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MONOTONE PATH-CONNECTEDNESS OF R-WEAKLY CONVEX SETS IN SPACES WITH LINEAR BALL EMBEDDING

A.R. Alimov

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Abstract. A subset M of a normed linear space X is called R-weakly convex ($R > 0$) if $(D_R(x, y) \setminus \{x, y\}) \cap M \neq \varnothing$ for any $x, y \in M$ satisfying $0 < \|x - y\| < 2R$. Here, $D_R(x, y)$ is the intersection of all closed balls of radius R containing x, y. The paper is concerned with the connectedness of R-weakly convex subsets of Banach spaces satisfying the linear ball embedding condition (BEL) (note that $C(Q)$ and $\ell^1(n) \in$ (BEL)). An R-weakly convex subset M of a space $X \in$ (BEL) is shown to be m-connected (Menger-connected) under the natural condition on the spread of points in M. A closed subset M of a finite-dimensional space $X \in$ (BEL) is shown to be R-weakly convex with some $R > 0$ if and only if M is a disjoint union of monotone path-connected suns in X, the Hausdorff distance between any connected components of M being less than $2R$. In passing we obtain a characterization of three-dimensional spaces with subequilateral unit ball.

1 Introduction

The theory of R-weakly convex sets is in active development at present (see [18], [7], [17], [4] and references given therein). The interest in R-weakly convex sets stems from their applications to the theory of extremal problems, problems of optimal control, theory of differential games, approximation theory and set-valued analysis [18].

In a recent paper [4] the author examined the problem of m-connectedness (Menger-connectedness) and monotone path-connectedness of R-weakly convex sets in the space $C(Q)$.

In the present paper we continue the study of this problem considering a more general setting of general normed linear spaces. In this context, the class (BEL) of spaces with linear embedding of balls (see (4) below) and the class of spaces with subequilateral (edge-antipodal) balls are quite natural; these spaces contain $C(Q)$-spaces and the space $\ell^1(n)$.

In Theorem 3.1 we characterize three-dimensional spaces $X \in$ (BEL). It turns out that these spaces are precisely the spaces whose unit ball is a cube or an octahedron. In Theorem 4.1 it is shown that an R-weakly convex subset M of a space $X \in$ (BEL) is m-connected (under the natural condition on the spread of points of M); if M is closed and X is finite-dimensional, it is shown that any connected component of M is monotone path-connected and is a sun in X (Theorem 4.2).

In what follows, X is a real normed linear space, X_n is a finite-dimensional space X of dimension n, $B(x, r)$ is a closed ball with centre x and radius r, $\mathring{B}(x, r)$ is an open ball, $B := B(0, 1)$ is the unit ball, S is the unit sphere.

2 Definitions and auxiliary results

Let $R > 0$ be fixed and let $x, y \in X$, $\|x - y\| < 2R$. The set

$$D_R(x, y) = \bigcap_{x,y \in B(z,R)} B(z, R) \tag{2.1}$$

is called an *R-strongly convex segment* [18], [7] (or C-spindle). A subset $\varnothing \neq M \subset X$ is called *R-weakly convex*[1] [7] if

$$(D_R(x, y) \setminus \{x, y\}) \cap M \neq \varnothing \qquad \forall\, x, y \in M, \quad 0 < \|x - y\| < 2R. \tag{2.2}$$

We note at once that (see [7, Lemma 3.13]):

$$D_R(x, y) \subset D_r(x, y) \quad \text{for} \;\; x, y \in X, \;\|x - y\| \leqslant 2r, \quad r < R. \tag{2.3}$$

For a bounded set $\varnothing \neq M \subset X$, the *Banach–Mazur* hull $\mathrm{m}(M)$ of M (see [12]) is defined as the intersection of all closed balls that contain M (as distinct from (2.1), here the radius of the balls is not assumed fixed).

A subset $\varnothing \neq M \subset X$ is called m-*connected* [12] (or *Menger-connected*) if $\mathrm{m}(\{x, y\}) \cap M \neq \{x, y\}$ for any two distinct points $x, y \in M$. In what follows, we write for compactness $\mathrm{m}(\{x, y\}) = \mathrm{m}(x, y)$.

