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TRIGONOMETRIC SERIES WITH LACUNARY-MONOTONE COEFFICIENTS

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Abstract. In this paper we study multiple trigonometric series with lacunary-monotone coefficients. We obtain necessary and sufficient conditions for the sum of such series to belong to L_p, $1 < p \leqslant \infty$ and the generalized Lipschitz spaces (Nikol'skii spaces).

1 Introduction

It is well known that trigonometric Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.1}$$

with special conditions on coefficients $\{a_n\}$ and $\{b_n\}$ possess many important properties. Such condition are, for example, monotonicity and lacunarity. In particular, for the series with monotone or lacunary coefficients the following problems can be solved completely: to find necessary and sufficient conditions on the Fourier coefficients for the sum of series to belong to the space L_p, $1 < p \leqslant \infty$, or to describe smoothness properties of the sum of series in terms of behavior of coefficients. Both problems are of great importance in Fourier analysis since the solution provides, e.g., an instrument to deal with functions with "limiting" smoothness properties (see, e.g., [13], [17]). Surveys on series with monotone or lacunary coefficients can be found in, e.g., [4] and [7].

Let the sequences $\{a_n\}$ and $\{b_n\}$ be monotone (briefly $\{a_n\}, \{b_n\} \in M$), then series (1.1) converges uniformly if and only if ([1, Ch. V, §30], [19, V, (1.3)])

$$\sum_{n=1}^{\infty} a_n < \infty, \qquad \lim_{n\to\infty} nb_n = 0. \tag{1.2}$$

Moreover, in this case for the sum f of series (1.1) Lorentz proved ([1, Ch. X, §9], [9]) that

$$f \in \text{ Lip } \alpha \quad \Longleftrightarrow \quad a_n, b_n = O\Big(\frac{1}{n^{\alpha+1}}\Big), \qquad 0 < \alpha < 1 \tag{1.3}$$

Here, the Lipschitz space Lip $\alpha :=$ Lip (α, ∞) is defined by

$$\text{Lip }(\alpha,p)=\{f\in L_p:\omega(f,\delta)_p=O\left(\delta^{\alpha}\right)\}, \qquad 0<\alpha\leqslant 1,$$

where $\omega(f,\delta)_p$ is the modulus of continuity of f in L_p, i.e.,

$$\omega(f,\delta)_p=\sup_{|h|\leqslant\delta}\|\Delta_h f(\cdot)\|_p, \qquad \Delta_h f(x)=f(x+h)-f(x).$$

Let $\mathbf{m}=\{m_k\}$ be a lacunary sequence of natural members, that is, satisfying the condition $\lambda=\inf\frac{m_{k+1}}{m_k}>1$ (λ is the degree of lacunarity), briefly $\{m_k\}\in\Lambda$.

If the sequences $\{a_n\}$ and $\{b_n\}$ are lacunary in the sense of Hadamard, i.e.,

$$a_n=0,\ b_n=0 \text{ if } n\notin\{m_k\},\ \text{where } \{m_k\}\in\Lambda$$

(briefly $\{a_n\},\{b_n\}\in\widetilde{\Lambda}$), then series (1.1) converges uniformly if and only if ([1, Ch. XI, §6])

$$\sum_{n=1}^{\infty}|a_n|+|b_n|<\infty. \tag{1.4}$$

Moreover ([1, Ch. XI, §6], [9]), we have

$$f\in \text{ Lip }\alpha \quad\Longleftrightarrow\quad a_n,b_n=O\Big(\frac{1}{n^{\alpha}}\Big), \qquad 0<\alpha<1. \tag{1.5}$$

The following two theorems (Hardy-Littlewood, Zygmund, and Konushkov, see [1, Ch. X, §3 and Ch. XI, §6], [8], and [19, Ch. V, formula (8.20)]) generalize the above results for the case L_p, $1<p<\infty$.

Theorem A. *Let series* (1.1) *be the Fourier series of an integrable function f and $\{a_n\}$, $\{b_n\}\in M$. Then the necessary necessary and sufficient condition for f to belong to L_p, $1<p<\infty$, is*

$$\sum_{k=1}^{\infty}(a_k^p+b_k^p)k^{p-2}<\infty. \tag{1.6}$$

Moreover,

$$f\in \text{ Lip }(\alpha,p) \quad\Longleftrightarrow\quad a_n,b_n=O\Big(n^{-\alpha+1/p-1}\Big), \qquad 0<\alpha<1. \tag{1.7}$$

Note that condition (1.6) is equivalent (see Lemma 2.3 below) to the following condition

$$\sum_{r=1}^{\infty}r^{-p-2}\left(\sum_{k=1}^{r}k(a_k+b_k)\right)^p<\infty;$$

see also [5].

The counterpart for series with lacunary coefficients looks as follows.

Theorem B. *Let series* (1.1) *be the Fourier series of integrable function* f *and* $\{a_n\}$, $\{b_n\} \in \widetilde{\Lambda}$. *Then the necessary and sufficient condition for* f *to belong to* L_p, $1 < p < \infty$, *is*

$$\sum_{k=1}^{\infty}(a_k^2 + b_k^2) < \infty. \tag{1.8}$$

Moreover,

$$f \in \text{ Lip } (\alpha, p) \quad \Longleftrightarrow \quad a_n, b_n = O\Big(n^{-\alpha}\Big), \qquad 0 < \alpha < 1. \tag{1.9}$$

Analogues of Theorems A and B for multiple series were investigated when coefficients are monotone-type (see, e.g., [6]) or lacunary (see, e.g., [3]).

In this paper we study the following double trigonometric series

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_{k,n}\psi_{m_k}(x)\psi_n(y), \tag{1.10}$$

where $\psi_k(x)$ are $\cos kx$, or $\sin kx$, or e^{ikx} for every k. Here the sequence $\{m_k\}$ is lacunary and for any fixed k the sequence $a_{k,n}$ is decreasing (non-increasing) with respect to n, i.e., $a_{k,n} \geqslant a_{k,n+1}$, $n \in \mathbb{N}$.

The paper is organized as follows. In Section 2 several auxiliary lemmas are given. In Section 3 we study conditions for the sum of lacunary-monotone series to be in L_p, $1 < p < \infty$. Note that if $f \in L_p(\mathbb{T}^2)$ and $1 < p < \infty$, then from [14] it follows that for any (partial or full) function $\tilde{f}$ conjugate to f (see, e.g., [18, Part 2, Ch.I]) we have $\tilde{f} \in L_p(\mathbb{T}^2)$. Then, considering trigonometric series (1.10) for any choice of ψ_k it suffices to study, for example, sine-sine series ($\psi_k \equiv \sin kx$). In particular, we prove the following analogue of Theorems A and B.

Theorem 1.1. *Let* $1 < p < \infty$ *and let series* (1.10) *be the Fourier series of an integrable on* $\mathbb{T}^2$ *function* f *such that* $\{m_k\} \in \Lambda$ *and for any fixed* k *the sequence* $a_{k,n}$ *is convex with respect to* n. *Then the necessary and sufficient condition for* f *to belong to* $L_p(\mathbb{T}^2)$ *is*

$$\sum_{r=1}^{\infty} r^{-p-2}\left(\sum_{k=1}^{\infty}\left(\sum_{n=1}^{r} na_{k,n}\right)^2\right)^{\frac{p}{2}} < \infty.$$

In Sections 4 and 5 we obtain Lorentz-Konushkov type results on smoothness properties of the sums of lacunary-monotone series in $L_p(\mathbb{T}^2)$ for the case $1 < p < \infty$ and for the case $p = \infty$. In particular, we prove

Theorem 1.2. *Let* $1 < p < \infty$ *and let series* (1.10) *be the Fourier series of an integrable on* $\mathbb{T}^2$ *function* f *such that* $\{m_k\} \in \Lambda$ *and for any fixed* k *the sequence* $a_{k,n}$ *is decreasing with respect to* n. *Then the necessary and sufficient condition for* f *to satisfy the* (α_1, α_2)*-Lipschitz condition, that is,*

$$\sup_{\substack{|t_1| \leqslant \delta_1 \\ |t_2| \leqslant \delta_2}} \Big\| f(x+t_1, y+t_2) - f(x+t_1, y) - f(x, y+t_2) + f(x,y) \Big\|_{L_p(\mathbb{T}^2)} = O\Big(\delta_1^{\alpha_1}\delta_2^{\alpha_2}\Big),$$

where $0<\alpha_1,\alpha_2<1,$ *is*

$$a_{k,n}=O\Big(m_k^{-\alpha_1}n^{-\alpha_2+1/p-1}\Big).$$

Throughout this paper, we denote by C, C_i, c positive constants that may be different on different occasions.

