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A REMARK ON SEMISTABILITY OF QUIVER BUNDLES

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Abstract. In this paper, we introduce a new semistability condition for quiver bundles which generalizes both the notion found by Álvarez-Cónsul and by the author. We construct moduli spaces for the semistable bundles, applying Geometric Invariant Theory.

1 Introduction

Let $(X, \mathscr{O}_X(1))$ be a polarized projective manifold over the complex numbers, $Q = (V, A, t, h)$ a quiver and $\underline{\mathscr{G}} = (\mathscr{G}_a, a \in A)$ a tuple of locally free sheaves on X. A *representation of* Q is a tuple $(\mathscr{E}_v, v \in V, \varphi_a, a \in A)$ in which $\mathscr{E}_v$ is a coherent $\mathscr{O}_X$-module, $v \in V$, and $\varphi_a \colon \mathscr{G}_a \otimes \mathscr{E}_{t(a)} \longrightarrow \mathscr{E}_{h(a)}$ is a homomorphism of $\mathscr{O}_X$-modules, $a \in A$. There is the obvious notion of an *isomorphism of representations of* Q. We would like to investigate the problem of classifying representations of Q up to isomorphism where the Hilbert polynomials of the participating sheaves are fixed. We will follow the path of defining semistability of representations and then constructing the moduli spaces with Geometric Invariant Theory.

This problem has already been considered before. A general semistability concept for quiver bundles was first discussed in the paper [2]. (Of course, some important examples such as Higgs bundles and holomorphic triples had been known before.) In that paper, semistable quiver bundles were related to solutions of certain differential equations and, by means of dimensional reduction, to semistable vector bundles on flag manifolds. The notion of semistability discussed in that paper is naturally a notion of **slope** semistability. If one is interested in constructing reasonable moduli spaces for quiver bundles with Geometric Invariant Theory, slope semistability is the right answer only in the case of curves. On higher dimensional manifolds, one should recur to a **Gieseker type** notion of semistability. In [13], the author introduced such a notion for quiver bundles and constructed the moduli spaces. More recently, Álvarez-Cónsul generalized in [1] his work with King [3] to quiver bundles and obtained another notion of semistability. As one would expect, it agrees with the one of the author in the case of curves but seems to be genuinely different on higher dimensional manifolds. Álvarez-Cónsul asks in [1] for a common generalization of the two semistability concepts. In this note, we provide such a generalization (in a rather straightforward manner). It turns out that the Geometric Invariant Theory construction given in [13] can be slightly modified to work also for the more general semistability concept. In this paper, we will also take the opportunity to fill in many details of the computations left out in [13].

The author thinks that this is a phenomenon worth studying: In the case of curves, semistability for principal bundles with extra structures is reasonably well understood. It can be derived from some basic principles (see [14]), it agrees in all known examples with the one coming from gauge theory ([4], [5], [11]) and can usually be explained in terms of line bundles on the moduli stack (e.g., [8]). As the example of quiver bundles shows, this understanding seems to be less thorough on higher dimensional base manifolds. Thus, it may be worthwhile investigating this topic a little more. Furthermore, variants of quiver representations on Calabi–Yau threefolds appear in Pandharipande–Thomas theory [15].

We fix

- a tuple $\underline{P} = (P_v, v \in V)$ of Hilbert polynomials,
- an integer $t \geq 0$,
- a tuple $\underline{\kappa} = (\kappa_v, v \in V)$ of positive **integral** polynomials of degree **exactly** t,
- a positive rational polynomial δ of degree **at most** $t + \dim(X) - 1$,
- and a tuple $\underline{\eta} = (\eta_v, v \in V)$ of rational numbers, subject to the condition $\sum_{v \in V} \eta_v \cdot r_v = 0$. Here, r_v is the rank determined by the Hilbert polynomial P_v, $v \in V$.

Furthermore, we define

- $\chi_v := \eta_v \cdot \delta$, $v \in V$, $\underline{\chi} = (\chi_v, v \in V)$,
- σ_v as the leading coefficient of κ_v, $v \in V$, $\underline{\sigma} = (\sigma_v, v \in V)$.

For a tuple $\underline{\mathscr{F}} = (\mathscr{F}_v, v \in V)$ of coherent sheaves on X, we set

$$P_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{F}}) := \sum_{v \in V} \big(\kappa_v \cdot P(\mathscr{F}_v) - \chi_v \cdot \mathrm{rk}(\mathscr{F}_v)\big)$$

and

$$\mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{F}}) := \sum_{v \in V} \sigma_v \cdot \mathrm{rk}(\mathscr{F}_v).$$

A representation $(\mathscr{E}_v, v \in V, \varphi_a, a \in A)$ with $P(\mathscr{E}_v) = P_v$, $v \in V$, is then called *(semi)stable*, if a) the sheaves $\mathscr{E}_v$, $v \in V$, are torsion free and b) for any collection of saturated subsheaves $\mathscr{F}_v \subset \mathscr{E}_v$, $v \in V$,[1] not all trivial and not all equal to $\mathscr{E}_v$, such that $\varphi_a(\mathscr{G}_a \otimes \mathscr{F}_{t(a)}) \subset \mathscr{F}_{h(a)}$ for all arrows $a \in A$, one has

$$\frac{P_{\underline{\kappa},\underline{\chi}}(\mathscr{F}_v, v \in V)}{\mathrm{rk}_{\underline{\sigma}}(\mathscr{F}_v, v \in V)} (\preceq) \frac{P_{\underline{\kappa},\underline{\chi}}(\mathscr{E}_v, v \in V)}{\mathrm{rk}_{\underline{\sigma}}(\mathscr{E}_v, v \in V)}.$$

The notation "$(\preceq)$" means that "$\prec$" is used for defining "stable" and "$\preceq$" for defining "semistable", and "$\prec$" and "$\preceq$" refer to the lexicographic ordering of polynomials.

[1] i.e., $\mathscr{E}_v/\mathscr{F}_v$ is again torsion free, $v \in V$

Finally, $(\mathscr{E}_v, v \in V, \varphi_a, a \in A)$ is called *polystable*, if it is a direct sum of stable representations $(\mathscr{E}_v^i, v \in V, \varphi_a^i, a \in A)$, $i = 1, ..., s$, with

$$\frac{P_{\underline{\kappa},\underline{\chi}}(\mathscr{E}_v^i, v \in V)}{\mathrm{rk}_{\underline{\sigma}}(\mathscr{E}_v^i, v \in V)} = \frac{P_{\underline{\kappa},\underline{\chi}}(\mathscr{E}_v^j, v \in V)}{\mathrm{rk}_{\underline{\sigma}}(\mathscr{E}_v^j, v \in V)}, \quad \text{for all } i, j = 1, ..., s.$$

Remark. *The above notion of semistability generalizes both the notion discussed in [13] and the notion introduced by Álvarez-Cónsul [1]. To recover the former notion we set $t = 0$, and for the latter notion we set $\eta_v = 0$, $v \in V$.*

The main result of this paper is the following

Main Theorem. i) *There is a quasi-projective moduli space $\mathscr{D} := \mathscr{D}(Q, \underline{\mathscr{G}})^{\mathrm{ss}}_{\underline{P}/\underline{\kappa}/\underline{\eta}/\delta}$ for polystable representations $(\mathscr{E}_v, v \in V, \varphi_a, a \in A)$ with $P(\mathscr{E}_v) = P_v$, $v \in V$. The points corresponding to stable representations form an open subset $\mathscr{D}^{\mathrm{s}}$.*

ii) *There are a vector space $\mathbb{D}$ and a projective morphism $H\colon \mathscr{D} \longrightarrow \mathbb{D}$, the* generalized Hitchin map.

Conventions

We will freely use the terminology of the paper [13]. For a vector bundle $\mathscr{E}$ on a scheme Y, we let $\mathbb{P}(\mathscr{E}) := \mathscr{P}roj\ (\mathscr{S}ym^{\star}(\mathscr{E}))$ be Grothendieck's projectivization and $P(\mathscr{E}) := \mathbb{P}(\mathscr{E}^{\vee})$. We write the vertex set as $V = \{ v_1, ..., v_t \}$.

2 Decorated Tuples of Sheaves

Following the strategy of [13], we will first deal with a much more general classification problem concerning decorated tuples of sheaves.