For example, in the spaces $C(Q)$ and $C_0(Q)$ the structure of $\mathrm{m}(M)$ is quite transparent: $\mathrm{m}(x, y) = \{z \in C(Q) \mid z(q) \in [x(q), y(q)],\ q \in Q\}$ (Q is a metrizable compact set).

Let $k(\tau)$, $0 \leqslant \tau \leqslant 1$, be a continuous curve in a normed linear space X. Following [11] we say that a curve $k(\cdot)$ is *monotone* if $f(k(\tau))$ is a monotone function in τ for any $f \in \operatorname{ext} S^*$; here $\operatorname{ext} S^*$ is the set of extreme points of the dual unit sphere S^*.

A subset $M \subset X$ is called *monotone path-connected* [2] if two arbitrary points in M can be connected by a continuous monotone curve $k(\cdot) \subset M$. It is readily verified that a monotone path-connected set is m-connected. The converse assertion is true for closed subsets of a finite-dimensional space (this follows from Lemma 2 of [2] in view of Straszewicz's theorem on the density of exposed points among the extreme points of the unit sphere); in infinite-dimensional case there is an example of a closed m-connected set consisting of two connected components (see [16], [2]).

Let us introduce the following class of normed linear spaces[2]

[1] In the terminology of Balashov and Ivanov [18], [7] such sets are called *weakly convex* (in the sense of Vial) *with constant* $R > 0$.

[2] BEL is derived from the phrase "linear balls embedding".

$$(\mathrm{BEL}): \quad \forall x, y \in B \ \ \exists z \in B: \ x, y \in B(z, \|x-y\|/2) \subset B. \tag{2.4}$$

It is easily verified that Euclidean spaces fail to lie in (BEL).

We shall show that the class (BEL) contains all the spaces $C(Q)$ and $\ell^1(n)$. Besides, Theorem 3.1 states that a three-dimensional space X lies in (BEL) if and only if the unit ball B of X is either a cube or octahedron (up to an affine transform); consequently, a two-dimensional space X lies in the class (BEL) if and only if its unit ball is a parallelogram.

Lemma 2.1. *Let $X \in$ (BEL), $x, y \in X$, $\|x-y\|/2 \leqslant r \leqslant R$. Then*

$$D_R(x, y) = D_r(x, y).$$

Proof of Lemma 2.1. There is no loss of generality in assuming that $R = 1$. By (2.3), $D_1(x, y) \subset D_r(x, y)$. We claim that $D_1(x, y) = D_r(x, y)$.

It is clear that $D_1(x, y) \subset D_\delta(x, y)$, where $\delta := \|x-y\|/2$. Assume, on the contrary, that $D_1(x, y) \subsetneqq D_\delta(x, y)$. Then $v \notin B(\xi, 1)$ for some $\xi \in X$ and $v \in D_\delta(x, y)$. However, since $X \in$ (BEL), there exists a point z such that $x, y \in B(z, \delta) \subset B(\xi, 1)$. But in this case $x, y \in B(z, \delta) \not\ni v$, which is impossible, because $v \in D_\delta(x, y)$ by the assumption. Hence $D_1(x, y) = D_\delta(x, y)$. Now $D_1(x, y) = D_r(x, y)$ by (2.3). $\square$.

As a corollary of Lemma 2.1, we have the following result.

Proposition 2.1. *Let $X \in$ (BEL) and $R > 0$. Then $D_R(x, y) = \mathrm{m}(x, y)$, whenever $x, y \in X$, $\|x-y\| \leqslant 2R$.*

The next result is clear.

Proposition 2.2. *Let $[a, b]$ be a one-dimensional face of the unit ball B. Then $\mathrm{m}(a, b) = [a, b]$.*

3 Spaces with linear embedding of balls. Subequilateral (edge-antipodal) balls

Recall that a set P is *antipodal* if, for any distinct x and y, there exist parallel (distinct) hyperplanes supporting $\operatorname{conv} P$ at x and y (such points x and y are called antipodal). A polytope P is called *antipodal* if its vertex set is antipodal. It is well known that an antipodal set (in a finite-dimensional space) is always finite. Moreover, if A is an antipodal set in $\mathbb{R}^n$, then $\operatorname{card} A \leqslant 2^n$, the equality attaining if and only if A is affine equivalent to the vertex set of the n-dimensional cube [15]. Three-dimensional antipodal sets are completely described in [20]. In particular, the following result is valid (see [20, § 5]).