2 Auxiliary results

Lemma 2.1. [11] *Let* $\{a_n\}_{n=1}^\infty$ *be a monotonic null-sequence,*

$$S_N(x)=\sum_{n=1}^{N}a_n\sin nx$$

for $N=1,2,...$ *and* $x\in(0,\pi),$ *and*

$$\alpha(x)=x\sum_{n=1}^{[\frac{\pi}{x}]}na_n.$$

Then for all N *one has*

$$|S_N(x)|\leqslant\alpha(x),\qquad x\in(0,\pi). \tag{2.1}$$

Lemma 2.2. [11, 15] *Let a null-sequence* $\{a_n\}_{n=1}^\infty$ *be convex, i.e.,*

$$\Delta^2a_n=a_n-2a_{n+1}+a_{n+2}\geqslant 0$$

for $n=1,2,...,$ *and*

$$f(x)=\sum_{n=1}^{\infty}a_n\sin nx.$$

Then for a positive constant $C>0$ *and all* $x\in(0,\frac{3\pi}{4})$ *one has*

$$f(x)\geqslant C\alpha(x). \tag{2.2}$$

The next result is a simple corollary of Hardy's inequality (see, e.g., [1, Add, §22]).

Lemma 2.3. *Let* $\{a_n\}_{n=1}^\infty$ *be a monotonic null-sequence and* $1<p<\infty$. *Then there exist positive constants* $C_1(p)$ *and* $C_2(p)$ *such that*

$$C_1(p)\sum_{n=1}^{\infty}a_n^pn^{p-2}\leqslant\sum_{r=1}^{\infty}r^{-p-2}\left(\sum_{n=1}^{r}na_n\right)^p\leqslant C_2(p)\sum_{n=1}^{\infty}a_n^pn^{p-2}.$$

Proof. We have

$$\sum_{r=1}^{\infty}r^{-p-2}\left(\sum_{n=1}^{r}na_n\right)^p\geqslant\sum_{r=1}^{\infty}r^{-p-2}a_r^p\left(\frac{r(r+1)}{2}\right)^p\geqslant\frac{1}{2^p}\sum_{r=1}^{\infty}a_r^pr^{p-2}.$$

On the other hand,

$$\begin{aligned}\sum_{r=1}^{\infty} r^{-p-2}\left(\sum_{n=1}^{r} n a_n\right)^p &\leqslant \sum_{r=1}^{\infty} r^{-p-2} r^{p-1} \sum_{n=1}^{r} n^p a_n^p \\ &= \sum_{n=1}^{\infty} a_n^p n^p \sum_{r=n}^{\infty} r^{-3} \leqslant \frac{3}{2} \sum_{n=1}^{\infty} a_n^p n^{p-2}.\end{aligned}$$

□

3 L_p-integrability of lacunary-monotone series, $1 < p < \infty$

As was mentioned in Introduction, it suffices to investigate the series

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_{k,n} \sin m_k x \sin ny. \tag{3.1}$$

Theorem 3.1. *Let $1 < p < \infty$ and let series (3.1) be such that $\mathbf{m} = \{m_k\} \in \Lambda$ and for any fixed k the sequence $a_{k,n}$ is decreasing with respect to n. If*

$$\sum_{r=1}^{\infty} r^{-p-2}\left(\sum_{k=1}^{\infty}\left(\sum_{n=1}^{r} n a_{k,n}\right)^2\right)^{\frac{p}{2}} < \infty. \tag{3.2}$$

then series (3.1) is the Fourier series of a function $f \in L_p(\mathbb{T}^2)$.

Proof. Note that condition (3.2) and Lemma 2.3 imply

$$\sum_{n=1}^{\infty} a_{k,n}^p n^{p-2} < \infty$$

for $k = 1, 2, \dots$. Therefore, by Theorem A,

$$f_k(y) = \sum_{n=1}^{\infty} a_{k,n} \sin ny \in L_p(\mathbb{T}).$$

Hence for any fixed k_0 the series

$$\sum_{k=1}^{k_0}\sum_{n=1}^{\infty} a_{k,n} \sin m_k x \sin ny$$

is the Fourier series of a function $f(k_0, x, y) \in L_p(\mathbb{T}^2)$ and therefore this series square converges to $f(k_0, x, y)$ in $L_p(\mathbb{T}^2)$.

Further, for fixed $\varepsilon > 0$, we choose (see (3.2)) an integer k_0 such that

$$\sum_{r=1}^{\infty} r^{-p-2}\left(\sum_{k=k_0+1}^{\infty}\left(\sum_{n=1}^{r} n a_{k,n}\right)^2\right)^{\frac{p}{2}} < \varepsilon. \tag{3.3}$$

Considering the series

$$\sum_{k=k_0+1}^{\infty}\sum_{n=1}^{\infty} a_{k,n}\sin m_k x\sin ny,$$

let us denote by $S_N(x,y)$ for $N=1,2,...$ its square partial sums. Denote also $k(N):=\max\{k: m_k\leqslant N\}$, where we will consider sufficiently large N so that $k(N)>k_0$.

Assuming first that $2\leqslant p<\infty$ and using Zygmind's theorem ([1, Ch. XI]), Lemma 2.1 and (3.2), we have

$$\begin{aligned}
\|S_N(x,y)\|_p^p &= \int_0^{\pi}\left(\int_0^{\pi}|S_N(x,y)|^p dx\right)dy\\
&\leqslant C(\mathbf{m},p)\int_0^{\pi}\left(\sum_{k=k_0+1}^{k(N)}\left(\sum_{n=1}^{N}a_{k,n}\sin ny\right)^2\right)^{\frac{p}{2}}dy\\
&\leqslant C(\mathbf{m},p)\int_0^{\pi}\left(\sum_{k=k_0+1}^{\infty}\left(y\sum_{n=1}^{[\frac{\pi}{y}]}na_{k,n}\right)^2\right)^{\frac{p}{2}}dy\\
&= C(\mathbf{m},p)\sum_{r=1}^{\infty}\int_{\frac{\pi}{r+1}}^{\frac{\pi}{r}}\left(\sum_{k=k_0+1}^{\infty}\left(y\sum_{n=1}^{r}na_{k,n}\right)^2\right)^{\frac{p}{2}}dy\\
&\leqslant C(\mathbf{m},p)\sum_{r=1}^{\infty}r^{-p-2}\left(\sum_{k=k_0+1}^{\infty}\left(\sum_{n=1}^{r}na_{k,n}\right)^2\right)^{\frac{p}{2}}<C(\mathbf{m},p)\,\varepsilon. \quad (3.4)
\end{aligned}$$

Since the function $f(k_0,x,y)$ belongs to $L_p(\mathbb{T}^2)$, by (3.4), the sequence of the square partial sums of series (3.1) is a Cauchy sequence in $L_p(\mathbb{T}^2)$. Then it converges to a function in $L_p(\mathbb{T}^2)$ and (3.1) is the Fourier series of this function.