The moduli functors of semistable decorated V-split sheaves

For our considerations, we fix the same parameters as in the introduction. In addition, we fix positive integers a, b and non-negative integers c, m. Recall that the parameters comprise the polynomials

$$\kappa_v(x) = \sigma_v \cdot x^t + \text{lower order terms}, \quad v \in V.$$

For a tuple $\underline{\mathscr{F}} = (\mathscr{F}_v, v \in V)$ of coherent $\mathscr{O}_X$-modules, called a *V-split sheaf*, we define

$$\mathscr{F}^{\mathrm{total}} := \bigoplus_{v \in V} \mathscr{F}^{\oplus \sigma_v}$$

and

$$\underline{\mathscr{F}}_{a,b,c} := \left((\mathscr{F}^{\mathrm{total}})^{\otimes a}\right)^{\oplus b} \otimes \det(\mathscr{F}^{\mathrm{total}})^{\otimes -c}.$$

A *decorated tuple of type* $(\underline{P}, a, b, c, m)$ is a tuple $(\underline{\mathscr{E}}, \varphi)$ which consists of a **torsion-free** V-split sheaf $\underline{\mathscr{E}} = (\mathscr{E}_v, v \in V)$ with $P(\mathscr{E}_v) = P_v$, $v \in V$, and a non-zero homomorphism

$$\varphi\colon \underline{\mathscr{E}}_{a,b,c} \longrightarrow \mathscr{O}_X(m).$$

Next, we define semistability. The test objects are *weighted filtrations* $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ of $\underline{\mathscr{E}}$, i.e., $\underline{\mathscr{E}}_\bullet$ is a tuple $(\underline{\mathscr{E}}_1, ..., \underline{\mathscr{E}}_s)$ of V-split sheaves $\underline{\mathscr{E}}_i = (\mathscr{E}_i^v, v \in V)$, $i = 1, ..., s$, such that

$$\mathscr{E}_\bullet^v : 0 \subseteq \mathscr{E}_1^v \subseteq \cdots \subseteq \mathscr{E}_s^v \subseteq \mathscr{E}_v$$

is a filtration of $\mathscr{E}_v$ by **saturated** subsheaves where equalities are allowed, $v \in V$, such that

$$\mathscr{E}_\bullet^{\text{total}} : 0 \subsetneq \mathscr{E}_1^{\text{total}} \subsetneq \cdots \subsetneq \mathscr{E}_s^{\text{total}} \subsetneq \mathscr{E}^{\text{total}}$$

is a filtration in which all inclusions are strict, and $\alpha_\bullet = (\alpha_1, ..., \alpha_s)$ is a tuple of positive rational numbers.

Given a V-split sheaf $\underline{\mathscr{E}} = (\mathscr{E}_v, v \in V)$ and a weighted filtration $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ of $\underline{\mathscr{E}}$, we set

$$M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}_\bullet, \alpha_\bullet) \ := \ \sum_{i=1}^{s} \alpha_i \cdot \Big(P_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}) \cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_i) - P_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}_i) \cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}) \Big).$$

We also introduce

$$R := \sum_{v \in V} \sigma_v \cdot R_v, \quad R_i := \mathrm{rk}(\mathscr{E}_i^{\text{total}}), \quad i = 1, ..., s,$$

and we let the vector $\underline{\gamma} = (\gamma_1, ..., \gamma_{s+1})$ consist of the integers occurring in the vector

$$\sum_{i=1}^{s} \alpha_i \cdot (\underbrace{R_i - R, ..., R_i - R}_{R_i \times}, \underbrace{R_i, ..., R_i}_{(R-R_i)\times})$$

in increasing order. If we are also given a decoration $\varphi \colon \underline{\mathscr{E}}_{a,b,c} \longrightarrow \mathscr{O}_X(m)$, we define the quantity

$$\mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \varphi) := - \min_{\substack{(i_1,...,i_a) \in \\ \{1,...,s+1\}^{\times a}}} \Big\{ \gamma_{i_1} + \cdots + \gamma_{i_a} \,|\, \varphi_{|(\mathscr{E}_{i_1}^{\text{total}} \otimes \cdots \otimes \mathscr{E}_{i_a}^{\text{total}})^{\oplus b}} \neq 0 \Big\}. \tag{2.1}$$

A decorated V-split sheaf $(\underline{\mathscr{E}}, \varphi)$ is called $(\underline{\kappa}, \underline{\eta}, \delta)$-*(semi)stable* or just *(semi)stable*, if for every weighted filtration $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ of $\underline{\mathscr{E}}$

$$M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}_\bullet, \alpha_\bullet) + \delta \cdot \mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \varphi)(\succeq)0.$$

Remark 2.1. *i) Let* $\underline{i}^0 = (i_1^0, ..., i_a^0) \in \{1, ..., s+1\}^{\times a}$ *be an index tuple which gives the minimum in* (2.1)*. Let*

$$\nu_i(\underline{i}^0) := \#\{ k = 1, ..., a \,|\, i_k^0 \leq i \}, \quad i = 1, ..., s.$$

Then,

$$\mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \varphi) = \sum_{i=1}^{s} \alpha_i \cdot (\nu_i(\underline{i}^0) \cdot R - a \cdot R_i). \tag{2.2}$$

ii) Since $\sum_{v\in V}\chi_v\cdot r_v\equiv 0$, *we may write*

$$M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}_\bullet,\alpha_\bullet)=\mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}})\cdot\sum_{i=1}^{s}\alpha_i\cdot\left(\sum_{v\in V}\chi_v\cdot\mathrm{rk}(\mathscr{E}_i^v)\right)\quad+$$
$$+\quad\underbrace{\sum_{i=1}^{s}\alpha_i\cdot\left(\left(\sum_{v\in V}\kappa_v\cdot P(\mathscr{E}_v)\right)\cdot\mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_i)-\left(\sum_{v\in V}\kappa_v\cdot P(\mathscr{E}_i^v)\right)\cdot\mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}})\right)}_{=:\,M_{\underline{\kappa}}(\underline{\mathscr{E}}_\bullet,\alpha_\bullet)}.$$

Finally, we define the functors

$$\underline{\mathrm{M}}^{(\underline{\kappa},\underline{\eta},\delta)\text{-(s)s}}_{\underline{P}/a/b/c/m}\colon\quad \underline{\mathrm{Sch}}_{\mathbb{C}}\longrightarrow\underline{\mathrm{Sets}}$$
$$S\longmapsto\left\{\begin{array}{c}\text{Equivalence classes of families of decorated}\\ (\underline{\kappa},\underline{\eta},\delta)\text{-(semi)stable } V\text{-split sheaves of}\\ \text{type } (\underline{P},a,b,c,m)\text{ parameterized by } S\end{array}\right\}.$$

Theorem 2.1. i) *There exist a projective scheme* $\mathscr{M}:=\mathscr{M}^{(\underline{\kappa},\underline{\eta},\delta)\text{-(s)s}}_{\underline{P}/a/b/c/m}$ *and a natural transformation* $\vartheta\colon\underline{\mathrm{M}}^{(\underline{\kappa},\underline{\eta},\delta)\text{-(s)s}}_{\underline{P}/a/b/c/m}\longrightarrow h_{\mathscr{M}}$, *such that for any other scheme* $\mathscr{M}'$ *and any other natural transformation* $\vartheta'\colon\underline{\mathrm{M}}^{(\underline{\kappa},\underline{\eta},\delta)\text{-(s)s}}_{\underline{P}/a/b/c/m}\longrightarrow h_{\mathscr{M}'}$, *there is a unique morphism* $\zeta\colon\mathscr{M}\longrightarrow\mathscr{M}'$ *with* $\vartheta'=h_\zeta\circ\vartheta$.

ii) *The space* $\mathscr{M}$ *contains an open subscheme* $\mathscr{M}^{\mathrm{s}}$ *which is a coarse moduli scheme for the functor* $\underline{\mathrm{M}}^{(\underline{\kappa},\underline{\eta},\delta)\text{-(s)s}}_{\underline{P}/a/b/c/m}$.

Boundedness

We first have to prove that the semistable objects move in bounded families. For this, we write

$$\begin{aligned}\delta &= \frac{1}{(\dim(X)-1)!}\cdot\overline{\delta}\cdot x^{t+\dim(X)-1}+\text{lower order terms},\\ \kappa_v &= \sigma_v\cdot x^t+\tau_v\cdot x^{t-1}+\text{lower order terms}\end{aligned}$$

and

$$\lambda_v:=\frac{\deg(X)}{\dim(X)}\cdot\tau_v-\eta_v\cdot\overline{\delta},\quad v\in V,\quad \underline{\lambda}=(\lambda_v,v\in V).$$

Recall that the Hirzebruch–Riemann–Roch theorem ([9], Theorem 21.1.1) gives the formula

$$\begin{aligned}P(\mathscr{F}) &= \frac{\deg(X)}{\dim(X)!}\cdot\mathrm{rk}(\mathscr{F})\cdot x^{\dim(X)}+\\ &\quad+\frac{1}{(\dim(X)-1)!}\cdot\left(\deg(\mathscr{F})+\frac{c_1(X)\cdot c_1(\mathscr{O}_X(1))^{\dim(X)-1}}{2}\cdot\mathrm{rk}(\mathscr{F})\right)\\ &\quad+\text{lower order terms}\end{aligned}$$

for a coherent $\mathscr{O}_X$-module $\mathscr{F}$ on X. Of course, $\deg(X)=c_1(\mathscr{O}_X(1))^{\dim(X)}$ and $\deg(\mathscr{F})=c_1(\mathscr{F})\cdot c_1(\mathscr{O}_X(1))^{\dim(X)-1}$. With this formula, we see that, for a V-split sheaf

$\underline{\mathscr{E}}$ and a weighted filtration $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ of $\underline{\mathscr{E}}$, the coefficient of $x^{t+\dim(X)}$ in $M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ is zero and the one of $x^{t+\dim(X)-1}$ is

$$\frac{1}{(\dim(X)-1)!} \cdot L_{\underline{\sigma},\underline{\lambda}}(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$$

with

$$L_{\underline{\sigma},\underline{\lambda}}(\underline{\mathscr{E}}_\bullet, \alpha_\bullet) = \sum_{i=1}^{s} \alpha_i \cdot \big(\deg_{\underline{\sigma},\underline{\lambda}}(\underline{\mathscr{E}}) \cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_i) - \deg_{\underline{\sigma},\underline{\lambda}}(\underline{\mathscr{E}}_i) \cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}})\big) \tag{2.3}$$

and

$$\deg_{\underline{\sigma},\underline{\lambda}}(\underline{\mathscr{F}}) := \sum_{v\in V} \big(\sigma_v \cdot \deg(\mathscr{F}_v) + \lambda_v \cdot \mathrm{rk}(\mathscr{F}_v)\big), \quad \underline{\mathscr{F}} \text{ a } V\text{-split sheaf.}$$

