.4) it suffices to show that

$$\tilde{A}_{m,0} \leqslant c\tilde{F}_{pq}, \quad c = c\,(m,p,q,a) \in R_+. \tag{3.13}$$

We insert (3.5) in (3.12), and obtain

$$\tilde{A}_{m,0} \leqslant a^m \left\{ \sum_{n\in\mathbb{Z}} V_p^s\,(\lambda_{n+1}) \left[ W_q^q\,(\lambda_{n+1}) - W_q^q\,(\lambda_n) \right]^{s/q} \right\}^{1/s}.$$

Next, we use the following inequality (see (3.24) below; recall that here $q \leqslant s$):

$$\left[ W_q^q\,(\lambda_{n+1}) - W_q^q\,(\lambda_n) \right]^{s/q} \leqslant c_{q,s}^s \left[ W_q^s\,(\lambda_{n+1}) - W_q^s\,(\lambda_n) \right].$$

Therefore,

$$\tilde{A}_{m,0} \leqslant a^m c_{q,s} \left\{ \sum_{n\in\mathbb{Z}} V_p^s\,(\lambda_{n+1}) \left[ W_q^s\,(\lambda_{n+1}) - W_q^s\,(\lambda_n) \right] \right\}^{1/s}. \tag{3.14}$$

Together with (2.7) this implies (3.13). Estimate (3.4) is proved for $l = 0$, $m \geqslant 2$.

Next, let $l \geqslant 1$. Inequality (3.9) remains true by similar arguments. Namely, we use equality (3.10) and then the triangle inequality in

$$\begin{aligned} A_{m,l} &= \left( \left\| \left\{ \sum_{j=0}^{l} \left( a^{-nq} \int\limits_{\Delta_{n-l+j}} \Psi^q\,(t, \lambda_{n+m})\,d\gamma \right) \right\} \right\|_{l_{s/q}} \right)^{1/q} \\ &\leqslant \left( \sum_{j=0}^{l} \left\| \left\{ a^{-nq} \int\limits_{\Delta_{n-l+j}} \Psi^q\,(t, \lambda_{n+m})\,d\gamma \right\} \right\|_{l_{s/q}} \right)^{1/q} \\ &= \left( \sum_{j=0}^{l} \left[ \sum_{n\in\mathbb{Z}} a^{-ns} \left( \int\limits_{\Delta_{n-l+j}} \Psi^q\,(t, \lambda_{n+m})\,d\gamma \right)^{s/q} \right]^{q/s} \right)^{1/q}. \end{aligned}$$

We replace here $n$ by $k = n - l + j$, and have

$$\begin{aligned} A_{m,l} &\leqslant \left( \sum_{j=0}^{l} a^{(j-l)q} \left[ \sum_{k\in\mathbb{Z}} a^{-ks} \left( \int\limits_{\Delta_k} \Psi^q\,(t, \lambda_{k+m+l-j})\,d\gamma \right)^{s/q} \right]^{q/s} \right)^{1/q} \\ &= \left( \sum_{j=0}^{l} a^{(j-l)q} A_{m+l-j,0}^q \right)^{1/q}. \end{aligned}$$

To each term we apply the already proved estimate (3.4) with $m+l-j$ instead of $m$, so that

$$A_{m+l-j,0} \leqslant c\left(A_{1,0} + \tilde{F}_{pq}\right).$$

As the result we obtain (3.4) for $m \in N, l \geqslant 1$. □

**Corollary 3.5.** *By notation (2.5) and (3.1) we see that*

$$\tilde{E}_{pq} = A_{2,0}; \tilde{E}^{0}_{pq} = A_{1,0}.$$

*Therefore, according to (3.4) we have,*

$$\tilde{E}^{0}_{pq} \leqslant \tilde{E}_{pq} \leqslant c\left(\tilde{E}^{0}_{pq} + \tilde{F}_{pq}\right). \tag{3.15}$$

**Remark 3.6.** *This implies estimates (3.2) thus proving Proposition 3.1.*

The following particular cases of Theorem 2.1 are of special interest.

**Proposition 3.7.** *Let, in addition to the hypotheses of Theorem 2.1, $V_p \circ \omega_p^{-1} \in \Delta_2$ that is*

$$D_a = \sup_{t \in R_+} \left[V_p\left(\omega_p^{-1}(t)\right) / V_p\left(\omega_p^{-1}(at)\right)\right] < \infty, \tag{3.16}$$

*for a given $a > 1$. Then,*

$$\tilde{E}^{0}_{pq} \leqslant a D_a \tilde{F}_{pq}, \tag{3.17}$$

*and, consequently,*

$$H_{\Omega_1}(B_\mu) \cong \tilde{F}_{pq}. \tag{3.18}$$

**Proof. 1.** The condotion (3.16) implies

$$V_p(\lambda_n) = V_p\left(\omega_p^{-1}(a^n)\right) \leqslant D_a V_p\left(\omega_p^{-1}\left(a^{n+1}\right)\right) = D_a V_p(\lambda_{n+1}). \tag{3.19}$$

For $p \leqslant q$ we have

$$\tilde{E}^{0}_{pq} = \sup_n \left\{ a^{-n} \left( \int_{\Delta_n} \Psi^q(t, \lambda_{n+1})\, d\gamma(t) \right)^{1/q} \right\}$$
$$\leqslant \sup_n \left\{ a^{-n} \Psi(\lambda_n, \lambda_{n+1}) \left( \int_{\Delta_n} d\gamma \right)^{1/q} \right\}. \tag{3.20}$$

Next, according to (2.45) (with $m = n$) and (3.19), we have

$$a^{-n}\Psi(\lambda_n, \lambda_{n+1}) \leqslant a V_p(\lambda_n) \leqslant a D_a V_p(\lambda_{n+1}); \tag{3.21}$$

and also,

$$\left( \int_{\Delta_n} d\gamma \right)^{1/q} \leqslant \left( \int_{(0,\lambda_{n+1})} d\gamma \right)^{1/q} = W_q(\lambda_{n+1}).$$

We insert these inequalities in (3.20) and obtain

$$\tilde{E}^0_{pq} \leqslant aD_a \sup_n \left\{ V_p\left(\lambda_{n+1}\right) W_q\left(\lambda_{n+1}\right) \right\} = aD_a \tilde{F}_{pq}.$$

Thus, (3.17) is proved for $p \leqslant q$.

**2.** Next, let $p > q$. Then,

$$\left(\tilde{E}^0_{pq}\right)^s = \sum_{n\in\mathbb{Z}} \left[ a^{-n} \left( \int_{\Delta_n} \Psi^q\left(t, \lambda_{n+1}\right) d\gamma\left(t\right) \right)^{1/q} \right]^s$$
$$\leqslant \sum_{n\in\mathbb{Z}} \left[ a^{-n} \Psi\left(\lambda_n, \lambda_{n+1}\right) \left( \int_{\Delta_n} d\gamma\left(t\right) \right)^{1/q} \right]^s$$
$$= \sum_{n\in\mathbb{Z}} \left[ a^{-ns} \Psi^s\left(\lambda_n, \lambda_{n+1}\right) \left[ W_q^q\left(\lambda_{n+1}\right) - W_q^q\left(\lambda_n\right) \right]^{s/q} \right]. \quad (3.22)$$