If $1<p<2$, then to estimate $\|S_N(x,y)\|_p^p$, we first apply Hölder's inequality

$$\|S_N(x,y)\|_p^p=\int_0^{\pi}\left(\int_0^{\pi}|S_N(x,y)|^p dx\right)dy\leqslant C(p)\int_0^{\pi}\left(\int_0^{\pi}|S_N(x,y)|^2 dx\right)^{\frac{p}{2}}dy,$$

and then repeat the previous calculations. □

Theorem 3.2. *Let* $1<p<\infty$ *and series* (3.1) *be the Fourier series of a function* $f\in L_p(\mathbb{T}^2)$ *such that* $\mathbf{m}=\{m_k\}\in\Lambda$ *and for any fixed* k *the sequence* $\{a_{k,n}\}$ *is convex with respect to* n*. Then*

$$\sum_{r=1}^{\infty}r^{-p-2}\left(\sum_{k=1}^{\infty}\left(\sum_{n=1}^{r}na_{k,n}\right)^2\right)^{\frac{p}{2}}<\infty.$$

Proof. If $2 \leqslant p < \infty$, then (see Lemma 2.2)

$$\begin{aligned}
\int_0^\pi \int_0^\pi |f(x,y)|^p dx dy &\geqslant C(p) \int_0^\pi \left(\int_0^\pi |f(x,y)|^2 dx \right)^{\frac{p}{2}} dy \\
&= C(p) \int_0^\pi \left(\sum_{k=1}^\infty \left| \sum_{n=1}^\infty a_{k,n} \sin ny \right|^2 \right)^{\frac{p}{2}} dy \\
&\geqslant C(p) \int_0^\pi y^p \left(\sum_{k=1}^\infty \left| \sum_{n=1}^{[\frac{\pi}{y}]} n a_{k,n} \right|^2 \right)^{\frac{p}{2}} dy \qquad (3.5) \\
&\geqslant C(p) \sum_{r=1}^\infty r^{-p-2} \left(\sum_{k=1}^\infty \left(\sum_{n=1}^r n a_{k,n} \right)^2 \right)^{\frac{p}{2}}. \qquad (3.6)
\end{aligned}$$

If $1 < p < 2$, then it is known that any function $g \in L_p(\mathbb{T})$, with the Fourier series

$$\sum_{k=1}^\infty b_k \sin m_k x,$$

should be a square integrable function. Moreover, there exists a constant $C = C(\mathbf{m}, p) > 0$ such that $\|g\|_2 \leqslant C \|g\|_p$. Then

$$\int_0^\pi \int_0^\pi |f(x,y)|^p dx dy \geqslant C(\mathbf{m}, p) \int_0^\pi \left(\int_0^\pi |f(x,y)|^2 dx \right)^{\frac{p}{2}} dy.$$

To finish the proof, we use estimates (3.5) and (3.6). □

Open question. *Find necessary and sufficient conditions for the sum of series* (1.10) *to belong to the space* $L_p(\mathbb{T}^2)$, $1 < p < \infty$, *in terms of coefficients* $\{a_{k,n}\}$ *in the case in which* $\{m_k\} \in \Lambda$ *and* $\{a_{k,n}\}_n \in M$ *for any fixed* k *(*$\{a_{k,n}\}_n$ *is not necessary convex).*

In connection with this question we give the following theorem on necessary conditions for the sum of series (1.10) to be in $L_p(\mathbb{T}^2)$.

Theorem 3.3. *Let* $1 < p < \infty$ *and let series* (3.1) *be the Fourier series of a function* $f \in L_p(\mathbb{T}^2)$ *such that* $\{m_k\} \in \Lambda$ *and* $\{a_{k,n}\}_n \in M$ *for any fixed* k. *Then*

$$\sum_{r=1}^\infty r^{-p-2} \left(\sum_{k=1}^\infty \left(\sum_{n=1}^r n^2 (a_{k,n} - a_{k,n+1}) \right)^2 \right)^{\frac{p}{2}} < \infty. \qquad (3.7)$$

Proof. Consider the function

$$h(x,y) = \frac{1}{2}(f(x,y) + f(x, \pi - y)) \in L_p(\mathbb{T}^2)$$

and its Fourier series

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_{k,2n-1}\sin m_k x\sin(2n-1)y.$$

The rest of the proof follows the same lines as the proof of Theorem 3.2 but using the next result (Lemma 3.1) instead of Lemma 2.2. □

Lemma 3.1. *Let $\{a_n\}_{n=1}^{\infty}$ be a monotonic null-sequence and*

$$h(y)=\sum_{n=1}^{\infty} a_{2n-1}\sin(2n-1)y.$$

Then there exists a constant $C>0$ such that for any $y\in(0,\frac{\pi}{2})$ one has

$$h(y)\geqslant Cy\sum_{k=1}^{[\frac{\pi}{y}]} k^2\Delta a_k.$$

Proof. Denote for $k=1,2,...$ and $y\in(0,\frac{\pi}{2})$,

$$E_k(y)=\sin y+\sin 3y+...+\sin(2k-1)y=\frac{\sin^2 ky}{\sin y}\geqslant 0.$$

Hence,

$$\begin{aligned} h(y)=\sum_{k=1}^{\infty}(a_{2k-1}-a_{2k+1})E_k(y) &\geqslant y\sum_{k=1}^{[\frac{\pi}{2y}]}\frac{4}{\pi^2}(a_{2k-1}-a_{2k+1})k^2\\ &\geqslant C\,y\sum_{k=1}^{[\frac{\pi}{y}]} k^2\Delta a_k. \end{aligned}$$

□

Example 1. *Let $1<p<\infty$ and $\alpha=1-\frac{1}{p}+\varepsilon$, $\varepsilon\in\mathbb{R}$. First, we consider series* (1.10) *with coefficients $a^{(1)}_{k,n}=(k\,(n+1))^{-\alpha}$. Then this series can be written as*

$$f_1(x)=\Big(\sum_{k=0}^{\infty}\frac{1}{k^\alpha}\psi_{m_k}(x)\Big)\Big(\sum_{n=0}^{\infty}\frac{1}{n^\alpha}\psi_n(y)\Big),$$

and, by Theorems A and B,

$$f_1\in L_p(\mathbb{T}^2)\qquad\Longleftrightarrow\qquad \varepsilon>\max\Big\{0,\frac{1}{p}-\frac{1}{2}\Big\}.$$

On the other hand, denoting the sum of series (1.10) *with coefficients $a^{(2)}_{k,n}=(k+n)^{-\alpha}$ by f_2, we have*

$$f_2\in L_p(\mathbb{T}^2)\qquad\Longleftrightarrow\qquad \varepsilon>\frac{1}{2}. \tag{3.8}$$

Proof. To prove (3.8), let first $\varepsilon > 1/2$. Then

$$\begin{aligned} I_r := \left(\sum_{k=1}^{\infty}\left(\sum_{n=1}^{r} n a_{k,n}^{(2)}\right)^2\right)^{\frac{p}{2}} &\leqslant C(p,\alpha)\left(\sum_{k=1}^{r}\left(\sum_{n=1}^{r} n^{1-\alpha}\right)^2\right)^{\frac{p}{2}} \\ &+ C(p,\alpha)\left(\sum_{k=r+1}^{\infty} k^{-2\alpha}\left(\sum_{n=1}^{r} n\right)^2\right)^{\frac{p}{2}} \\ &\leqslant C(p,\alpha)\left(\sum_{n=1}^{r} n^{1-\alpha}\right)^{p} r^{p/2} + C(p,\alpha) r^{5p/2-\alpha p}, \end{aligned}$$

since $\alpha > 1/2 + 1 - 1/p > 1/2$. Further, if $\alpha < 2$, then $I_r \leqslant C(p,\alpha) r^{5p/2-\alpha p}$ and if $\alpha \geqslant 2$, then $I_r \leqslant C(p,\alpha) r^{p/2} \ln^p(2r)$. Now it is easy to see that for $\varepsilon > \frac{1}{2}$

$$\sum_{r=1}^{\infty} r^{-p-2} I_r \leqslant C(p,\alpha) \sum_{r=1}^{\infty} r^{-p-2}\left(r^{5p/2-\alpha p} + r^{p/2}\ln^p(2r)\right) < \infty$$

and therefore Theorem 3.1 yields that $f_2 \in L_p(\mathbb{T}^2)$.