Remark 2.2. *Observing the remark at the bottom of page 28 in [13] about the condition $\sum_{v\in V} \eta_v \cdot r_v = 0$, we see that the notions of slope semistability one gets from the concepts of Gieseker semistability in [1], [13] and this paper are all the same.*

Proposition 2.1. *The set of isomorphism classes of $\mathscr{O}_X$-modules $\mathscr{E}'$, such that there exist a $(\underline{\kappa}, \underline{\eta}, \delta)$-semistable decorated V-split sheaf $(\underline{\mathscr{E}}, \varphi)$ of type $(\underline{P}, a, b, c, m)$ and a vertex $v \in V$ with $\mathscr{E}' \cong \mathscr{E}_v$ is bounded.*

Proof. The stated condition on $\mathscr{E}'$ leaves only finitely many options for the Hilbert polynomial of $\mathscr{E}'$ and implies that $\mathscr{E}'$ is torsion free. By Maruyama's boundedness theorem ([10], Theorem 3.3.7), it suffices to bound the slope of saturated subsheaves of $\mathscr{E}'$.

So, let $(\underline{\mathscr{E}}, \varphi)$ be a $(\underline{\kappa}, \underline{\eta}, \delta)$-semistable decorated V-split sheaf of type $(\underline{P}, a, b, c, m)$, $v_0 \in V$ a vertex and $\mathscr{F} \subset \mathscr{E}_{v_0}$ a saturated subsheaf. We choose the weighted filtration $(\underline{\mathscr{E}}_\bullet, (1))$ with $\underline{\mathscr{E}}_1 = (\mathscr{F}_v, v \in V)$ the tuple with $\mathscr{F}_v = 0$ for $v \neq v_0$ and $\mathscr{F}_{v_0} = \mathscr{F}$. By (2.3), we have

$$L_{\underline{\sigma},\underline{\lambda}}(\underline{\mathscr{E}}_\bullet, \alpha_\bullet) = \deg_{\underline{\sigma},\underline{\lambda}}(\underline{\mathscr{E}}) \cdot \sigma_{v_0} \cdot \mathrm{rk}(\mathscr{F}) - \big(\sigma_{v_0} \cdot \deg(\mathscr{F}) + \lambda_{v_0} \cdot \mathrm{rk}(\mathscr{F})\big) \cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_v).$$

The semistability condition yields the estimate

$$\frac{\deg(\mathscr{F})}{\mathrm{rk}(\mathscr{F})} \le \frac{\deg_{\underline{\sigma},\underline{\lambda}}(\underline{\mathscr{E}})}{\mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_v)} + \frac{\lambda_{v_0}}{\sigma_{v_0}} + \frac{\overline{\delta}}{\sigma_{v_0} \cdot \mathrm{rk}(\mathscr{F}) \cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_v)} \cdot \mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \varphi).$$

Formula (2.2) gives

$$\mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \varphi) \le a \cdot (R-1).$$

Altogether, we obtain an upper bound on $\mu(\mathscr{F})$ which depends only on the input data. □

3 Proof of Theorem 2.1

Now, we basically take over the construction of [13]. We need to make some small changes in the construction of the parameter space and have to be careful how to modify the linearization parameters.

The parameter space

We let $\mathfrak{A}_v$, $v \in V$, be the union of those components of $\mathrm{Pic}(X)$ which contain line bundles of the form $\det(\mathscr{E}_v)$ for a semistable decorated V-split sheaf $(\underline{\mathscr{E}}, \varphi)$ of type $(\underline{P}, a, b, c, m)$. We also set $\mathfrak{A} := \mathsf{X}_{v\in V} \mathfrak{A}_v$. By the usual boundedness arguments, we can find an l_0, such that for all $l \geq l_0$, all semistable decorated V-split sheaves $(\underline{\mathscr{E}}, \varphi)$ of type $(\underline{P}, a, b, c, m)$, all $v \in V$, all $[\mathscr{L}] \in \mathfrak{A}_v$, and all $\mathscr{N} = \bigotimes_{v\in V} \mathscr{L}_v^{\otimes \sigma_v}$ with $[\mathscr{L}_v] \in \mathfrak{A}_v$, $v \in V$,

- $\mathscr{E}_v(l)$ is globally generated and $H^i(\mathscr{E}_v(l)) = 0$ for all $i > 0$,
- $\mathscr{L}(r_v \cdot l)$ is globally generated and $H^i(\mathscr{L}(r_v \cdot l)) = 0$ for all $i > 0$,
- $\mathscr{N}^{\otimes c}(a \cdot l)$ is globally generated and $H^i(\mathscr{N}^{\otimes c}(a \cdot l)) = 0$ for all $i > 0$.

We fix such an l, and set $p_v := P_v(l)$, $v \in V$, and $p := \sum_{v\in V} \kappa_v(l) \cdot p_v$. Moreover, we choose a vector space W_v of dimension p_v and let $\mathfrak{Q}_v^0$ be the quasi-projective quot scheme parameterizing quotients $q\colon W_v \otimes \mathscr{O}_X(-l) \longrightarrow \mathscr{F}$ with $\mathscr{F}$ a torsion free coherent $\mathscr{O}_X$-module with Hilbert polynomial P_v and $H^0(q(l))$ an isomorphism, $v \in V$. Let $\mathfrak{E}_v$ be the universal quotient on $\mathfrak{Q}_v^0 \times X$, $v \in V$. With the universal quotients, we construct sheaves on $(\mathsf{X}_{v\in V} \mathfrak{Q}_v^0) \times X$:

$$\mathfrak{E}^{\text{total}} := \bigoplus_{v\in V} \pi^\star_{\mathfrak{Q}_v^0 \times X}\big(\mathfrak{E}_v^{\oplus \sigma_v}\big) \quad \text{and} \quad \widetilde{\mathfrak{E}}^{\text{total}} := \bigoplus_{v\in V} \pi^\star_{\mathfrak{Q}_v^0 \times X}\big(\mathfrak{E}_v^{\oplus \kappa_v(l)}\big).$$

Define

$$M := \bigoplus_{v\in V} W_v^{\oplus \sigma_v} \quad \text{and} \quad \widetilde{M} := \bigoplus_{v\in V} W_v^{\oplus \kappa_v(l)}.$$

Next, we set

$$\mathfrak{P} := P\Bigg(\big((M^{\otimes a})^{\oplus b}\big)^\vee \otimes \pi_{(\mathsf{X}_{v\in V} \mathfrak{Q}_v^0)\star}\Big(\det(\mathfrak{E}^{\text{total}})^{\otimes c} \otimes \pi_X^\star\big(\mathscr{O}_X(a\cdot l)\big)\Big)\Bigg).$$

This is a projective bundle over $\mathsf{X}_{v\in V} \mathfrak{Q}_v^0$, and the parameter space $\mathfrak{M}$ is constructed in the usual way as a closed subscheme of $\mathfrak{P}$. In particular, it is projective over $\mathsf{X}_{v\in V} \mathfrak{Q}_v^0$. Furthermore, $\mathfrak{M}$ comes with an action of the reductive group $(\mathsf{X}_{v\in V} \mathrm{GL}(W_v))/\mathbb{C}^\star$, $\mathbb{C}^\star$ being diagonally embedded. For $l \gg 0$, we certainly have $\kappa_v(l) \geq \sigma_v$, $v \in V$, so that we may fix surjections

$$M_v^{\oplus \kappa_v(l)} \longrightarrow M_v^{\oplus \sigma_v},$$

yielding a surjection

$$\widetilde{M} \longrightarrow M$$

of $(\mathsf{X}_{v\in V} \mathrm{GL}(W_v))$-modules. Using this surjection, we may construct $\mathfrak{M}$ as a $((\mathsf{X}_{v\in V} \mathrm{GL}(W_v))/\mathbb{C}^\star)$-invariant subscheme of

$$\widetilde{\mathfrak{P}} := P\Bigg(\big((\widetilde{M}^{\otimes a})^{\oplus b}\big)^\vee \otimes \pi_{(\mathsf{X}_{v\in V} \mathfrak{Q}_v^0)\star}\Big(\det(\mathfrak{E}^{\text{total}})^{\otimes c} \otimes \pi_X^\star\big(\mathscr{O}_X(a\cdot l)\big)\Big)\Bigg).$$

We define

$$\begin{aligned}\widetilde{G} &:= \Big(\bigtimes_{v\in V} \mathrm{GL}(W_v)\Big) \cap \mathrm{SL}(\widetilde{M}) \\ &= \Big\{ (h_1,\dots,h_t) \in \bigtimes_{v\in V} \mathrm{GL}(W_v) \,\Big|\, \det(h_1)^{\kappa_{v_1}(l)}\cdot \dots \cdot \det(h_t)^{\kappa_{v_t}(l)} = 1 \Big\}.\end{aligned}$$

The group $\widetilde{G}$ maps with finite kernel onto $(\bigtimes_{v\in V}\mathrm{GL}(W_v))/\mathbb{C}^\star$, so that we may restrict our attention to the action of $\widetilde{G}$.