Now, we use the following inequality: let $0 < b \leqslant d, s \geqslant q > 0$, then,

$$\left(d^q - b^q\right)^{1/q} \leqslant c_{q,s}\left(d^s - b^s\right)^{1/s}, \quad (3.23)$$

where $c_{q,s} \in R_+$ does not depend on $d$ and $b$. We set in it $d = W_q\left(\lambda_{n+1}\right), b = W_q\left(\lambda_n\right)$ and see that

$$\left[ W_q^q\left(\lambda_{n+1}\right) - W_q^q\left(\lambda_n\right) \right]^{s/q} \leqslant c^s_{q,s} \left[ W_q^s\left(\lambda_{n+1}\right) - W_q^s\left(\lambda_n\right) \right]. \quad (3.24)$$

We substitute inequalities (3.21) and (3.24) in (3.22) and obtain

$$\tilde{E}^0_{pq} \leqslant c \left\{ \sum_{n\in\mathbb{Z}} V_p^s\left(\lambda_{n+1}\right) \left[ W_q^s\left(\lambda_{n+1}\right) - W_q^s\left(\lambda_n\right) \right] \right\}^{1/s.} = c\tilde{F}_{pq}.$$

□

**Remark 3.8.** For the sake of completeness we present here the proof of inequality (3.23).

If $0 < b < d/2$, then $d^q - b^q \cong d^q, d^s - b^s \cong d^s$. This yields (3.23). If $d/2 \leqslant b \leqslant d$, then

$$d^q - b^q \cong d^{q-1}\left(d - b\right), \quad d^s - b^s \cong d^{s-1}\left(d - b\right),$$

and therefore, for $s \geqslant q$ we have

$$\left(d^q - b^q\right)^{1/q} \cong d^{1-1/q}\left(d - b\right)^{1/q} = d^{1-1/q}\left(d - b\right)^{1/q-1/s}\left(d - b\right)^{1/s}$$
$$\leqslant d^{1-1/q} d^{1/q-1/s}\left(d - b\right)^{1/s} = d^{1-1/s}\left(d - b\right)^{1/s} \cong \left(d^s - b^s\right)^{1/s}.$$

**Proposition 3.9.** *Let the hypotheses of Theorem 2.1 be satisfied. If for a given* $a > 1$,

$$\delta_a = \inf_{t\in R_+} \left[ W_q\left(\omega_p^{-1}\left(at\right)\right) / W_q\left(\omega_p^{-1}\left(t\right)\right) \right] > 1, \quad (3.25)$$

*then*

$$\tilde{F}_{pq} \leqslant c\tilde{E}_{pq}, \tag{3.26}$$

*with*

$$c = \delta_a^2 \left[ (a-1)(\delta_a - 1)\left(\delta_a^q - 1\right)^{1/q} \right]^{-1},$$

*and, consequently,*

$$H_{\Omega_1}(B_\mu) \cong \tilde{E}_{pq}. \tag{3.27}$$

**Proof.** From (2.6) and (2.7), we have

$$\tilde{F}_{pq} \leqslant \left\{ \sum_{n\in\mathbb{Z}} \left[ V_p(\lambda_{n+1}) W_q(\lambda_{n+1}) \right]^s \right\}^{1/s}.$$

Estimates (2.41) and (2.42) show that

$$V_p(\lambda_{n+1}) \leqslant \left(1 - a^{-1}\right)^{-1} \Phi_n, \quad \Phi_n = \left( \sum_{m\leqslant n} \varphi_m^\sigma \right)^{1/\sigma}, \quad \sigma \in (1, \infty],$$

where $\varphi_m$ were defined in (2.33). Thus,

$$\tilde{F}_{pq} \leqslant \left(1 - a^{-1}\right)^{-1} \left\{ \sum_{n\in\mathbb{Z}} \left[ W_q(\lambda_{n+1}) \left( \sum_{m\leqslant n} \varphi_m^\sigma \right)^{1/\sigma} \right]^s \right\}^{1/s}. \tag{3.28}$$

Let us note that according to (3.25),

$$W_q(\lambda_{n+1}) \geqslant \delta_a W_q(\lambda_n), \quad n \in Z. \tag{3.29}$$

Indeed,

$$W_q(\lambda_{n+1}) = W_q\left(\omega_p^{-1}\left(a^{n+1}\right)\right) \geqslant \delta_a W_q\left(\omega_p^{-1}(a^n)\right) = \delta_a W_q(\lambda_n).$$

It means that Proposition 2.3 is applicable to (3.28) with $W_n = W_q(\lambda_{n+1})$, $W = \delta_a > 1$. Therefore, we have

$$\tilde{F}_{pq} \leqslant c_a \left\{ \sum_{n\in\mathbb{Z}} \left[ \varphi_n W_q(\lambda_{n+1}) \right]^s \right\}^{1/s}, \quad c_a = a\delta_a \left[ (a-1)(\delta_a - 1) \right]^{-1}.$$

We substitute here formulas (2.33), and obtain

$$\tilde{F}_{pq} \leqslant c_a \left\{ \sum_{n\in\mathbb{Z}} \left[ a^{-(n+1)} \Psi(\lambda_{n+1}, \lambda_{n+2}) W_q(\lambda_{n+1}) \right]^s \right\}^{1/s}. \tag{3.30}$$

On the other hand, (3.1) gives

$$\tilde{E}_{pq} \geqslant \left\{ \sum_{n\in\mathbb{Z}} \left[ a^{-n} \Psi(\lambda_{n+1}, \lambda_{n+2}) \left( \int_{\Delta_n} d\gamma \right)^{1/q} \right]^s \right\}$$

$$= \left\{ \sum_{n\in\mathbb{Z}} \left[ a^{-n} \Psi(\lambda_{n+1}, \lambda_{n+2}) \left( W_q^q(\lambda_{n+1}) - W_q^q(\lambda_n) \right)^{1/q} \right]^s \right\}^{1/s}.$$

Now, we apply estimate (3.29), and obtain

$$\tilde{E}_{pq} \geqslant a\left(1-\delta_a^{-q}\right)^{1/q}\left\{\sum_{n\in\mathbb{Z}}\left[a^{-(n+1)}\Psi\left(\lambda_{n+1},\lambda_{n+2}\right)W_q\left(\lambda_{n+1}\right)\right]^s\right\}^{1/s}. \tag{3.31}$$

Estimates (3.30) and (3.31) imply (3.26). Finally, (3.26) and (3.3) yield (3.27). □

*Next, we present one equivalent criterion of the finiteness of $H_{\Omega_1}(B_\mu)$ in the discrete form slightly different from (2.8) and (3.3).*

For given $a>1$ we consider the function

$$\xi_a(t)=\omega_p^{-1}\left(a\omega_p(t)\right),\quad t\in R_+. \tag{3.32}$$

This definition is similar to (1.7), but with $a>1$ instead of $\alpha\in(0,1)$, so that $\xi_a(t)>t$. We define

$$\psi(t)=\Psi\left(t,\xi_a(t)\right),\quad t\in R_+, \tag{3.33}$$

$$\tilde{\varepsilon}_{pq}=\left\{\sum_{n\in\mathbb{Z}}\left[a^{-n}\left(\int_{\Delta_n}\psi^q d\gamma\right)^{1/q}\right]^s\right\}^{1/s}. \tag{3.34}$$

where, as always, $s=\infty$ for $p\leqslant q$, $s=pq/(p-q)$ for $p>q$.

