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ON OSCILLATION OF TWO TERMS LINEAR DIFFERENTIAL EQUATION WITH ALTERNATING POTENTIAL

L.K. Kussainova

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Abstract. Two terms high order linear differential equation are studied. For the case when potential alternats in any neighborhood of infinity, oscillation and nonoscillation conditions formulated in unified terms are obtained.

1 Introduction

Let us consider the equation

$$l[y] \equiv (-1)^n y^{(2n)} + q(x)y = 0 \quad (x > 0). \tag{1.1}$$

We are interested in oscillatory properties of equation (1.1) with a continuous potential $q(\cdot)$ alternating in any interval (t, ∞) as $t \to \infty$. We assume that the differential expression $l[y]$ is defined on the class $C^{2n}(0, \infty)$ of all functions m -times continuously differentiable on $(0, \infty)$. We will adhere to the definitions given in [1].

Definition 1. Equation (1.1) is called oscillatory (at infinity) if the following condition holds:

(OS) For any $t > 0$ there exists a nontrivial solution of equation (1.1) having at least two n-multiple zeros on (t, ∞).

In other words, if for any $t > 0$ there exist a nontrivial solution y of (1.1) and such points $t_2 > t_1 > t$ that

$$y^{(k)}(t_1) = y^{(k)}(t_2) = 0 \quad (k = 0, 1, ..., n-1).$$

Let L be one of self-adjoint operators generated by the differential expression $l[y]$. In fact condition (OS) is equivalent to infiniteness of the negative part of spectrum of L (see [1]).

For the second order equation

$$-y'' + q(x)y = 0 \quad (x > 0) \tag{1.2}$$

Definition 1 coincides with the classical definition. Namely, equation (1.2) is called oscillatory (at infinity) if every nontrivial solution of (1.2) has an infinite set of zeros x_k $(k = 1, 2, ...)$ such that $x_k \to \infty$ as $k \to \infty$(see [1], [2]).

Definition 2. Equation (1.1) is said to be non-oscillatory if the condition (OS) is not satisfied.

For equation (1.1) with negative potential $q(\cdot)$ a number of oscillation (nonoscillation) conditions, expressed in unified terms, was obtained.

Let $q(x) < 0$ for $x \geqslant x_0$. Here and hereafter x_0 is a sufficiantly large fixed positive number. Moreover, let

$$J_n(x, q) = x^{2n-1} \int\limits_x^\infty |q(t)|\, dt,$$

and $\alpha_n = (2n-1)!!/2^n$ be the Kneser constan. The integral criterion states that equation (1.1) is nonoscillatory if

$$\limsup_{x\to\infty} J_n(x, q) < \alpha_n^2/(2n-1),$$

and is oscillatory if

$$\limsup_{x\to\infty} J_n(x, q) > A_n^2.$$

Here

$$A_n^{-1} = \frac{\sqrt{2n-1}}{(n-1)!} \sum_{k=1}^{n} (-1)^{k-1}\, C_{k-1}^{n-1} (2n-k)^{-1} \ (n \geqslant 1)$$

(see [1, 2]). Note that $\alpha_1^2 = 1 = A_1$ (the case $n = 1$).

In this connection we formulate here one more oscillation condition proved in [3].

Equation (1.2) is oscillatory if

$$\liminf_{x\to\infty} \left(\frac{1}{x} \int\limits_0^x t^2\, |q(t)|\, dt. + x \int\limits_x^\infty |q(t)|\, dt > 1 \right). \tag{1.3}$$

Condition (1.3) was introduced in [4].

2 Main results

The purpose of the work is to describe in unified terms oscillation and nonoscillation conditios for equation (1.1) with sign-variable potential $q(\cdot)$. We study equation (1.1) with $q = u - \upsilon$, $u, \upsilon \in L_+^{loc}(I)$. $L_+^{loc}(I)$ denotes the space of all non-negative functions locally integrable on the interval $I = (x_0, \infty)$. We assume that u, υ are not degenerate, namely

$$\int_x^\infty \upsilon(t)dt > 0, \ \int_x^\infty u(t)dt > 0 \text{ for all } x \geqslant x_0.$$

We will use one modification of the Otelbaev function, in terms of which sophisticated spectral estimates for differential operators were obtaned (see [3, 5, 6]).