The linearization of the above group action will be induced via a Gieseker morphism to some other scheme. For this, we fix Poincaré line bundles $\mathscr{P}_v$ over $\mathfrak{A}_v \times X$, $v \in V$, and set

$$\mathfrak{G}_v := P\Bigg(\Big(\bigwedge^{r_v} W_v\Big)^\vee \otimes \pi_{\mathfrak{A}_v\star}\Big(\mathscr{P}_v \otimes \pi_X^\star\big(\mathscr{O}_X(r_v\cdot l)\big)\Big)\Bigg).$$

Choosing $\mathscr{P}_v$ appropriately, we may assume that $\mathscr{O}_{\mathfrak{G}_v}(1)$ is very ample for all $v \in V$. On $\mathfrak{A}\times X$, we get the line bundle

$$\mathscr{P} := \bigotimes_{v\in V} \pi^\star_{\mathfrak{A}_v\times X}\big(\mathscr{P}_v^{\otimes\sigma_v}\big).$$

Then, we define

$$\widetilde{\mathfrak{P}}' := P\Bigg(\big((\widetilde{M}^{\otimes a})^{\oplus b}\big)^\vee \otimes \pi_{\mathfrak{A}\star}\Big(\mathscr{P}^{\otimes c}\otimes \pi_X^\star\big(\mathscr{O}_X(a\cdot l)\big)\Big)\Bigg)$$

as a projective bundle over $\mathfrak{A}$. Again, $\mathscr{O}_{\widetilde{\mathfrak{P}}'}(1)$ can be assumed to be ample. We now have a $\widetilde{G}$-equivariant and injective morphism

$$\varGamma\colon \mathfrak{M} \longrightarrow \widetilde{\mathfrak{P}}' \times \bigtimes_{v\in V}\mathfrak{G}_v.$$

For given $\varrho \in \mathbb{Z}_{>0}$, and $\vartheta_v \in \mathbb{Z}_{>0}$, $v \in V$, there is a natural linearization of the $\widetilde{G}$-action on $\widetilde{\mathfrak{P}}' \times \bigtimes_{v\in V}\mathfrak{G}_v$ in the ample line bundle $\mathscr{O}(\varrho, \vartheta_v, v\in V)$. This may be altered by any character of $\bigtimes_{v\in V}\mathrm{GL}(W_v)$. Let $r := \sum_{v\in V}\kappa_v(l)\cdot r_v$, $d := \delta(l)$,

$$\widetilde{\chi}_v(l) := \chi_v(l) - \frac{p}{r}\cdot\kappa_v(l) + \frac{p}{R}\cdot\sigma_v + d\cdot\frac{a}{r}\cdot\kappa_v(l) - d\cdot\frac{a}{R}\cdot\sigma_v,$$

$$x_v := -\frac{\widetilde{\chi}_v(l)}{d}, \quad x'_v := \frac{r_v\cdot x_v}{p_v},$$

$$\varepsilon := \frac{p-a\cdot d}{r\cdot d}, \quad \varepsilon_v := \kappa_v(l) - \frac{x_v}{\varepsilon} = \kappa_v(l) + \frac{r\cdot\widetilde{\chi}_v(l)}{p-a\cdot d},$$

and

$$x''_v := \varepsilon\cdot\kappa_v(l)\cdot\left(\frac{r}{p} - \frac{r_v}{p_v}\right), \quad v\in V.$$

Remark 3.1. *i) The quantities ε and ε_v, $v \in V$, are functions in l. Since $p = P(l)$ is a positive polynomial of degree $t + \dim X$ and both δ and χ_v are polynomials of degree at most $t + \dim X - 1$, it is clear that ε will be positive for $l \gg 0$. Next we study*

$$r \cdot \widetilde{\chi}_v(l) = r \cdot \chi_v(l) - (p - a \cdot d) \cdot \kappa_v(l) + (p - a \cdot d) \cdot \frac{r}{R} \cdot \sigma_v. \tag{3.1}$$

The polynomial $r \cdot \chi_v(l)$ has degree at most $2t + \dim(X) - 1$, the leading coefficient of r/R is 1, so that the degree of $-(p - a \cdot d) \cdot \kappa_v(l) + (p - a \cdot d) \cdot (r/R) \cdot \sigma_v$ is at most $2t + \dim(X) - 1$. Thus, the polynomial in (3.1) has degree at most $2t + \dim(X) - 1$. Therefore, the expression

$$\frac{r \cdot \widetilde{\chi}_v(l)}{p - a \cdot d}$$

grows at most like a polynomial of degree $t - 1$. This means that ε_v will be positive for $l \gg 0$. So, the line bundle in which the action will be linearized will really be ample.

ii) Note that

$$\sum_{v \in V} p_v \cdot x'_v = \sum_{v \in V} r_v \cdot x_v = -\frac{1}{d} \cdot \left(\sum_{v \in V} r_v \cdot \chi_v(l) \right) - \frac{p - a \cdot d}{d \cdot r} \cdot r + \frac{p - a \cdot d}{d \cdot R} \cdot R = 0.$$

Now, we choose $\varrho \in \mathbb{Z}_{>0}$ and $\vartheta_v \in \mathbb{Z}_{>0}$, such that

$$\frac{\vartheta_v}{\varrho} = \varepsilon \cdot \varepsilon_v, \quad v \in V.$$

We modify the linearization of the $\widetilde{G}$-action on $\times_{v \in V} \mathfrak{G}_v$ in $\mathscr{O}(\vartheta_v, v \in V)$ by a character, such that $\mathbb{C}^{\star t} = \mathbb{C}^\star \cdot \mathrm{id}_{W_{v_1}} \times \cdots \times \mathbb{C}^\star \cdot \mathrm{id}_{W_{v_t}}$ acts via $(z_v, v \in V) \longmapsto \prod_{v \in V} z_v^{p_v \cdot e_v}$ with

$$e_v := \varrho \cdot (x'_v + x''_v), \quad v \in V.$$

Note that this character is just the restriction of the character

$$(m_1, ..., m_t) \longmapsto \det(m_1)^{e_{v_1}} \cdot \cdots \cdot \det(m_t)^{e_{v_t}}$$

of $\times_{v \in V} \mathrm{GL}(W_v)$ to the center $\mathscr{Z}$. We work with the resulting linearization of the $\widetilde{G}$-action on $\widetilde{\mathfrak{P}}' \times \times_{v \in V} \mathfrak{G}_v$ in $\mathscr{O}(\varrho, \vartheta_v, v \in V)$.

A weight formula

Let $(\mathscr{E}_v, v \in V)$ be a V-split sheaf, $l \in \mathbb{N}$, W_v a vector space of dimension $P(\mathscr{E}_v)(l)$ and $q_v \colon W_v \otimes \mathscr{O}_X \longrightarrow \mathscr{E}_v(l)$ a generically surjective homomorphism, $v \in V$. Suppose also that we are given a tuple $((\widehat{W}^v_\bullet, \underline{\gamma}^v), v \in V)$ of weighted filtrations of the W_v, $v \in V$,

$$\widehat{W}^v_\bullet : 0 \subsetneq \widehat{W}^v_1 \subsetneq \cdots \subsetneq \widehat{W}^v_{s_v} \subsetneq W_v,$$

$$\underline{\gamma}^v = (\gamma^v_1, ..., \gamma^v_{s_v+1}),$$

and filtrations

$$\widehat{\mathscr{E}}^v_\bullet : 0 \subseteq \widehat{\mathscr{E}}^v_1 \subseteq \cdots \subseteq \widehat{\mathscr{E}}^v_{s_v} \subseteq \mathscr{E}_v, \quad v \in V,$$

such that $q_v(\widehat{W}_v \otimes \mathscr{O}_X)$ generically generates $\widehat{\mathscr{E}}^v_i(l)$, $i = 1, ..., s_v$, $v \in V$.

Furthermore, we fix a tuple $(K_v, v \in V)$ of positive integers and set, for a V-split sheaf $(\mathscr{F}_v, v \in V)$,

$$\widetilde{\mathscr{F}}^{\mathrm{total}} := \bigoplus_{v \in V} \mathscr{F}_v^{\oplus K_v}.$$

Then, by the formalism described in [13], Section 3.2, we get a weighted filtration $(\underline{W}_\bullet, \gamma_\bullet)$, $\gamma_\bullet = (\gamma_1, ..., \gamma_s)$, of the V-split vector space $(W_v, v \in V)$ which, in turn, gives a filtration

$$\widetilde{W}^{\mathrm{total}}_\bullet : 0 \subsetneq \widetilde{W}^{\mathrm{total}}_1 \subsetneq \cdots \subsetneq \widetilde{W}^{\mathrm{total}}_s \subseteq \widetilde{W}^{\mathrm{total}}.$$

Suppose finally that we are given a filtration

$$\widetilde{\mathscr{E}}^{\mathrm{total}}_\bullet : 0 \subseteq \widetilde{\mathscr{E}}^{\mathrm{total}}_1 \subseteq \cdots \subseteq \widetilde{\mathscr{E}}^{\mathrm{total}}_s \subseteq \widetilde{\mathscr{E}}^{\mathrm{total}}$$

in which some of the inclusions may be equaltities, such that the image of $\widetilde{W}^{\mathrm{total}}_i \otimes \mathscr{O}_X$ under the homomorphism $\widetilde{W}^{\mathrm{total}} \otimes \mathscr{O}_X \longrightarrow \widetilde{\mathscr{E}}^{\mathrm{total}}(l)$ generically agrees with $\widetilde{\mathscr{E}}^{\mathrm{total}}_i(l)$, $i = 1, ..., s+1$.[2]