**Proposition 3.10.** *Under the hypotheses of Theorem 2.1 the following estimate holds*

$$H_{\Omega_1}(B_\mu)\cong\tilde{\varepsilon}_{pq}+\tilde{F}_{pq}. \tag{3.35}$$

**Proof.** First, we prove the two-sided estimate

$$\left(\int_{\Delta_n}\Psi^q\left(t,\lambda_{n+1}\right)d\gamma\right)^{1/q}\leqslant\left(\int_{\Delta_n}\psi^q d\gamma\right)^{1/q}\leqslant\left(\int_{\Delta_n}\Psi^q\left(t,\lambda_{n+2}\right)d\gamma\right)^{1/q}. \tag{3.36}$$

Let us note that

$$t\in\Delta_n=[\lambda_n,\lambda_{n+1})\Rightarrow a^{n+1}\leqslant a\omega_p(t)<a^{n+2}\Rightarrow\lambda_{n+1}\leqslant\xi_a(t)\leqslant\lambda_{n+2},$$

so that

$$\Psi\left(t,\lambda_{n+1}\right)\leqslant\Psi\left(t,\xi_a(t)\right)\leqslant\Psi\left(t,\lambda_{n+2}\right), \tag{3.37}$$

and for the integral

$$\int_{\Delta_n}\psi^q d\gamma=\int_{\Delta_n}\Psi^q\left(t,\xi_a(t)\right)d\gamma(t) \tag{3.38}$$

we obtain inequality (3.36). Therefore, (see (3.34), (2.5), and (3.1))

$$\tilde{E}^0_{pq}\leqslant\tilde{\varepsilon}_{pq}\leqslant\tilde{E}_{pq}, \tag{3.39}$$

and

$$\tilde{E}^0_{pq}+\tilde{F}_{pq}\leqslant\tilde{\varepsilon}_{pq}+\tilde{F}_{pq}\leqslant\tilde{E}_{pq}+\tilde{F}_{pq}. \tag{3.40}$$

Together with (3.2) and (3.3) this gives (3.35). □

**Remark 3.11.** *Introduce the quantities*

$$V_n = \left( \sum_{m \geqslant n} \left[ \psi\left(\lambda_m\right) a^{-m} \right]^{\sigma} \right)^{1/\sigma}, \tag{3.41}$$

*where* $\sigma = \infty$ *for* $p \in (0,1]\,;\sigma = p'$ *for* $p > 1$,

$$\tilde{f}_{pq} = \sup_n \left\{ V_n W_q\left(\lambda_n\right) \right\}, \quad p \leqslant q; \tag{3.42}$$

$$\tilde{f}_{pq} = \left\{ \sum_{n \in \mathbb{Z}} V_n^s \left[ W_q^s\left(\lambda_n\right) - W_q^s\left(\lambda_{n-1}\right) \right] \right\}^{1/s}, \quad p > q. \tag{3.43}$$

*Then,*

$$H_{\Omega_1}\left(B_\mu\right) \cong \tilde{\varepsilon}_{pq} + \tilde{f}_{pq}. \tag{3.44}$$

Indeed, $\psi\left(\lambda_m\right) = \Psi\left(\lambda_m, \lambda_{m+1}\right)$. Therefore, according to (2.33), (2.37), (2.40), and (2.41) we have $V_p\left(\lambda_n\right) \cong V_n$, so that

$$\tilde{f}_{pq} \cong \tilde{F}_{pq}, \tag{3.45}$$

and (3.44) follows by (3.35).

## 4 The criterion of the finiteness of $H_{\Omega_1}\left(B_\mu\right)$ in the continual form

Here, we prove a particular case of Theorem 1.1 on the cone of decreasing functions $\Omega_1$, see (2.1). All the notation of Sections 1, 2 is preserved here with $k\left(t\right) \equiv 1$.

**Theorem 4.1.** *Let* $\beta \in N_p\left(1\right)$, *and let the functions* $\omega_p$ *and* $W_q$ *be positive and continuous on* $R_+, \omega_p\left(+0\right) = 0$. *Then there exists* $c_1 = c_1\left(p,q,\alpha\right) \in [1,\infty)$ *such that*

$$c_1^{-1}\left(E_{pq} + F_{pq}\right) \leqslant H_{\Omega_1}\left(B_\mu\right) \leqslant c_1\left(E_{pq} + F_{pq}\right). \tag{4.1}$$

*Here, if* $p, q \in [\delta, \infty)$ *for a certain* $\delta \in R_+$ , *then* $1 \leqslant c_1\left(p,q,\alpha\right) \leqslant \bar{c}_1\left(\delta,\alpha\right) < \infty$.

**Remark 4.2.** *Assertions* (4.1) *remain true for* $p \leqslant q$ *if we replace there* $E_{pq}$ *by* $\dot{E}_{pq}$.

**Proof. 1.** First, we consider the case $p \leqslant q$. Let us show that

$$E_{pq} \leqslant \dot{E}_{pq} \leqslant H_{\Omega_1}\left(B_\mu\right) \tag{4.2}$$

(see (1.8), (1.24)). The first inequality is obvious. We prove the second one.

For all $p, q \in R_+$ we have

$$H_{\Omega_1}(B_\mu) = \sup_{f\in\Omega_1} \frac{\left(\int\limits_{R_+} (B_\mu f)^q \, d\gamma\right)^{1/q}}{\left(\int\limits_{R_+} f^p d\beta\right)^{1/p}} \geqslant \sup_{\tau>0} \frac{\left(\int\limits_{R_+} \left(B_\mu \chi_{(0,\tau)}\right)^q d\gamma\right)^{1/q}}{\left(\int\limits_{R_+} \left(\chi_{(0,\tau)}\right)^p d\beta\right)^{1/p}}$$

$$= \sup_{\tau>0} \frac{\left(\int\limits_{R_+} \left(\int\limits_{[t,\infty)} \chi_{(0,\tau)} d\mu\right)^q d\gamma(t)\right)^{1/q}}{\omega_p(\tau)}$$

$$= \sup_{\tau>0} \frac{\left(\int\limits_{(0,\tau)} \left(\int\limits_{[t,\tau)} d\mu\right)^q d\gamma(t)\right)^{1/q}}{\omega_p(\tau)} = \dot{E}_{pq}. \quad (4.3)$$

This means, in particular, that

$$E_{pq} \leqslant \dot{E}_{pq} \leqslant c\left[\tilde{E}^0_{pq} + \tilde{F}_{pq}\right] \quad (4.4)$$

because of estimate (2.8). Later, it will be proved that

$$a^{-2}\tilde{E}_{pq} \leqslant E_{pq}, \quad (4.5)$$

and also

$$\tilde{F}_{pq} \leqslant F_{pq} \leqslant c\left[\tilde{E}^0_{pq} + \tilde{F}_{pq}\right]. \quad (4.6)$$

Therefore, for $p \leqslant q$ we will have

$$E_{pq} + F_{pq} \cong \dot{E}_{pq} + F_{pq} \cong \tilde{E}^0_{pq} + \tilde{F}_{pq} \cong H_{\Omega_1}(B_\mu) \quad (4.7)$$

(the last assertion follows by (2.8)).