Proposition A. *A centrally-symmetric three-dimensional polytope is antipodal if and only if it is an octahedron or a cube* (*up to an affine transform*).

Remark 1. We do not know the structure of centrally-symmetric antipodal polytopes in spaces X_n of dimension $\geqslant 4$. In this connection we point out that in these spaces this condition is satisfied (together with a cube and octahedron (cross-polytope)) also by Hanner polytopes (see [19])— such a polytope is built using ℓ^∞- and ℓ^1-sums of the interval $[0, 1]$.

A convex polytope $P \subset \mathbb{R}^n$ is called *edge-antipodal* [22] if any two vertices that determine an edge of the polytope lie on distinct parallel supporting hyperplanes of P. An edge-antipodal 3-polytope is known [8] to have at most 8 vertices, the bound attaining only for the affine cubes. Also note [21] that the number of vertices of an edge-antipodal n-polytope is bounded by $(\frac{n}{2}+1)^n$.

It is clear that a two-dimensional edge-antipodal polytope is antipodal. For three-dimensional spaces, it is noted in [8] that the vertices of an edge-antipodal 3-polytope make up an antipodal set. However, for each $n \geqslant 4$, Talata [22] (see also [14, p. 202]) constructed an edge-antipodal n-polytope having a pair of non-antipodal vertices.

A polytope is called *equilateral* (with respect to some norm $\|\cdot\|$) if its vertices form an equidistant set. An equidistant set is well known to be antipodal. A polytope P is called *subequilateral* [21] if the length in the norm $\|\cdot\|_P$ of each of its edges equals the diameter of P (the norm $\|\cdot\|_P$ is defined by the unit ball $\frac{1}{2}(P-P)$). It is easily verified [21, § 2.2] that an edge-antipodal polytope P is subequilateral (in the norm $\|\cdot\|_P$). The converse assertion is also quite clear: any subequilateral polytope P is edge-antipodal (and as a corollary, is antipodal if $\dim X \leqslant 3$ or if the degree of any vertex thereof equals the dimension of the space [9]). It is also worth noting that the faces of an edge-antipodal polytope are edge-antipodal themselves and that if all two-dimensional faces of an edge-antipodal n-polytope P are parallelograms, then P is an n-cube [9].

In what follows, by an *subequilateral* ball of a space X_n we will understand a ball that is a subequilateral (or, equivalently, edge-antipodal) polytope. It is clear that a cube and an octahedron (cross-polytope) are subequilateral polytopes (in the norm they define).

Note that spaces with subequilateral balls naturally arise in the problem of the number of connected components in the complement to Chebyshev sets and suns (see [1], [6] and the references cited therein).

We have the following results.

Proposition 3.1. *The unit ball B of a space $X_n \in$ (BEL) is a subequilateral (edge-antipodal) polytope.*

Proposition 3.2. *The spaces $C(Q)$ (Q is a metrizable compact set) and $\ell^1(n)$ lie in the class* (BEL).

Proof of Proposition 3.1. First we assume that the unit ball B contains more than 2^n exposed points. Then [6, § 3] (see also [10, Theorem 9.11.1]) among them there are u and v such that $\|u-v\|/2 =: d < 1$. Let H_u be a supporting hyperplane of the ball B at the point u such that $H_u \cap B = \{u\}$, and let H_v be the similar supporting hyperplane at the point v.

We have $X_n \in$ (BEL). Hence, by the definition, for points $u, v \in B$, there exists a point $z \in B$ such that $u, v \in B(z,d) \subset B$. As a corollary, any hyperplane that supports the ball B at the point u (at the point v) is a supporting hyperplane to the ball $B(z,d)$ at the same point u (at the point v). Let $\hat{u}$ and $\hat{v}$ be the inverse images of the points u and v under the mapping $x \mapsto dx + z$, $x \in X$. Then $\hat{u}, \hat{v} \in S$, and by the above, $\hat{u} \in H_u$ and $\hat{v} \in H_v$. Now $u, \hat{u} \in H_u$, a contradiction with the fact that u is an exposed point of the ball B. Consequently, B contains at most 2^n exposed points. In particular, B is a polytope.