Proposition 3.1. *In the situation explained above, the following identity holds true*

$$\begin{aligned}
&\sum_{i=1}^{s} \frac{\gamma_{i+1} - \gamma_i}{P(\widetilde{\mathscr{E}}^{\mathrm{total}})(l)} \cdot \left(P(\widetilde{\mathscr{E}}^{\mathrm{total}})(l) \cdot \mathrm{rk}(\widetilde{\mathscr{E}}^{\mathrm{total}}_i) - \mathrm{rk}(\widetilde{\mathscr{E}}^{\mathrm{total}}) \cdot \left(\sum_{v \in V} K_v \cdot \dim_{\mathbb{C}}(\widehat{W}^v_i) \right) \right) \\
&\quad = \sum_{v \in V} K_v \cdot \left(\sum_{i=1}^{s_v} \frac{\gamma^v_{i+1} - \gamma^v_i}{P(\mathscr{E}_v)(l)} \cdot \left(P(\mathscr{E}_v)(l) \cdot \mathrm{rk}(\widehat{\mathscr{E}}^v_i) - \mathrm{rk}(\mathscr{E}_v) \cdot \dim_{\mathbb{C}}(\widehat{W}^v_i) \right) \right) \\
&\qquad - \sum_{v \in V} K_v \cdot \left(\frac{\mathrm{rk}(\mathscr{E}_v)}{P(\mathscr{E}_v)(l)} - \frac{\mathrm{rk}(\widetilde{\mathscr{E}}^{\mathrm{total}})}{P(\widetilde{\mathscr{E}}^{\mathrm{total}})(l)} \right) \cdot \left(\sum_{i=1}^{s_v+1} \gamma^v_i \cdot \left(\dim_{\mathbb{C}}(\widehat{W}^v_i) - \dim_{\mathbb{C}}(\widehat{W}^v_{i-1}) \right) \right).
\end{aligned}$$

Proof. By our assumption, the maps

$$\begin{aligned}
q_v \colon &\quad W_v \otimes \mathscr{O}_X \longrightarrow \mathscr{E}_v(l), \\
q_{v|\widehat{W}^v_i \otimes \mathscr{O}_X} \colon &\quad \widehat{W}^v_i \otimes \mathscr{O}_X \longrightarrow \widehat{\mathscr{E}}^v_i(l)
\end{aligned}$$

are generically surjective, $i = 1, ..., s_v$, $v \in V$. Thus, we restrict them to a general point of X and apply [13], Proposition 3.2.2. □

[2] This may constructed as follows: Given $i \in \{ 1, ..., s+1 \}$, set

$$s_v(i) := \max\{ j \in \{ 1, ..., s_v + 1 \} \mid \gamma^v_j \leq \gamma_i \}, \quad v \in V,$$

and

$$\widetilde{\mathscr{E}}^{\mathrm{total}}_i := \bigoplus_{v \in V} \bigoplus_{j=1}^{s_v(i)} \mathscr{E}_j^{v, \oplus K_v}.$$

GIT-Semistability implies semistability

Let $m=(q_v\colon W_v\otimes \mathscr{O}_X(-l)\longrightarrow \mathscr{E}_v, v\in V,\varphi)$ be a point in the parameter space $\mathfrak{M}$, such that $\Gamma(m)$ is (semi)stable with respect to the chosen linearization in $\mathscr{O}(\varrho,\vartheta_v,v\in V)$. We look at a weighted filtration $(\underline{\mathscr{E}}_\bullet,\alpha_\bullet)$ of $(\mathscr{E}_v,v\in V)$, such that

$$\mathscr{E}^v_i(l) \text{ is generically generated by } H^0\big(\mathscr{E}^v_i(l)\big),\quad i=1,...,s_v,\ v\in V.$$

Define $\underline{\gamma}=(\gamma_1,...,\gamma_{s+1})$ by the conditions

$$\frac{\gamma_{i+1}-\gamma_i}{p}=\alpha_i,\quad i=1,...,s,$$

and, setting $\widetilde{\mathscr{E}}^{\text{total}}_i:=\bigoplus_{v\in V}\mathscr{E}_i^{v,\oplus\kappa_v(l)}$, $i=1,...,s+1$,

$$\sum_{i=1}^{s+1}\gamma_i\cdot\Big(h^0\big(\widetilde{\mathscr{E}}^{\text{total}}_i(l)\big)-h^0\big(\widetilde{\mathscr{E}}^{\text{total}}_{i-1}(l)\big)\Big)=0.$$

Then, we obtain a weighted filtration $(\underline{\mathscr{E}}_\bullet,\gamma_\bullet)$ and, thus, weighted filtrations $(\widehat{\mathscr{E}}^v_\bullet,\gamma^v_\bullet)$ of the $\mathscr{E}_v$, $v\in V$ (see [13], Section 3.2). Next, we choose bases $\underline{w}^v=(w^v_1,...,w^v_{p_v})$ of the W_v with

$$\langle\, w^v_1,...,w^v_{h^0(\widehat{\mathscr{E}}^v_i(l))}\,\rangle=H^0\big(\widehat{\mathscr{E}}^v_i(l)\big),\quad i=1,...,s_v,\ v\in V,$$

and set

$$\widetilde{\underline{\gamma}}^v:=\big(\underbrace{\gamma^v_1,...,\gamma^v_1}_{h^0(\widehat{\mathscr{E}}^v_1(l))\times},...,\underbrace{\gamma^v_{s_v+1},...,\gamma^v_{s_v+1}}_{(p_v-h^0(\widehat{\mathscr{E}}^v_{s_v}(l)))\times}\big).$$

This yields the one parameter subgroup

$$\lambda:=\big(\lambda(\underline{w}^{v_1},\widetilde{\underline{\gamma}}^{v_1}),...,\lambda(\underline{w}^{v_t},\widetilde{\underline{\gamma}}^{v_t})\big)$$

of $\widetilde{G}$. Now, with $\Gamma(m)=([m'],[m_v],v\in V)$,

$$\begin{aligned}\frac{\mu\big(\lambda,\Gamma(m)\big)}{\varrho}&=\mu\big(\lambda,[m']\big)+\varepsilon\cdot A+B,\\ A&:=\sum_{v\in V}\varepsilon_v\cdot\mu\big(\lambda,[m_v]\big)-\\ &\quad-\sum_{v\in V}\kappa_v(l)\cdot\Big(\frac{r_v}{p_v}-\frac{r}{p}\Big)\cdot\Big(\sum_{i=1}^{s_v+1}\gamma^v_i\cdot\Big(h^0\big(\widehat{\mathscr{E}}^v_i(l)\big)-h^0\big(\widehat{\mathscr{E}}^v_{i-1}(l)\big)\Big)\Big),\\ B&:=\sum_{v\in V}\Big(x'_v\cdot\sum_{i=1}^{s_v+1}\gamma^v_i\cdot\Big(h^0\big(\widehat{\mathscr{E}}^v_i(l)\big)-h^0\big(\widehat{\mathscr{E}}^v_{i-1}(l)\big)\Big)\Big).\end{aligned}$$

Observe

$$\sum_{i=1}^{s_v+1}\gamma^v_i\cdot\Big(h^0\big(\widehat{\mathscr{E}}^v_i(l)\big)-h^0\big(\widehat{\mathscr{E}}^v_{i-1}(l)\big)\Big)=\gamma^v_{s_v+1}\cdot p_v-\sum_{i=1}^{s_v}\big(\gamma^v_{i+1}-\gamma^v_i\big)\cdot h^0\big(\widehat{\mathscr{E}}^v_i(l)\big).$$

Next,

$$\mu\big(\lambda,[m_v]\big) \;=\; \sum_{i=1}^{s_v} \frac{\gamma^v_{i+1}-\gamma^v_i}{p_v}\cdot\Big(p_v\cdot \mathrm{rk}(\widehat{\mathscr{E}}^v_i)-h^0\big(\widehat{\mathscr{E}}^v_i(l)\big)\cdot r_v\Big).$$