*Thus, Theorem 4.1 for $p \leqslant q$, as well as Remark 4.2 will be proved whenever (4.5) and (4.6) are established.*

Now, we prove estimate (4.5). To this end we fix the parameter of discretisation $a = \alpha^{-1/3} > 1$, where $\alpha \in (0,1)$ was introduced in (1.7), (1.8). According to (1.8), we obtain

$$E_{pq} = \sup_n \sup_{\tau\in\Delta_{n+2}} \left[\frac{1}{\omega_p(\tau)} \left(\int\limits_{[\xi_\alpha(\tau),\tau)} \Psi^q(t,\tau)\, d\gamma(t)\right)^{1/q}\right]$$

with $\xi_\alpha(\tau) = \omega_p^{-1}\left(a^{-3}\omega_p(\tau)\right)$. For $\tau \in \Delta_{n+2} = [\lambda_{n+2}, \lambda_{n+3})$ we have

$$\omega_p^{-1}\left(a^{-3}\omega_p(\tau)\right) \leqslant \omega_p^{-1}\left(a^{-3}\omega_p(\lambda_{n+3})\right) = \omega_p^{-1}(a^n) = \lambda_n,$$

so that

$$\Delta_n = [\lambda_n, \lambda_{n+1}) \subset [\xi_\alpha(\tau), \tau).$$

Therefore,

$$\int\limits_{[\xi_\alpha(\tau),\tau)} \Psi^q(t,\tau)\,d\gamma(t) \geqslant \int\limits_{\Delta_n} \Psi^q(t,\lambda_{n+2})\,d\gamma(t), \quad \tau\in\Delta_{n+2}, \tag{4.8}$$

and

$$E_{pq} \geqslant \sup_n \left[\left(\int\limits_{\Delta_n} \Psi^q(t,\lambda_{n+2})\,d\gamma(t)\right)^{1/q} \sup_{\tau\in\Delta_{n+2}} \omega_p(\tau)^{-1}\right]$$
$$= \sup_n \left[\left(\int\limits_{\Delta_n} \Psi^q(t,\lambda_{n+2})\,d\gamma(t)\right)^{1/q} a^{-(n+2)}\right] = a^{-2}\tilde{E}_{pq}.$$

(see (3.1) with $p\leqslant q$, i.e., with $s=\infty$). This proves (4.5).

The first inequality in (4.6) is evident; see (1.10) and (2.6). The second one was proved in [5; Sections 3.3, 3.4] by using the same notation as here. Therefore, we obtain inequalities (4.5) and (4.6) which completes the proof of Theorem 4.1 for $p\leqslant q$.

**2.** Next, we consider the case $p>q$. We fix the parameter of discretisation $a=\alpha^{-1/3}>1$, where $\alpha\in(0,1)$ was introduced in (1.7), (1.9). According to (1.9), we have

$$E_{pq}^s = \sum_{n\in\mathbb{Z}} \int\limits_{\Delta_{n+2}} \left(\int\limits_{[\xi_\alpha(\tau),\tau)} \Psi^q(t,\tau)\,d\gamma(t)\right)^{s/q} \left(-d\left[\frac{1}{\omega_p^s(\tau)}\right]\right). \tag{4.9}$$

We apply here estimate (4.8) and obtain

$$E_{pq}^s \geqslant \sum_{n\in\mathbb{Z}} \left(\int\limits_{\Delta_n} \Psi^q(t,\lambda_{n+2})\,d\gamma(t)\right)^{s/q} \int\limits_{\Delta_{n+2}} \left(-d\left[\frac{1}{\omega_p^s(\tau)}\right]\right)$$
$$\cong \sum_{n\in\mathbb{Z}} a^{-ns} \left(\int\limits_{\Delta_n} \Psi^q(t,\lambda_{n+2})\,d\gamma(t)\right)^{s/q} = \left(\tilde{E}_{pq}\right)^s. \tag{4.10}$$

This gives the first estimate in

$$c_1\tilde{E}_{pq} \leqslant E_{pq} \leqslant c_2\left(\tilde{E}_{pq}^0 + \tilde{F}_{pq}\right). \tag{4.11}$$

Let us prove the second one. Now, formula (1.9) is written otherwise:

$$E_{pq}^s = \sum_{n\in\mathbb{Z}} \int\limits_{\Delta_n} \left(\int\limits_{[\xi_\alpha(\tau),\tau)} \Psi^q(t,\tau)\,d\gamma(t)\right)^{s/q} \left(-d\left[\frac{1}{\omega_p^s(\tau)}\right]\right).$$

For $\tau \in \Delta_n = [\lambda_n, \lambda_{n+1})$ we have $\Psi(t,\tau) \leqslant \Psi(t, \lambda_{n+1})$,

$$\xi_\alpha(\tau) = \omega_p^{-1}\left(a^{-3}\omega_p(\tau)\right) \geqslant \omega_p^{-1}\left(a^{-3}\omega_p(\lambda_n)\right) = \omega_p^{-1}\left(a^{n-3}\right) = \lambda_{n-3},$$

so that

$$[\xi_\alpha(\tau), \tau) \subset [\lambda_{n-3}, \lambda_{n+1}),$$

and

$$E_{pq}^s \leqslant \sum_{n\in\mathbb{Z}} \int\limits_{\Delta_n} \left( \int\limits_{[\lambda_{n-3},\lambda_{n+1})} \Psi^q(t, \lambda_{n+1})\, d\gamma \right)^{s/q} \left( -d\left[\frac{1}{\omega_p^s(\tau)}\right]\right)$$

$$= \sum_{n\in\mathbb{Z}} \left( \int\limits_{[\lambda_{n-3},\lambda_{n+1})} \Psi^q(t, \lambda_{n+1})\, d\gamma \right)^{s/q} \int\limits_{\Delta_n} \left( -d\left[\frac{1}{\omega_p^s(\tau)}\right]\right)$$

$$\cong \sum_{n\in\mathbb{Z}} a^{-ns} \left( \int\limits_{[\lambda_{n-3},\lambda_{n+1})} \Psi^q(t, \lambda_{n+1})\, d\gamma \right)^{s/q} = (A_{1,3})^s.$$

(in the notation of Lemma 3.4). We apply now Lemma 3.4 with $m = 1, l = 3$ and take into account that $\tilde{E}_{pq}^0 = A_{1,0}$. Thus, the second estimate in (4.11) is proved.

Now, we need the estimate (4.6) in the case $p > q$. According to (1.11) and (2.7), we have

$$F_{pq}^s = \sum_{n\in\mathbb{Z}} \int\limits_{\Delta_n} V_p^s d\left(W_q^s\right) \geqslant \sum_{n\in\mathbb{Z}} V_p^s(\lambda_{n+1}) \int\limits_{\Delta_n} d\left(W_q^s\right) = \tilde{F}_{pq}^s.$$

This yields the first inequality in (4.6). The proof of the second one in the case $p > q$ is much more complicated. It was obtained in [5; Sections 3.5-3.7] by using the same notations as here (see [5; (3.14)]):

$$F_{pq} \leqslant c\left[\tilde{E}_{pq} + \tilde{F}_{pq}\right]$$

Finally, we recall (3.2) and arruve at the second inequality in (4.6).