Let $\omega \in L_+^{loc}(I)$, $x \geqslant x_0$, $h > 0$, $\Delta = [x, x+h)$.We denote

$$\mathcal{M}(\Delta|\omega) = \mathcal{M}(x,\ h|\omega) = h^{-1} \int\limits_{\Delta} \omega(t)dt.$$

We will use the following Otelbaev function υ^* defined by

$$\upsilon^*(x) = \inf_{h>0} \{h^{-2n} : h^{2n}\mathcal{M}(x, h|\upsilon) \leqslant 1\}.$$

It is easy to show that $0 < \upsilon^*(x) < \infty$. The function $h_x^* = \upsilon^*(x)^{-1/2n}$ is called the characteristic length. Let $\Delta^*(x) = [x, x + h_x^*)$, then the equality

$$\frac{\mathcal{M}(\Delta^*(x)|\upsilon)}{\upsilon^*(x)} = 1$$

holds (see [5]). We will also use the notation

$$\Delta_\delta^*(x) = [x + \delta h_x^*, x + (1-\delta)h_x^*)\,,\ 0 < \delta < 1.$$

Theorem 1. Let $q = \upsilon - u$, u, $\upsilon \in L_+^{loc}(I)$. Assume that for some $0 < \delta < 1$

$$\limsup_{x\to\infty} \frac{\mathcal{M}(\Delta_\delta^*(x)|u)}{\upsilon^*(x)} > A_{\delta,n},$$

where

$$A_{\delta,n} = 1 + 2A_n^2\delta^{-2n+1}.$$

Then equation (1.1) is oscillatory.

Theorem 2. Let $q = \upsilon - u$, u, $\upsilon \in L_+^{loc}(I)$.

a) If

$$\limsup_{x\to\infty} \frac{\mathcal{M}(\Delta_{1/4}^*(x)|u)}{\upsilon^*(x)} > 18$$

then equation (1.2) is oscillatory.

b) If

$$\limsup_{x\to\infty} \frac{\mathcal{M}(\Delta^*(x)|u)}{\upsilon^*(x)} \leqslant \frac{1}{10}$$

then equation (1.2) is nonoscillatory.

For the first time oscillation conditions in terms of the characteristic means of $q_+(\cdot) = \max\{q(\cdot), 0\}$ and $q_-(\cdot) = \min\{-q(\cdot), 0\}$ were obtained in [7]. However, those means are related only to second order equations.

Let $R^{(n)}(\Delta^*(x))$ be the set of all polynomials $R(t) = \sum\limits_{k=1}^{n-1} c_k t^k$ with

$$\int\limits_{\Delta^*(x)} |R(t)|^2 dt = 1.$$

We say that $\upsilon \in (R^{(n)})^*$ if there exist η, $0 < \eta < 1$, such that

$$\eta \mathcal{M}(\Delta^*(x)|\upsilon) \leqslant \inf \left\{ \int\limits_{\Delta^*(x)} |R(t)|^2 \upsilon(t) dt, \quad R \in \mathcal{R}^{(n)}(\Delta^*(x)) \right\}$$

for all $x \geqslant x_0$.

Let $c_{\infty,n}$ be the exact constant in the inequality

$$\max_{[0,1]} |\varphi| \leqslant c \left(\int_0^1 (|\varphi|^2 + \left|\varphi^{(n)}\right|^2) dt \right)^{1/2}.$$

Theorem 3. Assume $q = \upsilon - u$, $\upsilon \in (R^{(n)})^*$ with some η, $0 < \eta < 1$, $n > 1$, and $u \in L_+^{loc}(I)$. If

$$\limsup_{x \to \infty} \frac{\mathcal{M}(\Delta^*(x)|u)}{\upsilon^*(x)} < B_{\eta,n}^{-1} ,$$

where

$$B_{\eta,n} = c_{\infty,n}^2 (1 + 8\eta^{-1}),$$

then equation (1.1) is nonoscillatory.

3 Examples

Example 1. Assume that $\upsilon(\cdot) > 0$

$$\alpha < \upsilon(t)/\upsilon(x) < \beta \text{ if } 0 < t - x < h_x^*,$$

where the number $0 < \alpha < \beta$ are not dependet of x $(x \geqslant x_0)$. Then $\upsilon \in (R^{(n)})^*$ with $\eta = \alpha\beta^{-1}$.