Thus, for $v\in V$,

$$\begin{aligned}
&\varepsilon\cdot\varepsilon_v\cdot\mu\big(\lambda,[m_v]\big)-x'_v\cdot\sum_{i=1}^{s_v}\big(\gamma^v_{i+1}-\gamma^v_i\big)\cdot h^0\big(\widehat{\mathscr{E}}^v_i(l)\big)+x'_v\cdot\gamma^v_{s_v+1}\cdot p_v\\
=\;&\sum_{i=1}^{s_v}\frac{\gamma^v_{i+1}-\gamma^v_i}{p_v}\cdot\bigg(\varepsilon\cdot\varepsilon_v\cdot\Big(p_v\cdot\mathrm{rk}(\widehat{\mathscr{E}}^v_i)-h^0\big(\widehat{\mathscr{E}}^v_i(l)\big)\cdot r_v\Big)-x'_v\cdot p_v\cdot h^0\big(\widehat{\mathscr{E}}^v_i(l)\big)\bigg)\\
&+x'_v\cdot\gamma^v_{s_v+1}\cdot p_v\\
=\;&\sum_{i=1}^{s_v}\frac{\gamma^v_{i+1}-\gamma^v_i}{p_v}\cdot\bigg(\varepsilon\cdot\varepsilon_v\cdot\Big(p_v\cdot\mathrm{rk}(\widehat{\mathscr{E}}^v_i)-h^0\big(\widehat{\mathscr{E}}^v_i(l)\big)\cdot r_v\Big)-x_v\cdot r_v\cdot h^0\big(\widehat{\mathscr{E}}^v_i(l)\big)\bigg)\\
&+x_v\cdot\gamma^v_{s_v+1}\cdot r_v\\
=\;&\sum_{i=1}^{s_v}\frac{\gamma^v_{i+1}-\gamma^v_i}{p_v}\cdot\bigg(\varepsilon\cdot\kappa_v(l)\cdot\Big(p_v\cdot\mathrm{rk}(\widehat{\mathscr{E}}^v_i)-h^0\big(\widehat{\mathscr{E}}^v_i(l)\big)\cdot r_v\Big)-x_v\cdot p_v\cdot\mathrm{rk}(\widehat{\mathscr{E}}^v_i)\bigg)\\
&+x_v\cdot\gamma^v_{s_v+1}\cdot r_v\\
=\;&\varepsilon\cdot\kappa_v(l)\cdot\sum_{i=1}^{s_v}\bigg(\frac{\gamma^v_{i+1}-\gamma^v_i}{p_v}\cdot\Big(p_v\cdot\mathrm{rk}(\widehat{\mathscr{E}}^v_i)-h^0\big(\widehat{\mathscr{E}}^v_i(l)\big)\cdot r_v\Big)\bigg)-\\
&-\sum_{i=1}^{s_v}\Big(x_v\cdot(\gamma^v_{i+1}-\gamma^v_i)\cdot\mathrm{rk}(\widehat{\mathscr{E}}^v_i)\Big)+x_v\cdot\gamma^v_{s_v+1}\cdot r_v\\
=\;&\varepsilon\cdot\kappa_v(l)\cdot\sum_{i=1}^{s_v}\bigg(\frac{\gamma^v_{i+1}-\gamma^v_i}{p_v}\cdot\Big(p_v\cdot\mathrm{rk}(\widehat{\mathscr{E}}^v_i)-h^0\big(\widehat{\mathscr{E}}^v_i(l)\big)\cdot r_v\Big)\bigg)+\\
&+\sum_{i=1}^{s_v+1}x_v\cdot\gamma^v_i\cdot\big(\mathrm{rk}(\widehat{\mathscr{E}}^v_i)-\mathrm{rk}(\widehat{\mathscr{E}}^v_{i-1})\big).
\end{aligned}$$

By definition,

$$\sum_{v\in V}x_v\cdot\bigg(\sum_{i=1}^{s_v+1}\gamma^v_i\cdot\big(\mathrm{rk}(\widehat{\mathscr{E}}^v_i)-\mathrm{rk}(\widehat{\mathscr{E}}^v_i)\big)\bigg)=\sum_{i=1}^{s+1}\gamma_i\cdot\Bigg(\bigg(\sum_{v\in V}x_v\cdot\mathrm{rk}(\mathscr{E}^v_i)\bigg)-\bigg(\sum_{v\in V}x_v\cdot\mathrm{rk}(\mathscr{E}^v_{i-1})\bigg)\Bigg).$$

Since $\mathrm{rk}(\mathscr{E}^v_{s+1})=r_v$, $v\in V$, and $\sum_{v\in V}x_v\cdot r_v=0$, by Remark 3.1, ii), we may rewrite this quantity as

$$-\sum_{i=1}^{s}(\gamma_{i+1}-\gamma_i)\cdot\bigg(\sum_{v\in V}x_v\cdot\mathrm{rk}(\mathscr{E}^v_i)\bigg)=-p\cdot\sum_{i=1}^{s}\alpha_i\cdot\bigg(\sum_{v\in V}x_v\cdot\mathrm{rk}(\mathscr{E}^v_i)\bigg).$$

Using Proposition 3.1, with $K_v = \kappa_v(l)$, $v \in V$, we discover that $\varepsilon \cdot A + B$ equals

$$
\begin{aligned}
& \varepsilon \cdot \sum_{i=1}^{s} \alpha_i \cdot \Big(p \cdot \mathrm{rk}(\widetilde{\mathscr{E}}_i^{\mathrm{total}}) - h^0\big(\widetilde{\mathscr{E}}_i^{\mathrm{total}}(l)\big) \cdot r\Big) - p \cdot \sum_{i=1}^{s} \alpha_i \cdot \Big(\sum_{v \in V} x_v \cdot \mathrm{rk}(\mathscr{E}_i^v)\Big) \\
= & \sum_{i=1}^{s} \alpha_i \cdot \Bigg(\frac{p^2 \cdot \mathrm{rk}(\widetilde{\mathscr{E}}_i^{\mathrm{total}})}{r \cdot d} - \frac{p \cdot a \cdot \mathrm{rk}(\widetilde{\mathscr{E}}_i^{\mathrm{total}})}{r} - \frac{p \cdot h^0\big(\widetilde{\mathscr{E}}_i^{\mathrm{total}}(l)\big)}{d} + \\
& + a \cdot h^0\big(\widetilde{\mathscr{E}}_i^{\mathrm{total}}(l)\big)\Bigg) - p \cdot \sum_{i=1}^{s} \alpha_i \cdot \Big(\sum_{v \in V} x_v \cdot \mathrm{rk}(\mathscr{E}_i^v)\Big).
\end{aligned} \tag{3.2}
$$

Theorem 3.1. *The set of isomorphism classes of torsion free sheaves $\mathscr{E}'$, such that there exist an $l \in \mathbb{N}$, a point $m = (q_v \colon W_v \otimes \mathscr{O}_X(-l) \longrightarrow \mathscr{E}_v, v \in V, \varphi)$ in the parameter space $\mathfrak{M}$,[3] such that $\Gamma(m)$ is semistable with respect to the chosen linearization in $\mathscr{O}(\varrho, \vartheta_v, v \in V)$, and an index $v_0 \in V$ with $\mathscr{E}' \cong \mathscr{E}_{v_0}$ is bounded.*

Proof. The proof is similar to the one of Proposition 2.3.5.12 in [14]. Pick a vertex v_0, let $\mathscr{F}$ be a saturated subsheaf of $\mathscr{E}_{v_0}$ and $\widetilde{\mathscr{F}}$ the saturated subsheaf of $\mathscr{E}_{v_0}$ that is generically generated by $H^0(\mathscr{F}(l))$. Note that

$$
H^0\big(\widetilde{\mathscr{F}}(l)\big) = H^0\big(\mathscr{F}(l)\big). \tag{3.3}
$$

We define the weighted filtration $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ with $s = 1$, $\alpha_\bullet = (1)$ and

$$
\mathscr{E}_1^v := \begin{cases} \mathscr{E}_v, & \text{if } v \neq v_0 \\ \widetilde{\mathscr{F}}, & \text{if } v = v_0 \end{cases}.
$$

We also set

$$
\mathscr{F}^{\mathrm{total}} = \mathscr{F}^{\oplus \kappa_{v_0}(l)} \oplus \bigoplus_{v \neq v_0} \mathscr{E}_v^{\oplus \kappa_v(l)}
$$

and find the exact sequence

$$
0 \longrightarrow H^0\big(\mathscr{F}^{\mathrm{total}}(l)\big) \longrightarrow H^0\big(\widetilde{\mathscr{E}}^{\mathrm{total}}(l)\big) \longrightarrow H^0\big(\mathscr{Q}(l)\big)^{\oplus \kappa_{v_0}(l)} \tag{3.4}
$$

for the sheaf $\mathscr{Q} := \mathscr{E}_{v_0}/\mathscr{F}$. We estimate

$$
\mu\big(\lambda, [m']\big) \leq a \cdot \Big(p - h^0\big(\widetilde{\mathscr{E}}_1^{\mathrm{total}}(l)\big)\Big).
$$

Using this, equation (3.2),

$$
C := \sum_{v \in V} \widetilde{\chi}(l) \cdot \mathrm{rk}(\mathscr{E}_1^v)
$$

and semistability of $\Gamma(m)$, we find the inequality

$$
0 \leq \frac{p^2 \cdot \mathrm{rk}(\widetilde{\mathscr{E}}_1^{\mathrm{total}})}{r \cdot d} - \frac{p \cdot a \cdot \mathrm{rk}(\widetilde{\mathscr{E}}_1^{\mathrm{total}})}{r} - \frac{p \cdot h^0\big(\widetilde{\mathscr{E}}_1^{\mathrm{total}}(l)\big)}{d} + a \cdot p - p \cdot \frac{C}{d}.
$$

[3] Which depends on l!