Inequalities (4.6) and (4.11) imply Theorem 4.1, now for $p > q$. □

**Remark 4.3.** Let us note that under the hypotheses of Proposition 3.7, we can easily justify inequality (4.6) and show that

$$\tilde{F}_{pq} \leqslant F_{pq} \leqslant D_a \tilde{F}_{pq}. \tag{4.12}$$

Indeed, the first inequality in (4.12) is the same as in (4.6). Let us prove the second one. For $p \leqslant q$ we have by (1.10)

$$F_{pq} = \sup_n \sup_{t\in\Delta_n} \left[V_p(t) W_q(t)\right] \leqslant \sup_n V_p(\lambda_n) \sup_{t\in\Delta_n} W_q(t) \leqslant \sup_n V_p(\lambda_n) W_q(\lambda_{n+1}).$$

Now, we apply estimate (3.19) and obtain

$$F_{pq} \leqslant D_a \sup_n V_p(\lambda_{n+1}) W_q(\lambda_{n+1}) = D_a \tilde{F}_{pq}.$$

Similarly, for $p > q$, we have by (1.11) and (3.19),

$$F_{pq}^s = \sum_{n\in\mathbb{Z}} \int\limits_{\Delta_n} V_p^s dW_q^s \leqslant \sum_{n\in\mathbb{Z}} V_p^s(\lambda_n) \int\limits_{\Delta_n} dW_q^s$$
$$= \sum_{n\in\mathbb{Z}} V_p^s(\lambda_n)\left[W_q^s(\lambda_{n+1}) - W_q^s(\lambda_n)\right]$$
$$\leqslant D_a^s \sum_{n\in\mathbb{Z}} V_p^s(\lambda_{n+1})\left[W_q^s(\lambda_{n+1}) - W_q^s(\lambda_n)\right] = D_a^s \tilde{F}_{pq}^s,$$

and (4.12) follows. □

**4.2.** Next, we present one equivalent criterion of the finiteness of $H_{\Omega_1}(B_\mu)$ in the continual form slightly different from (4.1). We recall notation (3.32)-(3.34), and introduce the quantities

$$\varepsilon_{pq} = \sup_{\tau\in R_+}\left[\frac{1}{\omega_p(\tau)}\left(\int\limits_{(0,\tau)} \psi^q d\gamma\right)^{1/q}\right], \quad p \leqslant q, \tag{4.13}$$

$$\varepsilon_{pq} = \left\{\int\limits_{R_+}\left(\int\limits_{(0,\tau)} \psi^q d\gamma\right)^{s/q}\left(-d\left[\frac{1}{\omega_p^s(\tau)}\right]\right)\right\}^{1/s}, \quad p > q. \tag{4.14}$$

**Theorem 4.4.** *Under the hypotheses of Theorem 4.1 there exists* $c_2 = c_2(p,q,a) \in [1,\infty)$ *, such that*

$$c_2^{-1}(\varepsilon_{pq} + F_{pq}) \leqslant H_{\Omega_1}(B_\mu) \leqslant c_2(\varepsilon_{pq} + F_{pq}). \tag{4.15}$$

**Proof.** 1. Let us establish the following inequality

$$a^{-1}\left(1 - a^{-s}\right)^{1/s} \tilde{\varepsilon}_{pq} \leqslant \varepsilon_{pq} \leqslant \left(1 - a^{-s}\right)^{1/s}\left(1 - a^{-q}\right)^{1/q} \tilde{\varepsilon}_{pq}, \tag{4.16}$$

where as always $s = \infty$ for $p \leqslant q$ and $s = pq/(p-q)$ for $p > q$.

We have $R_+ = \bigcup\limits_n \Delta_n = \bigcup\limits_n \Delta_{n+1}$, and for $p \leqslant q$ (4.13) yields

$$\varepsilon_{pq} = \sup_n \sup_{\tau\in\Delta_{n+1}}\left[\frac{1}{\omega_p(\tau)}\left(\int\limits_{(0,\tau)} \psi^q d\gamma\right)^{1/q}\right]$$
$$\geqslant \sup_n \left(\int\limits_{(0,\lambda_{n+1})} \psi^q d\gamma\right)^{1/q} \sup_{\tau\in\Delta_{n+1}} \frac{1}{\omega_p(\tau)} = \sup_n\left[\left(\int\limits_{(0,\lambda_{n+1})} \psi^q d\gamma\right)^{1/q} a^{-(n+1)}\right]$$
$$\geqslant a^{-1}\sup_n\left[\left(\int\limits_{\Delta_n} \psi^q d\gamma\right)^{1/q} a^{-n}\right] = a^{-1}\tilde{\varepsilon}_{pq} \tag{4.17}$$

On the other hand,

$$\varepsilon_{pq} = \sup_n \sup_{\tau\in\Delta_n} \left[\frac{1}{\omega_p(\tau)}\left(\int\limits_{(0,\tau)} \psi^q d\gamma\right)^{1/q}\right] \leqslant \sup_n \left[a^{-n}\left(\int\limits_{(0,\lambda_{n+1})} \psi^q d\gamma\right)^{1/q}\right]$$
$$= \left[a^{-n}\left(\sum_{m\leqslant n}\int\limits_{\Delta_n} \psi^q d\gamma\right)^{1/q}\right] = \left[\sup_n\left\{a^{-nq}\left(\sum_{m\leqslant n}\int\limits_{\Delta_m} \psi^q d\gamma\right)\right\}\right]^{1/q}.$$

Now, we apply Proposition 2.3 with $W_n = a^{nq}, \sigma = \infty$ (see (2.12)) to the expression in the square brackets. Then,

$$\varepsilon_{pq} \leqslant \left[c\sup_n\left\{a^{-nq}\int\limits_{\Delta_n} \psi^q d\gamma\right\}\right]^{1/q} = c^{1/q}\tilde{\varepsilon}_{pq},$$

where

$$W = \inf_n \left(W_{n+1}W_n^{-1}\right) = a^q > 1, \quad c = \left(1 - W^{-1}\right)^{-1} = \left(1 - a^{-q}\right)^{-1}. \tag{4.18}$$

Inequalities (4.17) and (4.18) imply (4.16) for $p \leqslant q$.

For $p > q$ we have from (4.14) and (3.34),

$$\varepsilon_{pq}^s = \sum_{n\in\mathbb{Z}}\int\limits_{\Delta_{n+1}}\left(\int\limits_{(0,\tau)} \psi^q d\gamma\right)^{s/q}\left(-d\left[\frac{1}{\omega_p^s(\tau)}\right]\right)$$
$$\geqslant \sum_{n\in\mathbb{Z}}\left(\int\limits_{(0,\lambda_{n+1})} \psi^q d\gamma\right)^{s/q}\int\limits_{\Delta_{n+1}}\left(-d\left[\frac{1}{\omega_p^s(\tau)}\right]\right)$$
$$= a^{-s}\left(1 - a^{-s}\right)\sum_{n\in\mathbb{Z}}\frac{1}{a^{ns}}\left(\int\limits_{(0,\lambda_{n+1})} \psi^q d\gamma\right)^{s/q} \geqslant a^{-s}\left(1 - a^{-s}\right)\tilde{\varepsilon}_{pq}^s. \tag{4.19}$$

On the other hand, analogous arguments show that

$$\varepsilon_{pq}^{s}=\sum_{n\in\mathbb{Z}}\int\limits_{\Delta_n}\left(\int\limits_{(0,\tau)}\psi^q d\gamma\right)^{s/q}\left(-d\left[\frac{1}{\omega_p^s(\tau)}\right]\right)$$

$$\leqslant\sum_{n\in\mathbb{Z}}\left(\int\limits_{(0,\lambda_{n+1})}\psi^q d\gamma\right)^{s/q}\int\limits_{\Delta_n}\left(-d\left[\frac{1}{\omega_p^s(\tau)}\right]\right)$$

$$=\left(1-a^{-s}\right)\sum_{n\in\mathbb{Z}}\frac{1}{a^{ns}}\left(\int\limits_{(0,\lambda_{n+1})}\psi^q d\gamma\right)^{s/q}.$$

Therefore,

$$\varepsilon_{pq}\leqslant\left(1-a^{-s}\right)^{1/s}\left[\left\{\sum_{n\in\mathbb{Z}}\left(\frac{1}{a^{nq}}\sum_{m\leqslant n}\int\limits_{\Delta_m}\psi^q d\gamma\right)^{s/q}\right\}^{q/s}\right]^{1/q}.$$

Now, we apply Proposition 2.3 with $W_n=a^{nq},\sigma=s/q>1$ (see (2.12)) to the expression in square brackets. Then,

$$\varepsilon_{pq}\leqslant\left(1-a^{-s}\right)^{1/s}\left[c\left\{\sum_{n\in\mathbb{Z}}\left(\frac{1}{a^{nq}}\int\limits_{\Delta_n}\psi^q d\gamma\right)^{s/q}\right\}^{q/s}\right]^{1/q}. \tag{4.20}$$

Here, $c$ is defined by (4.18).