Conversely, if $f_2 \in L_p(\mathbb{T}^2)$, then, by Theorem 3.3, condition (3.7) holds. Then

$$\begin{aligned} \infty &> \sum_{r=1}^{\infty} r^{-p-2}\left(\sum_{k=r}^{2r}\left(\sum_{n=[r/2]}^{r} n^2\left(a_{k,n}^{(2)} - a_{k,n+1}^{(2)}\right)\right)^2\right)^{\frac{p}{2}} \\ &\geqslant C(p,\alpha)\sum_{r=1}^{\infty} r^{-p-2}\left(\sum_{k=r}^{2r}\left(\sum_{n=[r/2]}^{r} \frac{n^2}{(k+n+1)^{\alpha+1}}\right)^2\right)^{\frac{p}{2}} \geqslant C(p,\alpha)\sum_{r=1}^{\infty} r^{\frac{3p}{2}-\alpha p-2}, \end{aligned}$$

which converges only if $\varepsilon > \frac{1}{2}$. □

4 Smoothness properties of lacunary-monotone series in L_p, $1 < p < \infty$

Denote by $\omega_{\alpha_1,\alpha_2}(f;\delta_1,\delta_2)_p$ the mixed modulus of smoothness of a function $f \in L_p(\mathbb{T}^2)$ of orders $\alpha_1 \in \mathbb{N}$ and $\alpha_2 \in \mathbb{N}$ with respect to the variables x, y respectively, i.e.,

$$\omega_{\alpha_1,\alpha_2}(f;\delta_1,\delta_2)_p = \sup_{\substack{|h_1|\leqslant\delta_1 \\ |h_2|\leqslant\delta_2}} \|\Delta_{h_1}^{\alpha_1}(\Delta_{h_2}^{\alpha_2}(f))\|_p. \tag{4.1}$$

Here, the difference of order $\alpha_1 > 0$ with respect to the variable x and the difference of order $\alpha_2 > 0$ with respect to the variable y are defined as follows:

$$\Delta_{h_1}^{\alpha_1}(f) = \sum_{k_1=0}^{\alpha_1} (-1)^{k_1} \binom{\alpha_1}{k_1} f(x + (\alpha_1 - k_1)h_1, y)$$

and

$$\Delta_{h_2}^{\alpha_2}(f) = \sum_{k_2=0}^{\alpha_2} (-1)^{k_2} \binom{\alpha_2}{k_2} f(x, y + (\alpha_2 - k_2)h_2),$$

where, as usual,

$$\binom{\alpha}{\nu} = \frac{\alpha(\alpha-1)\cdots(\alpha-\nu+1)}{\nu!} \quad \text{for} \quad \nu > 1 \quad \text{and} \quad \binom{\alpha}{\nu} = \begin{cases} \alpha, & \text{for} \quad \nu = 1 \\ 1, & \text{for} \quad \nu = 0 \end{cases}.$$

We say that a function $\omega(t)$ is of class m_β $(\beta > 0)$ if ω is continuous on $[0,2]$ and satisfies the following conditions

$$\begin{aligned} 0 = \omega(0) \leqslant \omega(\mu) \leqslant \omega(\delta) \qquad & \text{for} \qquad 0 \leqslant \mu \leqslant \delta \leqslant 1, \\ \omega(\mu)\mu^{-\beta} \geqslant \omega(\delta)\delta^{-\beta} \qquad & \text{for} \qquad 0 < \mu \leqslant \delta \leqslant 1. \end{aligned}$$

Next, a function $\Omega(t_1, t_2)$ is of class $\mathcal{M}^{\alpha_1,\alpha_2}$ $(\alpha_1 > 0, \alpha_2 > 0)$ if Ω is continuous and nonnegative on $[0,2]^2$ and satisfies $\Omega(\cdot, \delta_2) \in m_{\alpha_1}$ for any fixed δ_2 and $\Omega(\delta_1, \cdot) \in m_{\alpha_2}$ for any fixed δ_1.

Also, a continuous nonnegative on $[0,2]^2$ function $\Omega(t_1, t_2)$ satisfies the two-dimensional Bary-Stechkin conditions (briefly $\Omega \in \mathcal{BS}^{\alpha_1,\alpha_2}$) if

$$\begin{aligned} \int_0^{u_1}\int_0^{u_2} \Omega(t_1,t_2)\frac{dt_1}{t_1}\frac{dt_2}{t_2} &= O(\Omega(u_1,u_2)), \\ u_1^{\alpha_1}\int_{u_1}^{2}\int_0^{u_2} \frac{\Omega(t_1,t_2)}{t_1^{\alpha_1}}\frac{dt_1}{t_1}\frac{dt_2}{t_2} &= O(\Omega(u_1,u_2)), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} u_2^{\alpha_2}\int_0^{u_1}\int_{u_2}^{2} \frac{\Omega(t_1,t_2)}{t_2^{\alpha_2}}\frac{dt_1}{t_1}\frac{dt_2}{t_2} &= O(\Omega(u_1,u_2)), \\ u_1^{\alpha_1}u_2^{\alpha_2}\int_{u_1}^{2}\int_{u_2}^{2} \frac{\Omega(t_1,t_2)}{t_1^{\alpha_1}t_2^{\alpha_2}}\frac{dt_1}{t_1}\frac{dt_2}{t_2} &= O(\Omega(u_1,u_2)) \end{aligned} \tag{4.3}$$

as $u_1, u_2 \to 0^+$.

We will use some notations in the two-dimensional case. Let $L_p^0(\mathbb{T}^2)$ $(1 < p < \infty)$ be the collection of all functions $f \in L_p(\mathbb{T}^2)$ such that

$$\int_{-\pi}^{\pi} f(x,y)dx = 0 \quad \text{for almost every} \quad y$$

and

$$\int_{-\pi}^{\pi} f(x,y)dy = 0 \quad \text{for almost every} \quad x.$$

Let us define the Nikol'skii classes of functions with dominating mixed modulus of smoothness. Denote by $\mathcal{H}_p^{\alpha_1,\alpha_2}(\Omega)$ the Nikol'skii class, i.e., the set of functions $f \in L_p^0(\mathbb{T}^2)$ such that

$$\omega_{\alpha_1,\alpha_2}\Big(f;\delta_1,\delta_2\Big)_p \quad\leqslant\quad C\,\Omega(\delta_1,\delta_2),$$

where $1 \leqslant p \leqslant \infty$, $\alpha_1 > 0, \alpha_2 > 0$ and $\Omega \in \mathcal{M}^{\alpha_1,\alpha_2}$.

Note that any function $f \in L_p(\mathbb{T}^2)$ can be represented as

$$f(x,y) = F(x,y) + \phi(x) + \psi(y) + C, \qquad \text{where} \qquad F \in L_p^0(\mathbb{T}^2).$$

This relation holds, for example, if

$$\phi(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)\,dy, \qquad \psi(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)\,dx,$$

$$C = -\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\,dxdy,$$

$$F(x,y) = f(x,y) - \phi(x) - \psi(y) - C.$$

Since $\omega_{\alpha_1,\alpha_2}(f;\delta_1,\delta_2)_p = \omega_{\alpha_1,\alpha_2}(F;\delta_1,\delta_2)_p$, below we will assume that $f \in L_p^0(\mathbb{T}^2)$.

We will also use the following notation. Denote the partial sums of the Fourier series of a function $f \in L_p(\mathbb{T}^2)$ as

$$S_{n,\infty}(f) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x+t,y)D_n(t)\,dt, \qquad S_{\infty,m}(f) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x,y+t)D_m(t)\,dt,$$

$$S_{n,m}(f) = \frac{1}{\pi^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} f(x+t_1,y+t_2)D_n(t_1)D_m(t_2)\,dt_1dt_2,$$

where D_m is the Dirichlet kernel, i.e., $D_m(x) = \frac{\sin(m+\frac{1}{2})x}{2\sin x/2}$.

As means of approximating a function $f \in L_p^0(\mathbb{T}^2)$ we use the best approximations by two-dimensional angles (see [12]):

$$Y_{n_1,n_2}(f)_p = \inf_{\substack{T_{n_1,\infty}\\ T_{\infty,n_2}}} \|f - T_{n_1,\infty} - T_{\infty,n_2}\|_p,$$

where the function $T_{n_1,\infty}(x,y)$ is a trigonometric polynomial of order at most n_1 in x, and the function $T_{\infty,n_2}(x,y)$ is a trigonometric polynomial of order at most n_2 in y.