Assume that a polytope B is not subequilateral. Then B contains an edge (a one-dimensional face) $[x, y]$ whose length is less than two. It is quite clear that condition (4) fails hold for such points x, y (see Proposition 2.2). □

Proof of Proposition 3.2. The assertion for $C(Q)$ is established in [4, Proposition 1]. Correspondingly, in what follows we will assume that $X = \ell^1(n)$. Let $u, v \in B$, $\|u - v\| = 2r < 2$. We need to show that $u, v \in B(\zeta, r) \subset B$ for some $\zeta \in B$.

The following property of the unit ball B in $X = \ell^1(n)$ will be of considerable value for us:

$$B = [\![\omega, \Omega]\!] := \{z \mid f(z) \in [f(\omega), f(\Omega)] \quad \forall f \in \operatorname{ext} S^*\}; \tag{3.1}$$

here ω, Ω are arbitrary opposite vertices of the unit ball (an octahedron or cross-polytope). Equality (3.1) is clear—it suffices to use a representation for the general bounded linear functional in X (from which it follows that $|f(\omega)| = 1$ for any $f \in \operatorname{ext} S^*$, and hence, $B \subset [\![\omega, \Omega]\!]$) and then invoke the fact that any point not lying in a closed ball can always be strictly separated from the ball by an extreme functional (see, for example, [5]).

Let $F = (f_i)_{i \in I}$ be all extreme (exposed) points of the dual unit sphere S^* (card $I = 2^n$). By (3.1), for any $i \in I$ we always have $|f_i(\omega)| = 1$ or $|f_i(\Omega)| = 1$. Notice that in any normed linear space

$$\|x\| = \sup_{f \in S^*} f(x) = \sup_{f \in \operatorname{ext} S^*} f(x). \tag{3.2}$$

Hence $|f_i(u) - f_i(v)| \leqslant 2r$ for all i, and also $|f_j(u) - f_j(v)| = 2r$ at least for one $j \in I$. For each $i \in I$, we set

$$\overline{m}_i = \max\{f_i(u), f_i(v)\}, \qquad \underline{m}_i = \min\{f_i(u), f_i(v)\}, \tag{3.3}$$

We have $-1 \leqslant \underline{m}_i \leqslant \overline{m}_i \leqslant 1$, $\overline{m}_i - \underline{m}_i \leqslant 2r$, also, $\overline{m}_j - \underline{m}_j = 2r$ for some j. Further, we set

$$\bar{\alpha}_i = 1 - \overline{m}_i, \qquad \underline{\alpha}_i = \underline{m}_i - 1$$

(it is clear that $\bar{\alpha}_i, \underline{\alpha}_i \geqslant 0$).

For any hyperstrip $-1 \leqslant f_i(z) \leqslant 1$ in (3.1), we defined the proper subhyperstrip as follows:

$$\begin{aligned} &\{z \mid f_i(z) \in [\overline{m}_i - r, \overline{m}_i]\} \qquad \text{if } \bar{\alpha}_i < \underline{\alpha}_i, \\ &\{z \mid f_i(z) \in [\underline{m}_i, \underline{m}_i + r]\} \qquad \text{if } \bar{\alpha}_i \geqslant \underline{\alpha}_i; \end{aligned} \tag{3.4}$$

the width of such a hyperstrip is $2r$. From the construction it is clear that $-1 \leqslant \overline{m}_i - r < \overline{m}_i \leqslant 1$, $-1 \leqslant \underline{m}_i < \underline{m}_i + r \leqslant 1$. Now we define Π as the intersection of all hyperstrips (3.4) over all i. From (3.1) it follows that Π is a ball of radius r; we we also have $\Pi \subset B$, since any hyperstrip (3.4) is contained in the hyperstrip $-1 \leqslant f_i(z) \leqslant 1$. Finally, from the construction it follows that any of the points u, v lies in any hyperstrip of the form (3.4), which forces $u, v \in \Pi$. The proof of Proposition 3.2 is complete. □

The next result, which puts forward a characterization of subequilateral three-dimensional spaces, has some independent interest.