Estimates (4.19) and (4.20) give (4.16) for $p>q$.

**2.** We see from estimates (4.6) and (3.38) that

$$\tilde{F}_{pq}\leqslant F_{pq}\leqslant c_0\left(\tilde{\varepsilon}_{pq}+\tilde{F}_{pq}\right). \tag{4.21}$$

Together with (4.16) it gives

$$F_{pq}+\varepsilon_{pq}\cong\tilde{F}_{pq}+\tilde{\varepsilon}_{pq}\cong H_{\Omega_1}(B_\mu).$$

The last assertion is based on Proposition 3.10. □

## 5 Proofs of Theorems 1.1 and 1.2. Some equivalent criteria

### 5.1 Proof of Theorem 1.1.

**1.** The proof of Theorem 1.1 for $H_{\Omega_k}(B_\mu)$ will be obtained by reducing this theorem to its particular case, Theorem 4.1. We will preserve the abbreviated notation (1.1)–(1.11) and use also the following full variants of these notation. Namely, denote

$$H_{\Omega_k}(B_\mu)\equiv H_{\Omega_k}(B_\mu;p,q,\beta,\gamma), \tag{5.1}$$

$$\omega_p(t) \equiv \omega_p(k,\beta;t), \quad \Psi(t,\tau) \equiv \Psi(k,\mu;t,\tau), \tag{5.2}$$

$$V_p(t) \equiv V_p(k,\beta,\mu;t), \quad W_q(t) \equiv W_q(\gamma,t), \tag{5.3}$$

$$E_{pq} \equiv E_{pq}(k,\beta,\gamma,\mu), \quad F_{pq} \equiv F_{pq}(k,\beta,\gamma,\mu). \tag{5.4}$$

For measures $\beta$ and $\mu$ we denote by $\beta_{kp}$ and $\mu_k$ the measures on $R_+$ defined by the formulas

$$d\beta_{kp}(\xi) = k(\xi)^p \, d\beta(\xi), \quad d\mu_k(\xi) = k(\xi)\, d\mu(\xi). \tag{5.5}$$

Let us show that the following relations hold, which allows one to reduce the problems formulated on the cone $\Omega_k$ to the cone of decreasing functions $\Omega_1$:

$$H_{\Omega_k}(B_\mu;p,q,\beta,\gamma) = H_{\Omega_1}(B_{\mu_k};p,q,\beta_{kp},\gamma), \tag{5.6}$$

$$E_{pq}(k,\beta,\gamma,\mu) = E_{pq}(1,\beta_{kp},\gamma,\mu_k), \tag{5.7}$$

$$F_{pq}(k,\beta,\gamma,\mu) = F_{pq}(1,\beta_{kp},\gamma,\mu_k), \tag{5.8}$$

$$\beta \in N_p(k) \Leftrightarrow \beta_{kp} \in N_p(1). \tag{5.9}$$

We have (see (1.2) and (1.4))

$$f \in \Omega_k \quad \Leftrightarrow \quad f = k\varphi, \varphi \in \Omega_1; \quad \int_{R_+} f^p d\beta = \int_{R_+} \varphi^p d\beta_{kp}; \tag{5.10}$$

$$(B_\mu f)(t) = \int_{[t,\infty)} \varphi k d\mu = (B_{\mu_k}\varphi)(t). \tag{5.11}$$

Substituting these formulas in (5.1), (1.1) we obtain (5.6). Similarly (see (1.4)–(1.6), and (5.2)–(5.4)

$$\omega_p(k,\beta;t) = \omega_p(1,\beta_{kp};t), \quad \Psi(k,\mu;t,\tau) = \Psi(1,\mu_k;t,\tau), \tag{5.12}$$

$$V_p(k,\beta,\mu;t) = V_p(1,\beta_{kp},\mu_k;t). \tag{5.13}$$

This implies equalities (5.7)-(5.9).

**2.** Now, suppose that the hypotheses of Theorem 1.1 are satisfied. Then the hypotheses of Theorem 4.1 are satisfied for $H_{\Omega_1}(B_{\mu_k};p,q,\beta_{kp},\gamma)$ and we obtain

$$H_{\Omega_1}(B_{\mu_k};p,q,\beta_{kp},\gamma) \cong E_{pq}(1,\beta_{kp},\gamma,\mu_k) + F_{pq}(1,\beta_{kp},\gamma,\mu_k) \tag{5.14}$$

with constants independent of measures $\beta_{kp},\gamma,\mu_k$. We substitute formulas (5.6)-(5.8) into (5.14) and obtain estimate (1.13). □

**5.2. Proof of Theorem 1.2.**

It can be reduced to Theorem 1.1. For a Borel set $e \in R_+ = (0,\infty)$ we denote

$$e^{-1} = \left\{t \in R_+ : t^{-1} \in e\right\}, \tag{5.15}$$

and introduce the measures $\tilde{\beta},\tilde{\gamma},\tilde{\mu}$ by the formulas

$$\tilde{\beta}(e) = \beta\left(e^{-1}\right), \tilde{\gamma}(e) = \gamma\left(e^{-1}\right), \tilde{\mu}(e) = \mu\left(e^{-1}\right). \tag{5.16}$$

Then, for each Borel function $f$ on $R_+$ we have

$$\int_{e} f(\xi)\, d\tilde{\beta} = \int_{e^{-1}} f\left(\xi^{-1}\right) d\beta. \tag{5.17}$$

Similar formulas hold for the measures $\gamma$ and $\mu$. Note that,

$$f \in \Omega^m \Leftrightarrow \varphi(t) = f\left(t^{-1}\right) \in \Omega_k, k(t) = m\left(t^{-1}\right). \tag{5.18}$$

Next (see (1.3) and (5.17)),

$$(A_\mu f)(t) = \int_{(0,t]} f d\mu = \int_{[t^{-1},\infty)} \varphi d\tilde{\mu} = (B_{\tilde{\mu}}\varphi)\left(t^{-1}\right). \tag{5.19}$$

Therefore,

$$\int_{R_+} (A_\mu f)^q d\gamma = \int_{R_+} (B_{\tilde{\mu}}\varphi)^q d\tilde{\gamma}, \quad \int_{R_+} f^p d\beta = \int_{R_+} \varphi^p d\tilde{\beta}. \tag{5.20}$$

Hence (see (1.1)-(1.3)),

$$H_{\Omega^m}(A_\mu; p, q, \beta, \gamma) = H_{\Omega_k}\left(B_{\tilde{\mu}}; p, q, \tilde{\beta}, \tilde{\gamma}\right), \tag{5.21}$$

where $k(t) = m(t^{-1})$. Moreover, (5.17) implies the equivalence: $\beta \in \bar{N}_p(m) \Leftrightarrow \tilde{\beta} \in N_p(k)$. Thus, we can apply Theorem 1.1 to the right–hand side of (5.21) and obtain

$$H_{\Omega_k}\left(B_{\tilde{\mu}}; p, q, \tilde{\beta}, \tilde{\gamma}\right) \cong E_{pq}\left(k, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}\right) + F_{pq}\left(k, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}\right). \tag{5.22}$$