For large l, we may assume that $p \geq a \cdot d$. With $\mathrm{rk}(\widetilde{\mathscr{E}}_1^{\mathrm{total}}) \leq \mathrm{rk}(\mathscr{F}^{\mathrm{total}})$, this yields the estimate

$$0 \leq \frac{p^2 \cdot \mathrm{rk}(\mathscr{F}^{\mathrm{total}})}{r \cdot d} - \frac{p \cdot a \cdot \mathrm{rk}(\mathscr{F}^{\mathrm{total}})}{r} - \frac{p \cdot h^0\big(\mathscr{F}^{\mathrm{total}}(l)\big)}{d} + a \cdot p - p \cdot \frac{C}{d}.$$

After multiplication by $r \cdot d/p$, this inequality transforms into

$$p \cdot \mathrm{rk}(\mathscr{F}^{\mathrm{total}}) - r \cdot h^0\big(\mathscr{F}^{\mathrm{total}}(l)\big) + d \cdot a \cdot \big(r - \mathrm{rk}(\mathscr{F}^{\mathrm{total}})\big) - r \cdot C \geq 0.$$

By (3.4), we have $h^0(\mathscr{F}^{\mathrm{total}}(l)) \geq p - \kappa_{v_0}(l) \cdot h^0(\mathscr{Q}(l))$, so that

$$r \cdot \kappa_{v_0}(l) \cdot h^0\big(\mathscr{Q}(l)\big) - p \cdot \kappa_{v_0}(l) \cdot \mathrm{rk}(\mathscr{Q}) + d \cdot a \cdot \kappa_{v_0}(l) \cdot \mathrm{rk}(\mathscr{Q}) - r \cdot C \geq 0$$

and finally

$$\frac{h^0\big(\mathscr{Q}(l)\big)}{\mathrm{rk}(\mathscr{Q})} \geq \frac{p}{r} - \frac{a \cdot d}{r} + \frac{C}{\kappa_{v_0}(l) \cdot \mathrm{rk}(\mathscr{Q})}.$$

The right hand is a rational function of degree[4] $\dim(X)$ in l which takes positive values for $l \gg 0$. Asymptotically, it looks like

$$\begin{aligned} &\frac{\deg(X)}{\dim(X)!} \cdot l^{\dim(X)} + \\ +\ &\frac{l^{\dim(X)-1}}{(\dim(X)-1)!} \cdot \left(\frac{\deg_{\underline{\sigma},\underline{\lambda}}(\mathscr{E})}{R} + l - \frac{a \cdot \overline{\delta}}{R} + \frac{\sum_{v \in V} \eta_v \cdot \overline{\delta} \cdot \mathrm{rk}(\mathscr{E}_1^v)}{\sigma_{v_0} \cdot \mathrm{rk}(\mathscr{Q})} + \frac{\deg(T_X)}{2} \right) + \\ +\ &\text{lower order terms.} \end{aligned}$$

We may easily find a constant C' which depends only on the input data with

$$\frac{\sum_{v \in V} \eta_v \cdot \overline{\delta} \cdot \mathrm{rk}(\mathscr{E}_1^v)}{\sigma_{v_0} \cdot \mathrm{rk}(\mathscr{Q})} \geq C'.$$

Now, we assume that $\mathscr{Q}$ is the minimal destabilizing quotient of $\mathscr{E}_{v_0}$. This is a semistable sheaf. Then, we may apply the LePotier–Simpson estimate ([10], Corollary 3.3.3) to it:

$$\frac{\deg(X)}{\dim(X)!} \cdot \left[\frac{\mu(\mathscr{Q})}{\deg(X)} + l + \frac{\mathrm{rk}(\mathscr{Q}) - 1}{2} \right]_+^{\dim(X)} \geq \frac{h^0\big(\mathscr{Q}(l)\big)}{\mathrm{rk}(\mathscr{Q})}.$$

We see that for large l, a lower estimate for $\mu_{\min}(\mathscr{E}_{v_0}) = \mu(\mathscr{Q})$ which depends only on the input data must be satisfied. Therefore, $\mathscr{E}_{v_0}$ belongs to some bounded family. □

In order to show that a GIT-semistable point in the parameter space corresponds to a semistable V-split sheaf, we have to reformulate semistability. This will be done in the following lemmas and Theorem 3.2.

[4] i.e., degree of the numerator minus degree of the denominator

Lemma 3.1. *Let $\mathfrak{S}_v$ be a bounded family of torsion free sheaves with Hilbert polynomial P_v, $v \in V$. There, there is a constant $A \in \mathbb{N}$, such that a decorated V-split sheaf $(\underline{\mathscr{E}}, \varphi)$ of type $(\underline{P}, a, b, c, m)$ with $[\mathscr{E}_v] \in \mathfrak{S}_v$, $v \in V$, is $(\underline{\kappa}, \underline{\eta}, \delta)$-(semi)stable, if and only if it satisfies the (semi)stability condition for all weighted filtrations $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ in which $\underline{\alpha} = (\alpha_1, ..., \alpha_s)$ is a vector of* ***integers****, such that $\alpha_i \le A$, $i = 1, ..., s$.*

Proof. We refer to [12], Theorem 3.3, [6], Lemma 3.4.4, or [14], page 153. □

Lemma 3.2. *Let $(\underline{\mathscr{E}}, \varphi)$ be a decorated V-split sheaf $(\underline{\mathscr{E}}, \varphi)$ of type $(\underline{P}, a, b, c, m)$ and $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ a weighted filtration of $\underline{\mathscr{E}}$. Write*

$$\{\, 1, ..., s \,\} = \{\, i_1, ..., i_{s_1} \,\} \sqcup \{\, j_1, ..., j_{s_2} \,\}$$

and define the weighted filtrations $(\underline{\mathscr{E}}^1_\bullet, \alpha^1_\bullet)$ and $(\underline{\mathscr{E}}^2_\bullet, \alpha^2_\bullet)$ by

$$\underline{\mathscr{E}}^1_\bullet = (\underline{\mathscr{E}}_{i_1}, ..., \underline{\mathscr{E}}_{i_{s_1}}),\ \ \alpha^1_\bullet = (\alpha_{i_1}, ..., \alpha_{i_{s_1}})$$

and

$$\underline{\mathscr{E}}^2_\bullet = (\underline{\mathscr{E}}_{j_1}, ..., \underline{\mathscr{E}}_{j_{s_2}}),\ \ \alpha^2_\bullet = (\alpha_{j_1}, ..., \alpha_{j_{s_2}}).$$

Then, one has

$$M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}_\bullet, \alpha_\bullet) = M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}^1_\bullet, \alpha^1_\bullet) + M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}^2_\bullet, \alpha^2_\bullet)$$

and

$$\mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \varphi) \ge \mu(\underline{\mathscr{E}}^1_\bullet, \alpha^1_\bullet, \varphi) - a \cdot (R-1) \cdot \sum_{g=1}^{s_2} \alpha_{j_g}.$$

Proof. The equality is immediate from the definitions, the inequality follows as Lemma 1.8, ii), in [12] or Lemma 3.4.5 in [6]. □

Lemma 3.3. *Let $\mathfrak{S}_v$ be a bounded family of torsion free sheaves with Hilbert polynomial P_v, $v \in V$, A, $C > 0$ constants and $(\underline{\mathscr{E}}, \varphi)$ a decorated V-split sheaf of type $(\underline{P}, a, b, c, m)$ with $[\mathscr{E}_v] \in \mathfrak{S}_v$, $v \in V$. Then, there is a constant C', such that the following holds for every weighted filtration $(\underline{\mathscr{E}}, \alpha_\bullet)$: If $\alpha_\bullet = (\alpha_1, ..., \alpha_s)$ consists of positive* ***integers*** *with $\alpha_i \le A$, $i = 1, ..., s$, and if there are an index $i_0 \in \{\, 1, ..., s \,\}$ and a vertex $v_0 \in V$, such that*

$$\mu(\mathscr{E}^{v_0}_{i_0}) < C',$$

then

$$M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}_\bullet, \alpha_\bullet) > C \cdot x^{t+\dim(X)-1}.$$

Proof. This is trivial, because the boundedness assumption implies that there is a constant C'' which depends only on $\underline{P}$, such that

$$\mu(\mathscr{F}) \le C''$$

for every vertex $v \in V$ and every saturated subsheaf $\mathscr{F} \subset \mathscr{E}_v$, and there are only finitely many possibilities for $\alpha_\bullet$. □

Theorem 3.2. *Let $\mathfrak{S}_v$ be a bounded family of torsion free sheaves with Hilbert polynomial P_v, $v \in V$. Then, there is a natural number l_0, such that for every $l \geq l_0$ and every decorated V-split sheaf $(\underline{\mathscr{E}}, \varphi)$ of type $(\underline{P}, a, b, c, m)$ with $[\mathscr{E}_v] \in \mathfrak{S}_v$, $v \in V$, the following holds true: If*

$$M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)(l) + \delta(l) \cdot \mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \varphi)(\geq)0 \tag{3.5}$$

holds for every weighted filtration $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ of $\underline{\mathscr{E}}$, such that

$$\mathscr{E}_i^v(l) \text{ is globally generated and } h^j\big(\mathscr{E}_i^v(l)\big) = 0,\ j > 0,\ i = 1, ..., s,\ v \in V,$$

then $(\underline{\mathscr{E}}, \varphi)$ is $(\underline{\kappa}, \underline{\eta}, \delta)$-(semi)stable.

Proof. We first invoke Lemma 3.1. With the constant A from that lemma, we define

$$C := a \cdot A \cdot (R-1)^2 \cdot \frac{\overline{\delta} + 1}{(\dim(X) - 1)!}.$$

Next, we apply Lemma 3.3. It gives a certain constant C'. We introduce two sets $\mathfrak{S}'$ and $\mathfrak{S}''$ of isomorphy classes of sheaves on X: The isomorphy class $[\mathscr{F}]$ of a coherent sheaf $\mathscr{F}$ on X belongs to $\mathfrak{S}'$, if and only if $\mu(\mathscr{F}) < C'$, and to $\mathfrak{S}''$, if and only if $\mu(\mathscr{F}) \geq C'$ and there are an index $v \in V$ and a torsion free coherent $\mathscr{O}_X$-module $\mathscr{E}$ with $[\mathscr{E}] \in \mathfrak{S}_v$, such that $\mathscr{F}$ is isomorphic to a **saturated** subsheaf of $\mathscr{E}$. By [10], Lemma 1.7.9, the set $\mathfrak{S}''$ is bounded.