Theorem 3.1. *Let X be a three-dimensional normed linear space. Then the following conditions are equivalent:*

a) *the space X lies in the class* (BEL)*; that is,*

$$\forall\, x, y \in B \;\; \exists z \in B \;:\; x, y \in B(z, \|x - y\|/2) \subset B;$$

b) *the unit ball B of X is a cube or octahedron* (*up to an affine transform*);
c) *the unit ball B is a subequilateral* (*edge-antipodal*) *polytope*;
d) *the unit ball B is an antipodal polytope.*

Proof of Theorem 3.1. Implication b)⇒c) is clear: the length of any edge of a cube (or an octahedron) equals 2 in the norm it defines.

c)⇒b) Let P be a centrally-symmetric subequilateral three-dimensional polytope. In [21] it is noted, for an n-dimensional polytope, that the property of subequilaterality is equivalent to edge-antipodality. Further, in [8] it is shown that the number of vertices of a three-dimensional edge-antipodal polytope P is at most 8, the equality attaining only in the case when P is an (affine) cube. On the other hand, any polytope that defines a norm on a three-dimensional space has at least 6 vertices (the lower bound is attained at an octahedron).

Implication b)⇒a) is contained in Proposition 3.2.

c)⇒d) Above it was noted (see [8]) that the vertices of a subequilateral (edge-antipodal) 3-polytope form an antipodal set.

For implication d)⇔b), see Proposition A; that a)⇒c) is secured by Proposition 3.1.

4 The main results

Below we put forward results on the monotone path-connectedness and m-connectedness of R-weakly convex subsets of spaces of the class (BEL) (and in particular, in $C(Q)$ and $\ell^1(n)$).

Recall that the connectedness of R-weakly convex sets was examined by Vial, Balashov and Ivanov (see [23], [18], [7], [17]), this problem was being studied mostly for sufficiently smooth or finite-dimensional spaces. We mention the following results (respectively, Lemmas 4.1, 4.13 and 4.17 in [7]; see also [17, Theorem 4.1]), which we summarize as follows.

Theorem A (Balashov–Ivanov). *Let X be a Banach space and let $M \subset X$ be an R-weakly convex set for some $R > 0$. Also let $x, y \in M$, $\|x - y\| < 2R$. Then:*

a) *if $M \cap D_R(x, y)$ is compact, then it is connected*;

b) *if X is uniformly convex or* $\dim X < \infty$*, and if M is closed, then $M \cap D_R(x, y)$ is path-connected.*

Further, [7] contains the following results (Theorems 2.7 and 2.9, respectively).

Theorem B (Balashov–Ivanov). *Let X be a Banach space that is uniformly convex or finite-dimensional. Also let $M \subset X$ be closed and R-weakly convex for some $R > 0$. Then:*

a) *$M \cap \mathring{B}(x, r)$ connected for any $r \in (0, R]$ and $x \in X$;*

b) *any connected component A of M is path-connected, any two points of A can be joined by a rectifiable curve lying in A.*

The main results of the present paper are Theorems 4.1 and 4.2, in which we partially refine Theorems A and B for spaces (BEL), thereby obtaining results on m-connectedness and monotone path-connectedness of R-weakly convex sets. Similar results for $C(Q)$ were obtained by the author in [4].

Theorem 4.1. *Let* $X \in$ (BEL) *and let* $\varnothing \neq M \subset X$ *be an* R*-weakly convex set for some* $R > 0$. *Then*:

a) $M \cap \mathring{B}(x, r)$ *is* m*-connected for all* $r \in (0, R]$ *and* $x \in X$;
b) $M \cap B(x, r')$ *is* m*-connected for all* $r' \in (0, R)$ *and* $x \in X$.

Suppose in addition that M *is closed and* $\dim X < \infty$. *Then*:

1. *Any connected component of* M *is monotone path-connected, the Hausdorff distance between any connected components being at least* $2R$;
2. *Each of the following sets is monotone path-connected:*

c) $M \cap \mathring{B}(x, r)$ *for all* $r \in (0, R]$ *and* $x \in X$;
d) $M \cap B(x, r')$ *for all* $r' \in (0, R)$ *and* $x \in X$;
e) $M \cap A$, *where* $A \subset X$, $\operatorname{diam} A < 2R$, *is such that* $A = \mathrm{m}(A)$ (*in particular, the set* $M \cap \mathrm{m}(x, y)$ *is monotone path-connected for any* x, y, $\|x - y\| < 2R$).