Lemma 4.1. [12] *If $f \in L_p^0(\mathbb{T}^2)$, $1 < p < \infty$, then*

$$Y_{n_1,n_2}(f)_p \asymp \Big\|f - \Big(S_{n_1,\infty}(f) + S_{\infty,n_2}(f) - S_{n_1,n_2}(f)\Big)\Big\|_p.$$

Lemma 4.2. [12] *If $f \in L_p^0(\mathbb{T}^2)$, $1 < p < \infty$, $\alpha_1, \alpha_2 \in \mathbb{N}$, then*

$$Y_{n_1,n_2}(f)_p \leqslant C\omega_{\alpha_1,\alpha_2}\Big(f; \frac{1}{n_1}, \frac{1}{n_2}\Big)_p \leqslant \frac{C}{n_1^{\alpha_1} n_2^{\alpha_2}} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} k_1^{\alpha_1-1} k_2^{\alpha_2-1} Y_{k_1,k_2}(f)_p.$$

Let, for $\alpha > 0$, BS_α denote the class of all function ω continuous on [0, 2] for which

$$\int_0^u \omega(t)\frac{dt}{t} = O(\omega(u)), \qquad u^\alpha \int_u^1 \frac{\omega(t)}{t^\alpha}\frac{dt}{t} = O(\omega(u)) \qquad \text{as} \quad u \to 0.$$

Lemma 4.3. *Let a function Ω defined on $[0,2]^2$ be such that $\Omega \in \mathcal{M}^{\alpha_1,\alpha_2} \cap \mathcal{BS}^{\alpha_1,\alpha_2}$, where $\alpha_1, \alpha_2 > 0$. Then*

(A) *For any fixed t_1 the function $\Omega(t_1,\cdot) \in BS_{\alpha_2}$ and for any fixed t_2 the function $\Omega(\cdot,t_2) \in BS_{\alpha_1}$.*

(B) *There exists $\varepsilon > 0$ such that, for any t_1, $x^{-\varepsilon}\Omega(t_1,x) \leqslant C\, y^{-\varepsilon}\Omega(t_1,y)$ for any $x \leqslant y$. Similarly, there exists $\varepsilon > 0$ such that, for any t_2, $x^{-\varepsilon}\Omega(x,t_2) \leqslant C\, y^{-\varepsilon}\Omega(y,t_2)$ for any $x \leqslant y$.*

(C)
$$\Omega(2t_1,t_2) \leqslant C\Omega(t_1,t_2) \qquad \textit{and} \qquad \Omega(t_1,2t_2) \leqslant C\Omega(t_1,t_2).$$

(D) *If the sequence $\{m_k\}$ is lacunary, then for any positive β*

$$\sum_{k=r}^\infty \Omega^\beta(m_k^{-1}, l^{-1}) \leqslant C\,\Omega^\beta(m_r^{-1}, l^{-1}), \qquad \sum_{l=n}^\infty \frac{\Omega^\beta(m_k^{-1}, l^{-1})}{l} \leqslant C\,\Omega^\beta(m_k^{-1}, n^{-1}).$$

Proof. Proof of part (A) follows immediately by conditions (4.2)-(4.3). Indeed, conditions (4.2) and $\Omega \in \mathcal{M}^{\alpha_1,\alpha_2}$ yield (for any fixed u_2)

$$\int_0^{u_1} \Omega(t_1,u_2)\frac{dt_1}{t_1} \leqslant C \int_0^{u_1}\int_{u_2/2}^{u_2} \Omega(t_1,t_2)\frac{dt_1}{t_1}\frac{dt_2}{t_2} \leqslant C\Omega(u_1,u_2)$$

and

$$u_1^{\alpha_1}\int_{u_1}^2 \frac{\Omega(t_1,u_2)}{t_1^{\alpha_1}}\frac{dt_1}{t_1} \leqslant Cu_1^{\alpha_1}\int_{u_1}^2\int_{u_2/2}^{u_2} \frac{\Omega(t_1,t_2)}{t_1^{\alpha_1}}\frac{dt_1}{t_1}\frac{dt_2}{t_2} \leqslant C\Omega(u_1,u_2),$$

i.e., $\Omega(\cdot,t_2) \in BS_{\alpha_1}$. Similarly, conditions (4.3) and $\Omega \in \mathcal{M}^{\alpha_1,\alpha_2}$ give $\Omega(t_1,\cdot) \in BS_{\alpha_2}$.

Further, inequalities of parts (B) and (C) immediately follow by (A) and [2]. Finally, (D) follows by (B). For example,

$$\sum_{k=r}^\infty \Omega^\beta(m_k^{-1}, l^{-1}) \;\leqslant\; C\,m_r^{\varepsilon\beta}\Omega^\beta(m_r^{-1}, l^{-1})\sum_{k=r}^\infty m_k^{-\varepsilon\beta} \leqslant C\,\Omega^\beta(m_r^{-1}, l^{-1}),$$

where the inequality $\sum_{k=r}^\infty m_k^{-\varepsilon\beta} \leqslant C\,m_r^{-\varepsilon\beta}$ (for any fixed $\beta > 0$) holds because of lacunarity of the sequence $\{m_k\}$ (see also [16, Cor. 4.10]). □

Lemma 4.4. [10, 1.5.2] *Let the series*

$$\sum_{s_1=1}^{\infty}\sum_{s_2=1}^{\infty} a_{s_1,s_2}\sin s_1x\sin s_2y$$

be the Fourier series of a function $f\in L_p^0(\mathbb{T}^2)$, $1<p<\infty$. *Let* $\{n_k\}\in\Lambda$. *Then*

$$\|f\|_p \asymp \left(\int_{\mathbb{T}^2}\left[\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\triangle_{k_1,k_2}^2\right]^{\frac{p}{2}}dx\,dy\right)^{\frac{1}{p}},$$

where

$$\triangle_{k_1,k_2} := \sum_{s_1=n_{k_1}}^{n_{k_1+1}-1}\sum_{s_2=n_{k_2}}^{n_{k_2+1}-1} a_{s_1s_2}\sin s_1x\,\sin s_2y, \quad k_1,k_2=0,1,2,\cdots$$

The main result of this section is the following Lorentz-Konushkov type theorem.

Theorem 4.1. *Let* $1<p<\infty$ *and* $\Omega\in\mathcal{M}^{\alpha_1,\alpha_2}\cap\mathcal{BS}^{\alpha_1,\alpha_2}$, *where* $\alpha_1,\alpha_2\in\mathbb{N}$. *Let* $\{m_k\}\in\Lambda$ *and let for any fixed* k *the sequence* $\{a_{k,n}\}_{n=1}^{\infty}$ *be decreasing with respect to* n. *Then the necessary and sufficient condition for the sum of series* (3.1) *to be the Fourier series of a function* $f\in\mathcal{H}_p^{\alpha_1,\alpha_2}(\Omega)$ *is*

$$a_{k,n} \leqslant C\,\Omega\Big(\frac{1}{m_k},\frac{1}{n}\Big)\,n^{\frac{1}{p}-1}. \tag{4.4}$$

Proof. We remark that, by $\Omega\in\mathcal{BS}^{\alpha_1,\alpha_2}$ and Lemma 4.2, the condition

$$Y_{k_1,k_2}(f)_p \leqslant C\,\Omega\Big(\frac{1}{k_1},\frac{1}{k_2}\Big) \tag{4.5}$$

is necessary and sufficient for the series (3.1) to be the Fourier series of a function $f\in\mathcal{H}_p^{\alpha_1,\alpha_2}(\Omega)$.

Let us show that condition (4.5) implies condition (4.4). First, for the sequence $\mathbf{m}=\{m_k\}\in\Lambda$, we construct a lacunary sequence $\{m_k^*\}$ so that $\{m_k\}\subset\{m_k^*\}$ and $1<\lambda_1<m_{k+1}^*/m_k^*\leqslant 2$.