Example 2. The function

$$\upsilon(x) = \frac{9}{16} x^{-4} \sin^2 x \quad (x > 0) \tag{3.1}$$

also belongs to $(R^{(n)})^*$ on the interval I. This follows from the two facts: 1) the function υ in (3.1) possesses the following property: there exist $0 < \delta, \tau < 1$ such thatΔ

$$\int\limits_e \upsilon \leqslant \tau \int\limits_{\Delta^*(x)} \upsilon \quad if \quad e \subset \Delta^*(x) \text{ and } \operatorname{meas}(e) \leqslant \delta h_x^*, \tag{3.2}$$

2) since $R^{(n)}(\Delta^*(x))$ is a compact subset in $C[x, x + h_x^*]$ the following uniform estimate holds

$$\operatorname{meas}\{t \in \Delta^*(x) : |R(t)| \leqslant \gamma h_x^{*-1/2}\} \leqslant \delta h_x^* \quad (R \in \mathcal{R}^n(\Delta^*(x))), \tag{3.3}$$

where $\gamma \in (0,1)$ depends only on δ. Here $C[x, x + h_x^*]$ denotes the space of all functions continuous on $[x, x + h_x^*]$.

Using (3.2), (3.3) we can show that υ defined by (3.1) belongs to $(R(^{(n)})^*$ with $\eta = \gamma_0^2(1 - \tau)$, where γ_0 is the maximal constant in (3.3) for $\delta = 9/20$, $1 - \tau =$

$\beta^2(1-2\delta)(a/(1+a))^4$, $a = 2.21$, and β is the solution of the equation $2.2 \arcsin\beta = \pi\delta$. For the Otelbaev function υ^* the following estimates hold

$$(2.22\ x)^{-4} < \upsilon^*(x) < (1.15\ x)^{-4}. \tag{3.4}$$

Example 3. Let us consider the equation

$$y^{IV} + \upsilon(x)y - u(x)y = 0, \tag{3.5}$$

where υ is defined by (3.1), and $u \in L_+^{loc}(I)$. Then: a) equation (3.5) is nonoscillatory if

$$\limsup_{x\to\infty} x^3 \int\limits_x^{3.21x} u(t)dt < c_{\infty,2}^{-2}(1 + 74\gamma_0^2)^{-1},$$

b) equation (3.5) is oscillatory if

$$\limsup_{x\to\infty} x^3 \int\limits_{1.45x}^{1.63x} u(t)dt > 172.$$

The proofs of Theorems 1-3 are based on the oscillation criterion for the equations of the type $\widetilde{l}[y] = 0$, where $\widetilde{l}[y]$ is a self-adjoint differential expression of order $2n$ $(n \geqslant 1)$. See [1, Section 10]. We alsou use estimats of exact constants in inequalities of local embeddings of weighted Sobolev spaces on the characteristic interval $\Delta^*(x)$. Basic techniques of proofs can be found in [5, 6, 8].

References

[1] I.M. Glazman, *Direct methods of qualitative analysis of singular differential operators.* Jerusalim, 1965.

[2] P. Hartman, *Ordinary differential equations.* Wiley. New York, 1964.

[3] M. Otelbaev, *Estimates of the spectrum of the Sturm-Liouville operator.* Gylym, Alma-Ata, 1990 (in Russian).

[4] R. Oinarov, S.Y. Rakhimova, *Oscillation and nonoscillation of two terms linear and half-linear equations of higher order.* E.J. Qualitative Theory of Diff. Equ., Hungary, no. 49 (2010), 1-15.

[5] M. Otelbaev, L. Kussainova, A. Bulabaev, *Estimate of the spactrum for a certain class of differential operators* Sbornik pratz. Instituta Matematiki NAN Ukraina 6-1 (2009), 165-190 (in Russian).

[6] K.T. Mynbaev, M.O. Otelbaev, *Weighted functional spaces and differential operator spectrum.* Moscow, Nauka, 1988 (in Russian).

[7] R. Oinarov, K. R. Myrzatayeva, *Nonoscillation of second-order half-linear differential equation.* Mathimatical J., Almaty, 2(24) (2007), 72-82 (in Russian).

[8] L.K. Kussainova, *Embedding of weighted Sobolev space $W_p^l(\Omega; \upsilon)$ in the space $L_p(\Omega; \omega)$.* Mat. Sb., 191:2 (2000), 132-148 (in Russian).

Leili Kabidenovna Kussainova
Faculty of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
5 Munaitpasov St, 010008 Astana, Kazakhstan
E-mail: leili2006@mail.ru

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