Recall that a subset M of X is called a *sun* (see, for example, [24]) if, for any point $x \in X \setminus M$, there exists a point $y \in P_M x$ such that $y \in P_M[(1-\lambda)y + \lambda x]$ for all $\lambda \geqslant 0$ (here $P_M x$ is the set of all nearest points to x in M). A set M is called a *strict sun* if $P_M x \neq \varnothing$ for all $x \in X$ and $y \in P_M[(1-\lambda)y + \lambda x]$ for all $x \in X$ and $y \in P_M x$.

Theorem 4.2 (characterization of R-weakly convex sets). *Let* $X_n \in$ (BEL). *Then a closed subset* M *of* X_n *is* R*-weakly convex for some* $R > 0$ *if and only if* M *is a disjoint union of monotone path-connected suns in* X_n *with the Hausdorff distance between any pair of connected components of* M *at least* $2R$.

Since a monotone path-connected subset $C(Q)$ is R-weakly convex for any $R > 0$ and since a boundedly compact strict sun in $C(Q)$ is monotone path-connected [2], it follows that a *boundedly compact strict sun* (*in particular, a boundedly compact Chebyshev set*) *in* $C(Q)$ *is* R*-weakly convex for any* $R > 0$. It is worth noting that this is not so in $\ell^1(3)$. The space $\ell^1(3)$ is known [13] to contain a non-monotone path-connected sun; nevertheless, it is unknown whether any Chebyshev set in $\ell^1(3)$ is monotone path-connected (see also remarks in [3]).

Proof of Theorem 4.1. Assertion a). Let $u, v \in M \cap \mathring{B}(x, r)$. Using Lemma 2.1 and Proposition 2.1,

$$D_r(u, v) = D_R(u, v) = \mathrm{m}(u, v). \tag{4.1}$$

Since, by the hypotheses, M is R-weakly convex, it follows from (2.2) that $M \cap \mathring{B}(x, r)$ is m-connected. Assertion b) is proved similarly.

1. Let M be closed and let M', M'' be two arbitrary distinct connected components of M. Assume that there were $x \in M'$, $y \in M''$, such that $\|x - y\| < 2R$. As before, we have $D_R(x, y) = \mathrm{m}(x, y)$. By the assumption M is R-weakly convex, and hence, by (2.2), the intersection $M \cap \mathrm{m}(x, y)$ is m-connected, and hence, by Lemma 2 of [2], is monotone path-connected. This contradicts the fact that x and y lie in different connected components of the set M.

Now consider an arbitrary connected component M' of the set M. By the above, the distance between any two connected components is at least $2R$. By the hypotheses, M is R-weakly convex, and hence the component M' is also R-weakly convex, and in view of (4.1), is m-connected. Now the monotone path-connectedness of M' is secured by the lemma [2, Lemma 2].

2. By what has been proved above it follows that if $M \cap \mathring{B}(x,r) \neq \varnothing$ (for some $r \in (0,R])$, then $M \cap \mathring{B}(x,r) = M' \cap \mathring{B}(x,r)$ for some connected component M' of M. By the above, the component M' is monotone path-connected (and hence is m-connected). To complete the proof we invoke the following fact [16, Proposition 5.1]: Let a set $\varnothing \neq A \subset X$ be such that $A = \mathrm{m}(A)$ and let M be m-connected, $M \cap A \neq \varnothing$. Then $M \cap A$ is m-connected and, since it is closed, is monotone path-connected. Theorem 4.1 is proved. □

Proof of Theorem 4.2. By Theorem 4.1, each connected component of M is monotone path-connected, the Hausdorff distance between any connected components being at least $2R$. For any connected component we apply the following arguments: by Brown's theorem [12, Theorem 3.6], a closed m-connected subset of X_n is P-acyclic, and further, by Vlasov's theorem [24, Theorem 4.4], any P-acyclic boundedly compact subset of a Banach space is a sun.

Conversely. By (4.1), any monotone path-connected set is R-weakly convex. Clearly, a disjoint union of such sets is again an R-weakly convex set, provided that connected components of the union have distance at least $2R$ from each other. Theorem 4.2 is proved. □

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Alexey Rostislavovich Alimov
Faculty of Mechanics and Mathematics
M.V. Lomonosov Moscow State University
Moscow 119899, Russia
E-mail: alexey.alimov@gmail.com

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