Secondly, define $b_{m_k,l}:=a_{k,l}$ for $k\geqslant 1$ and $b_{s,l}:=0$ for $s\neq m_k$. Then using Lemma 4.1 (let $z\in\mathbb{N}$ be such that $m_s=m_z^*$), we get

$$Y_{m_s-1,m_t^*-1}(f)_p = Y_{m_z^*-1,m_t^*-1}(f)_p \asymp \Big\|\sum_{k=m_z^*}^{\infty}\sum_{l=m_t^*}^{\infty} b_{k,l}\sin kx\sin ly\Big\|_p.$$

Now applying the Littlewood-Paley theorem (see Lemma 4.4) for this series and for the sequence $\{m_k^*\}$, we estimate

$$\begin{aligned}Y_{m_s-1,m_t^*-1}(f)_p &\geqslant C\Big\|\sum_{k=m_z^*}^{m_{z+1}^*-1}\sum_{l=m_t^*}^{m_{t+1}^*-1} b_{k,l}\sin kx\sin ly\Big\|_p\\ &\geqslant C\left(\int_0^\pi\int_0^{\pi/(4m_t^*)}\Big|\sum_{l=m_t^*}^{m_{t+1}^*-1} a_{s,l}\sin m_s x\sin ly\Big|^p dx\,dy\right)^{\frac1p}\\ &\geqslant C\left(\Big(\sum_{l=m_t^*}^{m_{t+1}^*-1} l a_{s,l}\Big)^p\int_0^\pi|\sin m_s x|^p\,dx\int_0^{\pi/(4m_t^*)} y^p\,dy\right)^{\frac1p}.\end{aligned}$$

Making use of inequality (4.5) and monotonicity of the sequence $\{a_{k,n}\}_n$, we have $(m_{t+1}^*\leqslant l<m_{t+2}^*)$

$$\Omega\Big(\frac{1}{m_s-1},\frac{4}{l}\Big)\geqslant C\,Y_{m_s-1,m_t^*-1}(f)_p \;\geqslant\; C\Big((m_t^*)^{(p-1)}a^p_{s,m_{t+1}^*}\Big)^{\frac1p}\geqslant C\,l^{1-\frac1p}a_{s,l}.$$

Thus, using condition of the function Ω, we arrive at (4.4).

Next, let condition (4.4) hold. To show (4.5), by Lemma 4.1, it suffices to check that

$$J^p:=\Big\|\sum_{k=s}^{\infty}\sum_{l=n}^{\infty} a_{k,l}\sin m_k x\sin ly\Big\|_p^p\leqslant C\,\Omega^p\Big(\frac{1}{m_s},\frac1n\Big). \tag{4.6}$$

First,

$$\begin{aligned}J^p &\leqslant 4^p\int\limits_{\frac1n}^{\pi}\int\limits_0^{\pi}\Big|\sum_{k=s}^{\infty}\sum_{l=n}^{\infty} a_{k,l}\sin ly\sin m_k x\Big|^p dx\,dy\\ &+ 4^p\int\limits_0^{\frac1n}\int\limits_0^{\pi}\Big|\sum_{k=s}^{\infty}\sum_{l=[\frac1y]+1}^{\infty} a_{k,l}\sin ly\sin m_k x\Big|^p dx\,dy\\ &+ 4^p\int\limits_0^{\frac1n}\int\limits_0^{\pi}\Big|\sum_{k=s}^{\infty}\sum_{l=n}^{[\frac1y]} a_{k,l}\sin ly\sin m_k x\Big|^p dx\,dy =: 4^p\big(I_1+I_2+I_3\big).\end{aligned}$$

Secondly, denote for any $k\geqslant s$

$$\begin{aligned}f_k(y)&:=\sum_{l=n}^{\infty} a_{k,l}\sin ly \qquad\text{for}\qquad y\in\Big[\frac1n,\pi\Big],\\ g_k(y)&:=\sum_{l=[\frac1y]+1}^{\infty} a_{k,l}\sin ly \qquad\text{for}\qquad y\in\Big[0,\frac1n\Big],\\ h_k(y)&:=\sum_{l=n}^{[\frac1y]} a_{k,l}\sin ly \qquad\text{for}\qquad y\in\Big[0,\frac1n\Big].\end{aligned}$$

Then, by the Abel transformation, we have

$$|f_k(y)| \leqslant C\frac{a_{k,n}}{y} \leqslant C\,\frac{\Omega(m_k^{-1}, n^{-1})}{n^{1-\frac{1}{p}}y}$$

for $k \geqslant s$ and $y \in [\frac{1}{n}, \pi]$. Using this inequality, Zygmund's theorem (see Theorem B), and Lemma 4.3 (D), we get

$$\begin{aligned}
I_1 &\leqslant C(p,\mathbf{m}) \int\limits_{\frac{1}{n}}^{\pi} \left(\int\limits_0^{\pi} \left| \sum_{k=s}^{\infty} f_k(y) \sin m_k x \right|^2 dx \right)^{\frac{p}{2}} dy \\
&\leqslant C(p,\mathbf{m}) \int\limits_{\frac{1}{n}}^{\pi} \left(\sum_{k=s}^{\infty} f_k^2(y) \right)^{\frac{p}{2}} dy \\
&\leqslant C(p,\mathbf{m}) n^{1-p} \left(\sum_{k=s}^{\infty} \Omega^2(m_k^{-1}, n^{-1}) \right)^{\frac{p}{2}} \int\limits_{\frac{1}{n}}^{\pi} \frac{dy}{y^p} \\
&\leqslant C(p,\mathbf{m}) \left(\sum_{k=s}^{\infty} \Omega^2(m_k^{-1}, n^{-1}) \right)^{\frac{p}{2}} \leqslant C(p,\mathbf{m}) \Omega^p(m_s^{-1}, n^{-1}).
\end{aligned}$$

Further, for $k \geqslant s$, $l \geqslant n$, and $y \in [\frac{1}{l+1}, \frac{1}{l}]$ the following inequality holds

$$|g_k(y)| \leqslant C\,\frac{\Omega(m_k^{-1}, l^{-1})}{l^{1-\frac{1}{p}}y}.$$

Then similarly to the estimate of I_1, we have

$$\begin{aligned}
I_2 &\leqslant \sum_{l=n}^{\infty} \int\limits_{\frac{1}{l+1}}^{\frac{1}{l}} \left(\int\limits_0^{\pi} \left| \sum_{k=s}^{\infty} g_k(y) \sin m_k x \right|^2 dx \right)^{\frac{p}{2}} dy \\
&\leqslant C(p,\mathbf{m}) \sum_{l=n}^{\infty} l^{1-p} \left(\sum_{k=s}^{\infty} \Omega^2(m_k^{-1}, l^{-1}) \right)^{\frac{p}{2}} \int\limits_{\frac{1}{l+1}}^{\frac{1}{l}} \frac{dy}{y^p} \\
&\leqslant C(p,\mathbf{m}) \sum_{l=n}^{\infty} l^{1-p} \Omega^p(m_s^{-1}, l^{-1}) l^{p-2} \leqslant C(p,\mathbf{m}) \Omega^p(m_s^{-1}, n^{-1}).
\end{aligned}$$

In the last two inequalities we used Lemma 4.3 (D).