Recall notation (1.4)–(1.11) and the detailed version (5.1)–(5.4) and note that these quantities can now be rewritten by using (5.17) as

$$\omega_p\left(k, \tilde{\beta}; \tau\right) = \left(\int_{(0,\tau)} m^p\left(\xi^{-1}\right) d\tilde{\beta}\right)^{1/p} = \left(\int_{(\tau^{-1},\infty)} m^p d\beta\right)^{1/p} = \bar{\omega}_p\left(\tau^{-1}\right); \tag{5.23}$$

(see notation (1.14)-(1.17))

$$\Psi(k, \tilde{\mu}; t, \tau) = \int_{[t,\tau)} m\left(\xi^{-1}\right) d\tilde{\mu} = \int_{(\tau^{-1},t^{-1}]} m d\mu = \Phi\left(\tau^{-1}, t^{-1}\right), \tag{5.24}$$

so that

$$V_p(t) = V_p\left(k, \tilde{\beta}, \tilde{\mu}; t\right) = V_p^{(0)}\left(t^{-1}\right). \tag{5.25}$$

Indeed, when $p \in (0,1]$, relations (1.5), (5.23), and (5.24) imply

$$V_p\left(k, \tilde{\beta}, \tilde{\mu}; t\right) = \sup_{\tau \in (t,\infty)} \left[\frac{\Phi\left(\tau^{-1}, t^{-1}\right)}{\bar{\omega}_p\left(\tau^{-1}\right)}\right] = \sup_{\tau \in (0,t^{-1})} \left[\frac{\Phi\left(\tau, t^{-1}\right)}{\bar{\omega}_p(\tau)}\right],$$

which yields according to (1.15), equality (5.25). Similarly, for $p > 1$, we obtain (5.25) by using (1.6), (5.23), and (5.24). Then (see (1.7) and (1.17)), we have

$$W_q\left(\tilde{\gamma}, t\right) = \left( \int\limits_{(0,t)} d\tilde{\gamma} \right)^{1/q} = \left( \int\limits_{(t^{-1},\infty)} d\gamma \right)^{1/q} = \bar{W}_q\left(t^{-1}\right). \qquad (5.26)$$

Therefore, for $p \leqslant q$ we obtain from (1.10) and (1.20) that

$$F_{pq}\left(k, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}\right) = \sup_{t \in R_+} \left[V_p\left(t\right) W_q\left(\tilde{\gamma}, t\right)\right] = \sup_{t \in R_+} \left[V_p^{(0)}\left(t^{-1}\right) \bar{W}_q\left(t^{-1}\right)\right] = F_{pq}^{(0)}.$$

Similarly, from (1.11) and (1.21) we conclude that such equality holds for $p > q$. Thus, for all $p, q$ the equality

$$F_{pq}\left(k, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}\right) = F_{pq}^{(0)}, \qquad (5.27)$$

holds.

Now, we have,

$$\int\limits_{[\xi_\alpha(\tau),\tau)} \Psi^q\left(k, \tilde{\mu}; t, \tau\right) d\tilde{\gamma}\left(t\right) = \int\limits_{[\xi_\alpha(\tau),\tau)} \Phi^q\left(\tau^{-1}, t^{-1}\right) d\tilde{\gamma}\left(t\right)$$
$$= \int\limits_{(\tau^{-1}, 1/\xi_\alpha(\tau)]} \Phi^q\left(\tau^{-1}, t\right) d\gamma\left(t\right). \qquad (5.28)$$

Therefore, for $p \leqslant q$, we obtain by (1.8) and (5.23) that

$$E_{pq}\left(k, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}\right) = \sup_{\tau \in R_+} \left[ \frac{1}{\omega_p\left(k, \tilde{\beta}; \tau\right)} \left( \int\limits_{[\xi_\alpha(\tau),\tau)} \Psi^q\left(k, \tilde{\mu}; t, \tau\right) d\tilde{\gamma}\left(t\right) \right)^{1/q} \right]$$
$$= \sup_{\tau \in R_+} \left[ \frac{1}{\bar{\omega}_p\left(\tau^{-1}\right)} \left( \int\limits_{(\tau^{-1}, 1/\xi_\alpha(\tau)]} \Phi^q\left(\tau^{-1}, t\right) d\gamma\left(t\right) \right)^{1/q} \right]. \qquad (5.29)$$

Let us note that

$$\xi_\alpha\left(\tau\right) = 1/\varsigma_\alpha\left(\tau^{-1}\right). \qquad (5.30)$$

Indeed, it follows by (5.23) that $\omega_p^{-1}\left(t\right) = 1/\bar{\omega}_p^{-1}\left(t\right)$, therefore,

$$\xi_\alpha\left(\tau\right) = \omega_p^{-1}\left(\alpha\omega_p\left(\tau\right)\right) = 1/\bar{\omega}_p^{-1}\left(\alpha\omega_p\left(\tau\right)\right) = 1/\bar{\omega}_p^{-1}\left(\alpha\bar{\omega}_p\left(\tau^{-1}\right)\right),$$

and (5.30) follows. Now, we substitute (5.30) into (5.29) and obtain

$$E_{pq}\left(k, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}\right) = \sup_{\tau \in R_+} \left[ \frac{1}{\bar{\omega}_p\left(\tau^{-1}\right)} \left( \int\limits_{(\tau^{-1}, \varsigma_\alpha(\tau^{-1})]} \Phi^q\left(\tau^{-1}, t\right) d\gamma\left(t\right) \right)^{1/q} \right].$$

Together with (1.18), this proves the equality

$$E_{pq}\left(k,\tilde{\beta},\tilde{\gamma},\tilde{\mu}\right)=E_{pq}^{(0)} \tag{5.31}$$

for $p\leqslant q$. Similarly, from (1.9), (5.28), and (5.30) we obtain for $p>q$

$$E_{pq}\left(k,\tilde{\beta},\tilde{\gamma},\tilde{\mu}\right)$$
$$=\left\{\int\limits_{R_+}\left(\int\limits_{(\tau^{-1},\varsigma_\alpha(\tau^{-1})]}\Phi^q\left(\tau^{-1},t\right)d\gamma\left(t\right)\right)^{s/q}\left(-d\left[\frac{1}{\bar{\omega}_p^s\left(\tau^{-1}\right)}\right]\right)\right\}.$$

Together with (1.19), this proves the equality (5.31) for $p>q$.

Finally, we substitute (5.27) and (5.31) into (5.22) and obtain, taking into account (5.21), the needed estimate (1.23). □

**5.3. Equivalent criterion for the finiteness of $H_{\Omega_k}(B_\mu)$.**

Here we establish the general variant of Theorem 4.4 on the cone $\Omega_k$. We preserve the notation (3.32)–(3.34), and (4.13), (4.14), but emphasize that now, unlike to Sections 2–4, is assumed that in (1.4)–(1.7) $k$ means a general positive continuous function not necessary equal to 1.