To estimate I_3, we should obtain a pointwise bound of $|h_k(y)|$. By Lemma 4.3, there exists $\alpha > 0$ such that the function $\Omega(x,y)y^{-\alpha}$ is almost increasing. Take $\alpha < \frac{1}{p}$. Then for $k \geqslant s$ and $y \in [0, \frac{1}{n}]$ we get

$$|h_k(y)| \leqslant \sum_{l=n}^{[\frac{1}{y}]} a_{k,l} \leqslant C \sum_{l=n}^{[\frac{1}{y}]} \frac{\Omega(m_k^{-1}, l^{-1}) l^{\alpha}}{l^{\frac{p-1}{p}+\alpha}} \leqslant C(p,\alpha)\, \Omega(m_k^{-1}, n^{-1}) \frac{n^{\alpha}}{y^{\frac{1}{p}-\alpha}}.$$

Hence,

$$
\begin{aligned}
I_3 &\leqslant \int\limits_0^{\frac{1}{n}} \left(\int\limits_0^{\pi} \left| \sum_{k=s}^{\infty} h_k(y) \sin m_k x \right|^2 dx \right)^{\frac{p}{2}} dy \leqslant C(p,\mathbf{m}) \int\limits_0^{\frac{1}{n}} \left(\sum_{k=s}^{\infty} h_k^2(y) \right)^{\frac{p}{2}} dy \\
&\leqslant C(p,\alpha,\mathbf{m}) \left(\sum_{k=s}^{\infty} \Omega^2(m_k^{-1}, n^{-1}) \right)^{\frac{p}{2}} n^{p\alpha} \int\limits_0^{\frac{1}{n}} \frac{dy}{y^{1-p\alpha}} \\
&\leqslant C(p,\alpha,\mathbf{m})\, \Omega^p(m_s^{-1}, n^{-1}).
\end{aligned}
$$

Thus, inequality (4.6) follows by the estimates of I_1, I_2, and I_3. □

5 Smoothness properties of lacunary-monotone series in L_∞

In this section we prove several results on functions with lacunary-monotone Fourier series belonging to the class $\mathcal{H}_p^{\alpha_1,\alpha_2}(\Omega)$, where $p = \infty$. These results are supplements to relations (1.3) and (1.5), which were mentioned in Introduction. For the sake of simplicity, we consider only the case when $\alpha_1 = \alpha_2 = 1$ and denote $\mathcal{H}(\Omega) := \mathcal{H}_\infty^{1,1}(\Omega)$.

Theorem 5.1. *Let $\Omega \in \mathcal{M}^{1,1} \cap \mathcal{BS}^{1,1}$ and $\{m_k\} \in \Lambda$. If*

$$
|a_{k,n}| \leqslant C \frac{\Omega(m_k^{-1}, n^{-1})}{n}, \tag{5.1}
$$

then series (3.1) *converges uniformly to a function $f \in \mathcal{H}(\Omega)$.*

Proof. Without loss of generality let $m_0 = \frac{1}{2}$ and $2m_1 > \lambda$, where λ is the degree of lacunarity of $\{m_k\}$. Note that for any integer l and any $v \in (0,1)$, by Lemma 4.3, we have

$$
\int\limits_{m_l^{-1}}^{m_{l-1}^{-1}} \frac{\Omega(u,v)}{u^2}\, du \geqslant \int\limits_{m_l^{-1}}^{\lambda m_l^{-1}} \frac{\Omega(u,v)}{u^2}\, du \geqslant C(\lambda)\, m_l^2 m_l^{-1} \Omega(m_l^{-1}, v) = C(\lambda)\, m_l\, \Omega(m_l^{-1}, v) \tag{5.2}
$$

Similarly, for any integer l and for any $v \in (0,1)$ we get

$$
\int\limits_{m_{l+1}^{-1}}^{m_l^{-1}} \frac{\Omega(u,v)}{u}\, du \geqslant \int\limits_{\lambda^{-1} m_l^{-1}}^{m_l^{-1}} \frac{\Omega(u,v)}{u}\, du \geqslant C(\lambda)\, m_l m_l^{-1}\, \Omega(\lambda^{-1} m_l^{-1}, v) \geqslant C_1(\lambda)\, \Omega(m_l^{-1}, v) \tag{5.3}
$$

Then

$$
\begin{aligned}
\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |a_{k,n}| \leqslant C \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\Omega(m_k^{-1}, n^{-1})}{n} &\leqslant C(\lambda) \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int\limits_{m_{l+1}^{-1}}^{m_l^{-1}} \int\limits_{(n+1)^{-1}}^{n^{-1}} \frac{\Omega(u,v)}{uv}\, du dv \\
&= C(\lambda) \int\limits_0^{m_1^{-1}} \int\limits_0^{1} \frac{\Omega(u,v)}{uv} du dv < \infty.
\end{aligned}
$$

Therefore, series (3.1) is the Fourier series of a function $f \in C(\mathbb{T}^2)$.

Letting $0 < t, h < \frac{1}{2}$ and denoting $k_0 := \max\{k : m_k \leqslant \frac{1}{t}\}$ and $n_0 = [\frac{1}{h}]$, we have for any x, y

$$\begin{aligned} I &:= |f(x+t, y+h) - f(x+t, y) - f(x, y+h) + f(x, y)| \\ &\leqslant Cth \sum_{k=1}^{k_0} \sum_{n=1}^{n_0} m_k \Omega(m_k^{-1}, n^{-1}) + Ct \sum_{k=1}^{k_0} \sum_{n=n_0+1}^{\infty} m_k \frac{\Omega(m_k^{-1}, n^{-1})}{n} \\ &+ Ch \sum_{k=k_0+1}^{\infty} \sum_{n=1}^{n_0} \Omega(m_k^{-1}, n^{-1}) + C \sum_{k=k_0+1}^{\infty} \sum_{n=n_0+1}^{\infty} \frac{\Omega(m_k^{-1}, n^{-1})}{n}. \end{aligned}$$

Making use of (5.2) and (5.3) and taking into account that $\Omega \in \mathcal{M}^{1,1} \cap \mathcal{BS}^{1,1}$, we estimate

$$\begin{aligned} I &\leqslant C(\lambda)\, th \sum_{k=1}^{k_0} \sum_{n=1}^{n_0} \int_{m_k^{-1}}^{m_{k-1}^{-1}} \int_{(n+1)^{-1}}^{n^{-1}} \frac{\Omega(u,v)}{u^2 v^2} du dv \\ &+ C(\lambda)\, t \sum_{k=1}^{k_0} \sum_{n=n_0+1}^{\infty} \int_{m_k^{-1}}^{m_{k-1}^{-1}} \int_{(n+1)^{-1}}^{n^{-1}} \frac{\Omega(u,v)}{u^2 v} du dv \\ &+ C(\lambda)\, h \sum_{k=k_0+1}^{\infty} \sum_{n=1}^{n_0} \int_{m_{k+1}^{-1}}^{m_k^{-1}} \int_{(n+1)^{-1}}^{n^{-1}} \frac{\Omega(u,v)}{u v^2} du dv \\ &+ C(\lambda) \sum_{k=k_0+1}^{\infty} \sum_{n=n_0}^{\infty} \int_{m_{k+1}^{-1}}^{m_k^{-1}} \int_{(n+1)^{-1}}^{n^{-1}} \frac{\Omega(u,v)}{u v} du dv \\ &\leqslant C(\lambda)\, th \int_{m_{k_0}^{-1}}^{2} \int_{(n_0+1)^{-1}}^{1} \frac{\Omega(u,v)}{u^2 v^2} du dv + C(\lambda)\, t \int_{m_{k_0}^{-1}}^{2} \int_{0}^{n_0^{-1}} \frac{\Omega(u,v)}{u^2 v} du dv \\ &+ C(\lambda)\, h \int_{0}^{m_{k_0+1}^{-1}} \int_{(n_0+1)^{-1}}^{1} \frac{\Omega(u,v)}{u v^2} du dv + C(\lambda) \int_{0}^{m_{k_0+1}^{-1}} \int_{0}^{n_0^{-1}} \frac{\Omega(u,v)}{u v} du dv \\ &\leqslant C(\lambda)\, \Omega(t, h). \end{aligned}$$

□

Theorem 5.2. *Let $\Omega \in \mathcal{M}^{1,1} \cap \mathcal{BS}^{1,1}$, $\{m_k\} \in \Lambda$ and let for any fixed k the sequence $\{a_{k,n}\}_{n=1}^{\infty}$ be a decreasing null-sequence as $n \to \infty$. Let also series (3.1) be the Fourier series of a function $f \in \mathcal{H}(\Omega)$. Then for some positive C and all integers k and n condition (5.1) holds.*

Proof. We fix $k \geqslant 1$ and $n \geqslant 1$ and put $t = \frac{1}{2m_k}$ and $h = \frac{1}{2n}$. Let $\sigma(x,y) = \sigma_{2m_k,2n}(x,y)$ be the Cesàro $(1,1)$ means. Consider

$$I = \sigma(t,h) - \sigma(t,0) - \sigma(0,h) + \sigma(0,0).$$

Then

$$\begin{aligned} |I| &= \frac{1}{\pi^2}\left|\int_{\mathbb{T}}\int_{\mathbb{T}} \Big(f(x+t,y+h) - f(x+t,y) - f(x,y+h) + f(x,y)\Big) K_{2m_k}(x)K_{2n}(y)dxdy\right| \\ &\leqslant \frac{1}{\pi^2}\int_{\mathbb{T}}\int_{\mathbb{T}} \Big|f(x+t,y+h) - f(x+t,y) - f(x,y+h) + f(x,y)\Big| K_{2m_k}(x)K_{2n}(y)dxdy \\ &\leqslant C\Omega(t,h), \end{aligned} \tag{5.4}$$

where $K_i(u)$ is the Fejér kernel. Also, if $k_0 = \max\{s : m_s \leqslant 2m_k\}$, then

$$\begin{aligned} I = \sigma(t,h) - 0 - 0 + 0 &= \sum_{r=1}^{k_0}\sum_{l=1}^{2n} \frac{2m_k - m_r + 1}{2m_k + 1}\,\frac{2n - l + 1}{2n+1}\, a_{r,l} \sin m_r t\, \sin lh \\ &\geqslant C \sum_{l=[n/2]+1}^{n} a_{k,l} \geqslant C_1 n a_{k,n}. \end{aligned} \tag{5.5}$$

Thus, the statement of the theorem follows by (5.4) and (5.5). □

Let us now consider the partial moduli of continuity. Denote by $\omega_{1,0}(f,t)$ and $\omega_{0,1}(f,t)$ the partial moduli with respect to x, y respectively. We present two results for the cosine-cosine series

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty} \alpha_n a_{k,n} \cos m_k x \cos ny, \tag{5.6}$$

where $\alpha_0 = \frac{1}{2}$ and $\alpha_n = 1$ for $n \geqslant 1$.

Theorem 5.3. *Let $\omega(\cdot) \in m_1 \cap BS^1$ and $\{m_k\} \in \Lambda$. Let also $\{a_{k,n}\}_{k=1,n=0}^{\infty,\infty}$ be such that*

$$\sum_{k=1}^{\infty} |a_{k,n}| \leqslant C \frac{\omega(\frac{1}{n+1})}{n+1} \tag{5.7}$$

for any $n \geqslant 0$ and

$$\sum_{n=0}^{\infty} |a_{k,n}| \leqslant C\omega(\frac{1}{m_k}) \tag{5.8}$$

for any $k \geqslant 1$. Then series (5.6) is the Fourier series of a function $f \in C(\mathbb{T}^2)$ and, moreover, its partial moduli of continuity satisfy the inequalities

$$\omega_{1,0}(f,t) \leqslant C\omega(t) \quad \text{and} \quad \omega_{0,1}(f,h) \leqslant C\omega(h)$$

for some $C > 0$ and any $t, h \in (0, 1/2)$.

Proof. First we note that

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}|a_{k,n}| \leqslant C\sum_{n=0}^{\infty}\frac{\omega(\frac{1}{n+1})}{n+1} \leqslant C_1\int\limits_0^2 \frac{\omega(t)}{t}\,dt < \infty.$$

This implies that series (5.6) is the Fourier series of a continuous function f. Further, similarly to estimates (5.2) and (5.3), we have (here as in Theorem 5.1, $m_0 = 1/2$)

$$\int\limits_{m_l^{-1}}^{m_{l-1}^{-1}} \frac{\omega(u)}{u^2}du \geqslant C(\lambda)m_l\omega(m_l^{-1})$$

and

$$\int\limits_{m_{l+1}^{-1}}^{m_l^{-1}} \frac{\omega(u)}{u}du \geqslant C(\lambda)\omega(m_l^{-1}).$$

Next, if $(x, y) \in \mathbb{T}^2$ and $t \in (0, \frac{1}{2})$, then we put $k_0 := \max\{k : m_k \leqslant \frac{1}{t}\}$. Then

$$\begin{aligned}
|f(x+t,y) - f(x,y)| &\leqslant \sum_{k=1}^{\infty}\left(\sum_{n=0}^{\infty}\alpha_n\big|a_{k,n}\big|\right)\big|\cos m_k(x+t) - \cos m_k x\big| \\
&\leqslant Ct\sum_{k=1}^{k_0} m_k\omega(\frac{1}{m_k}) + C\sum_{k=k_0+1}^{\infty}\omega(\frac{1}{m_k}) \\
&\leqslant C_1(\lambda)t\int\limits_{\frac{1}{m_{k_0}}}^{2}\frac{\omega(u)}{u^2}du + C_1(\lambda)\int\limits_0^{\frac{1}{m_{k_0+1}}}\frac{\omega(u)}{u}du \leqslant C_2(\lambda)\omega(t).
\end{aligned}$$

Similarly, if $(x, y) \in \mathbb{T}^2$ and $h \in (0, \frac{1}{2})$, then we put $n_0 = [\frac{1}{h}]$. Then

$$\begin{aligned}
|f(x,y+h) - f(x,y)| &\leqslant \sum_{n=0}^{\infty}\alpha_n\left(\sum_{k=1}^{\infty}\big|a_{k,n}\big|\right)|\cos n(y+h) - \cos ny| \\
&\leqslant Ch\sum_{n=0}^{n_0}\omega(\frac{1}{n+1}) + C\sum_{n=n_0+1}^{\infty}\frac{\omega(\frac{1}{n+1})}{n+1} \\
&\leqslant C_1(\lambda)h\int\limits_{\frac{1}{n_0}}^{2}\frac{\omega(u)}{u^2}du + C_1(\lambda)\int\limits_0^{\frac{1}{n_0+1}}\frac{\omega(u)}{u}du \leqslant C_2(\lambda)\omega(h).
\end{aligned}$$

Thus, both partial moduli of continuity of f have the order $O(\omega(u))$ as $u \to {}_+0$. □

Note that the full modulus of continuity of f has the same order.

Theorem 5.4. *Let $\omega(\cdot) \in m_1$ and a function $f \in C(\mathbb{T}^2)$ be such that its partial moduli of continuity satisfy $\omega_{1,0}(f,t) \leqslant C\omega(t)$ and $\omega_{0,1}(f,h) \leqslant C\omega(h)$ for some $C > 0$ and for any $t, h \in (0, 1/2)$. Let the Fourier series of f be given by (5.6), where $\{m_k\} \in \Lambda$. Let also for any fixed k a sequence $\{a_{k,n}\}_{n=0}^{\infty}$ be decreasing. Then for some $C > 0$ conditions (5.7) and (5.8) hold for any $n \geqslant 0$.*

Proof. By means of continuity of f, we get

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} a_{k,n} < \infty.$$

Consider the function $g(x) = f(x, 0)$, where $x \in \mathbb{T}$. Then $\omega(g, \delta) = O(\omega(\delta))$ and the Fourier coefficients of g satisfy $a_n(g) = 0$ for $n \notin \{m_k\}_{k=1}^{\infty}$ and

$$a_{m_k}(g) = \sum_{n=0}^{\infty} \alpha_n a_{k,n}, \qquad \text{for} \quad k = 1, 2, \cdots$$

On the other hand, it is well known that the Fourier coefficients are bounded by modulus of continuity, that is,

$$a_{m_k}(g) \leqslant C\omega(\frac{1}{m_k}).$$

This gives inequality (5.8).

Condition (5.7) follows by considering the function $q(y) = f(0, y)$, where $y \in \mathbb{T}$. Indeed, $\omega(q, \delta) = O(\omega(\delta))$, and the Fourier coefficients of q satisfy the equality

$$a_n(q) = \sum_{k=1}^{\infty} a_{k,n}$$

and are decreasing. By Theorem A for the modulus of continuity with a general majorant (see [1, Ch. X, §9]), we have

$$a_n(q) \leqslant C\,\frac{\omega(1/(n+1))}{n+1},$$

which yields the desired inequality. □

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