**Theorem 5.1.** *Let the hypotheses of Theorem 1.1 be satisfied. Then, there exists $c_0=c_0(p,q,a)\in[1,\infty)$ such that*

$$c_0^{-1}\left(\varepsilon_{pq}+F_{pq}\right)\leqslant H_{\Omega_k}\left(B_\mu\right)\leqslant c_0\left(\varepsilon_{pq}+F_{pq}\right). \tag{5.32}$$

**Proof.** The scheme of the proof is essentially the same as in Subsection 5.1. As in (5.1)-(5.4) we use the following full variant of notation (4.13), (4.14)

$$\varepsilon_{pq}=\varepsilon_{pq}\left(k,\beta,\gamma,\mu\right). \tag{5.33}$$

The reduction of the problem initially formulated on the cone $\Omega_k$ to the cone $\Omega_1$ is realized by assertions (5.6)-(5.9), and by the similar equality for $\varepsilon_{pq}$, namely

$$\varepsilon_{pq}\left(k,\beta,\gamma,\mu\right)=\varepsilon_{pq}\left(1,\beta_{kp},\gamma,\mu_k\right). \tag{5.34}$$

Now, if the hypotheses of Theorem 1.1 are satisfied, then the hypotheses of Theorem 4.4 (the same as in Theorem 4.1) are satisfied too, and we obtain by (4.15) that

$$H_{\Omega_1}\left(B_{\mu_k};p,q,\beta_{kp},\gamma\right)\cong\varepsilon_{pq}\left(1,\beta_{kp},\gamma,\mu_k\right)+F_{pq}\left(1,\beta_{kp},\gamma,\mu_k\right). \tag{5.35}$$

Now, we substitute formulas (5.6)–(5.8), and (5.34) in (5.35) and obtain (5.32). □

**5.4. Equivalent criterion for the boundedness of $H_{\Omega^m}(A_\mu)$ .**

We preserve the notation (1.14)–(1.17), and (1.20)–(1.22), but now we consider the function

$$\varsigma_a\left(\tau\right)=\bar{\omega}_p^{-1}\left(a\bar{\omega}_p\left(\tau\right)\right),\quad \tau\in R_+, \tag{5.36}$$

in (1.17), where $a>1$ is the same as in (3.32). This implies that $\varsigma_a(\tau)<\tau$.

Define,

$$\varphi(t) = \Phi(\varsigma_a(t), t), \quad t \in R_+; \tag{5.37}$$

$$\varepsilon_{pq}^{(0)} = \sup_{\tau \in R_+} \left[ \frac{1}{\bar{\omega}_p(\tau)} \left( \int_{(\tau,\infty)} \varphi^q d\gamma \right)^{1/q} \right], \quad p \leqslant q; \tag{5.38}$$

$$\varepsilon_{pq}^{(0)} = \left\{ \int_{R_+} \left( \int_{(\tau,\infty)} \varphi^q d\gamma \right)^{s/q} d\left[ \frac{1}{\bar{\omega}_p^s(\tau)} \right] \right\}, \quad p > q. \tag{5.39}$$

**Theorem 5.2.** *Let the hypotheses of Theorem 1.2 be satisfied. Then,*

$$c_0^{-1}\left(\varepsilon_{pq}^{(0)} + F_{pq}^{(0)}\right) \leqslant H_{\Omega^m}(A_\mu) \leqslant c_0\left(\varepsilon_{pq}^{(0)} + F_{pq}^{(0)}\right), \tag{5.40}$$

*with* $c_0 \in [1, \infty)$ *the same as in Theorem 5.1.*

**Proof.** Theorem 5.2 is reduced to Theorem 5.1 similarly to how Theorem 1.2 was reduced to Theorem 1.1. We have assertions (5.15)- (5.21), so that

$$H_{\Omega^m}(A_\mu) = H_{\Omega_k}\left(B_{\tilde{\mu}}; p, q, \tilde{\beta}, \tilde{\gamma}\right). \tag{5.41}$$

To estimate $H_{\Omega_k}\left(B_{\tilde{\mu}}; p, q, \tilde{\beta}, \tilde{\gamma}\right)$ we apply Theorem 5.1 in corresponding notations, and obtain

$$H_{\Omega_k}\left(B_{\tilde{\mu}}; p, q, \tilde{\beta}, \tilde{\gamma}\right) \cong \varepsilon_{pq}\left(k, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}\right) + F_{pq}\left(k, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}\right). \tag{5.42}$$

For the second term in (5.42), equality (5.27) holds. Thus, our aim is to prove that

$$\varepsilon_{pq}\left(k, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}\right) = \varepsilon_{pq}^{(0)}. \tag{5.43}$$

For $p \leqslant q$ we have by (4.13)

$$\varepsilon_{pq}\left(k, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}\right) = \sup_{\tau \in R_+} \left[ \frac{1}{\omega_p\left(k, \tilde{\beta}; \tau\right)} \left( \int_{(0,\tau)} \psi^q(k, \tilde{\mu}; t)\, d\tilde{\gamma}(t) \right)^{1/q} \right].$$

According to (5.23) and (5.17), this equality yields

$$\varepsilon_{pq}\left(k, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}\right) = \sup_{\tau \in R_+} \left[ \frac{1}{\bar{\omega}_p(\tau^{-1})} \left( \int_{(\tau^{-1},\infty)} \psi^q\left(k, \tilde{\mu}; t^{-1}\right) d\gamma(t) \right)^{1/q} \right]$$

$$= \sup_{\tau \in R_+} \left[ \frac{1}{\bar{\omega}_p(\tau)} \left( \int_{(\tau,\infty)} \psi^q\left(k, \tilde{\mu}; t^{-1}\right) d\gamma(t) \right)^{1/q} \right]. \tag{5.44}$$

Let us note that

$$\psi\left(k,\tilde{\mu};t^{-1}\right)=\varphi\left(t\right),\quad t\in R_{+}. \tag{5.45}$$

Indeed, according to (5.24)

$$\psi\left(k,\tilde{\mu};t^{-1}\right)=\Psi\left(k,\tilde{\mu};t^{-1},\xi_a\left(t^{-1}\right)\right)=\Phi\left(\frac{1}{\xi_a\left(t^{-1}\right)},t\right).$$

Now, we apply equalities (5.30), and (5.37), and obtain (5.45).

By (5.45) and (5.44), it follows that (5.43) holds for $p\leqslant q$. Similarly, (5.43) may be established for $p>q$. Namely, as well as in (5.44), we have by (4.14),

$$\varepsilon_{pq}\left(k,\tilde{\beta},\tilde{\gamma},\tilde{\mu}\right)=\left\{\int\limits_{R_+}\left(\int\limits_{(\tau^{-1},\infty)}\psi^q\left(k,\tilde{\mu};t^{-1}\right)d\gamma\left(t\right)\right)^{s/q}\left(-d\left[\frac{1}{\bar{\omega}_p^s\left(\tau^{-1}\right)}\right]\right)\right\}^{1/s}.$$

We apply here (5.45), and obtain,

$$\varepsilon_{pq}\left(k,\tilde{\beta},\tilde{\gamma},\tilde{\mu}\right)=\left\{\int\limits_{R_+}\left(\int\limits_{(\tau^{-1},\infty)}\varphi^q\left(t\right)d\gamma\left(t\right)\right)^{s/q}\left(-d\left[\frac{1}{\bar{\omega}_p^s\left(\tau^{-1}\right)}\right]\right)\right\}.$$

This equality together with (5.39) yields (5.43) for $p>q$.

Finally, we substitute (5.43) in (5.42), and in (5.41); therefore, (5.40) follows. □

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Mikhail L'vovich Goldman
Department of Nonlinear Analysis and Optimization
Peoples' Friendship University of Russia
Moscow 117198, Miklukho-Maklaya str. 6, Russia
E-mail: seulydia@yandex.ru


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