Let $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ be a weighted filtration of $\underline{\mathscr{E}}$, such that $\alpha_\bullet = (\alpha_1, ..., \alpha_s)$ is a vector of integers with $\alpha_i \leq A$, $i = 1, ..., s$. Define

$$\{\, i_1, ..., i_{s_1} \,\} := \Big\{\, i \in \{\, 1, ..., s \,\} \,\Big|\, \forall v \in V:\ [\mathscr{E}_i^v] \in \mathfrak{S}'' \,\Big\}$$

and

$$\{\, j_1, ..., j_{s_2} \,\} := \Big\{\, i \in \{\, 1, ..., s \,\} \,\Big|\, \exists v \in V:\ [\mathscr{E}_i^v] \in \mathfrak{S}' \,\Big\}.$$

Using Lemma 3.2 and 3.3, we find

$$\begin{aligned}
&M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}_\bullet, \alpha_\bullet) + \delta \cdot \mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \varphi) \\
&\geq M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}^1_\bullet, \alpha^1_\bullet) + \delta \cdot \mu(\underline{\mathscr{E}}^1_\bullet, \alpha^1_\bullet, \varphi) + M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}^2_\bullet, \alpha^2_\bullet) - \delta \cdot a \cdot (R-1) \cdot \sum_{g=1}^{s_2} \alpha_{j_g} \\
&\geq M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}^1_\bullet, \alpha^1_\bullet) + \delta \cdot \mu(\underline{\mathscr{E}}^1_\bullet, \alpha^1_\bullet, \varphi) + M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}^2_\bullet, \alpha^2_\bullet) - C \cdot x^{t+\dim(X)-1} \\
&\geq M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}^1_\bullet, \alpha^1_\bullet) + \delta \cdot \mu(\underline{\mathscr{E}}^1_\bullet, \alpha^1_\bullet, \varphi).
\end{aligned}$$

We see that it is enough to check the (semi)stability condition for weighted filtrations $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$, such that $\alpha_\bullet = (\alpha_1, ..., \alpha_s)$ is a vector of integers with $\alpha_i \leq A$, $i = 1, ..., s$, and $[\mathscr{E}_v^i] \in \mathfrak{S}''$, $v \in V$, $i = 1, ..., s$. Since there are only finitely many options for $\alpha_\bullet$, there are only finitely many possible values of $\mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \varphi)$. In addition, the fact that $\mathfrak{S}''$ is a bounded family implies that there only finitely many possible polynomials of the form

$$M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}_\bullet, \alpha_\bullet) + \delta \cdot \mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \varphi)$$

for a weighted filtration $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ with the stated properties. So, there is a natural number l_0, such that, for $l \geq l_0$, a polynomial of the above finite list is positive or non-negative in the lexicographic ordering of polynomials, if and only if its value at l is positive or non-negative, respectively. □

Theorem 3.3. *There exists an $l_0 \in \mathbb{N}$, such that the following holds true: If $l \geq l_0$ and $m = (q_v \colon W_v \otimes \mathscr{O}_X(-l) \longrightarrow \mathscr{E}_v, v \in V, \varphi)$ is a point in the parameter space $\mathfrak{M}$, such that $\Gamma(m)$ is (semi)stable with respect to the chosen linearization in $\mathscr{O}(\varrho, \vartheta_v, v \in V)$, then $(\underline{\mathscr{E}}, \varphi)$ is $(\underline{\kappa}, \underline{\eta}, \delta)$-(semi)stable.*

Proof. We apply the criterion of Theorem 3.2. Let $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ be a weighted filtration of $\underline{\mathscr{E}}$, such that

$$\mathscr{E}_i^v(l) \text{ is globally generated and } h^j\big(\mathscr{E}_i^v(l)\big) = 0,\ j > 0,\ i = 1, ..., s,\ v \in V.$$

We continue the computations started before Theorem 3.1. In order to conclude, we have to compute $\mu(\lambda, [m'])$. Under the identification of $\widetilde{M}$ with the space $H^0(\widetilde{\mathscr{E}}^{\text{total}}(l))$, we define

$$\operatorname{gr}_i(\widetilde{M}) := H^0\Big(\big(\widetilde{\mathscr{E}}_i^{\text{total}}/\widetilde{\mathscr{E}}_{i-1}^{\text{total}}\big)(l)\Big), \quad i = 1, ..., s+1.$$

The basis $\underline{m}$ of $\widetilde{M}$ induced by the bases $\underline{w}^v$ for the W_v, $v \in V$, yields a natural isomorphism

$$\widetilde{M} \cong \bigoplus_{i=1}^{s+1} \operatorname{gr}_i(\widetilde{M}).$$

For an index tuple $\underline{\iota} \in J^a := \{1, ..., s+1\}^{\times a}$, we define $\widetilde{M}_{\underline{\iota}} := \operatorname{gr}_{\iota_1}(\widetilde{M}) \otimes \cdots \otimes \operatorname{gr}_{\iota_a}(\widetilde{M})$, and for $k \in \{1, ..., b\}$, we let $\widetilde{M}_{\underline{\iota}}^k$ be $\widetilde{M}_{\underline{\iota}}$ embedded into the k-th copy of $\widetilde{M}^{\otimes a}$ in $(\widetilde{M}^{\otimes a})^{\oplus b}$. If we denote $P(\widetilde{\mathscr{E}}_i^{\text{total}}(l)) = h^0(\widetilde{\mathscr{E}}_i^{\text{total}}(l))$ by m_i, $i = 1, ..., s$, then $\lambda = \sum_{i=1}^s \alpha_i \cdot \lambda(\underline{m}, \underline{\gamma}_p^{(m_i)})$ as a one parameter subgroup of $\operatorname{SL}(\widetilde{M})$. Therefore,

$$\mu(\lambda, [m']) = -\min\Big\{ \sum_{i=1}^s \alpha_i \cdot \big(a \cdot m_i - \nu_i(\underline{\iota}) \cdot p\big) \,\Big|\, k = 1, ..., b,\ \underline{\iota} \in J^a \colon \widetilde{M}_{\underline{\iota}}^k \not\subset \ker(m') \Big\}.$$

Here,

$$\nu_i(\underline{\iota}) = \#\{\, \iota_i \leq i \mid \underline{\iota} = (\iota_1, ..., \iota_a),\ i = 1, ..., a \,\}. \tag{3.6}$$

Let $\iota_0 \in J^a$ be an index which realizes the precise value of $\mu(\lambda, [m'])$. Then, by (3.2), we find

$$\sum_{i=1}^s \alpha_i \cdot \left(\frac{p^2 \cdot \operatorname{rk}(\widetilde{\mathscr{E}}_i^{\text{total}})}{r \cdot d} - \frac{p \cdot a \cdot \operatorname{rk}(\widetilde{\mathscr{E}}_i^{\text{total}})}{r} - \frac{p \cdot P\big(\widetilde{\mathscr{E}}_i^{\text{total}}(l)\big)}{d} \right) + \\ + p \cdot \sum_{i=1}^s \alpha_i \cdot \nu_i(\underline{\iota_0}) - p \cdot \sum_{i=1}^s \alpha_i \cdot \left(\sum_{v \in V} \big(x_v \cdot \operatorname{rk}(\widetilde{\mathscr{E}}_i^v)\big) \right)$$

as the value for $\mu(\lambda, \Gamma(m))/\varrho$. We multiply this by $r \cdot d/p$ and get

$$\sum_{i=1}^s \alpha_i \cdot \Big(p \cdot \operatorname{rk}(\widetilde{\mathscr{E}}_i^{\text{total}}) - r \cdot P\big(\widetilde{\mathscr{E}}_i^{\text{total}}(l)\big) \Big) + \\ + d \cdot \left(\sum_{i=1}^s \alpha_i \cdot \big(\nu_i(\underline{\iota_0}) \cdot r - a \cdot \operatorname{rk}(\widetilde{\mathscr{E}}_i^{\text{total}})\big) \right) + r \cdot \sum_{i=1}^s \alpha_i \cdot \left(\sum_{v \in V} \widetilde{\chi}_v(l) \cdot \operatorname{rk}(\mathscr{E}_i^v) \right).$$

In the next step, we plug in the definition of $\chi_v(l)$, $v \in V$. This leads to the expression:

$$\sum_{i=1}^{s} \alpha_i \cdot \Bigl(p \cdot \frac{r}{R} \cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_i) - r \cdot P\bigl(\widetilde{\mathscr{E}}_i^{\mathrm{total}}(l)\bigr)\Bigr) +$$
$$+ d \cdot \Biggl(\sum_{i=1}^{s} \alpha_i \cdot \Bigl(\nu_i(\underline{\iota_0}) \cdot r - a \cdot \frac{r}{R} \cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_i)\Bigr)\Biggr) + r \cdot \sum_{i=1}^{s} \alpha_i \cdot \Bigl(\sum_{v \in V} \chi_v(l) \cdot \mathrm{rk}(\mathscr{E}_i^v)\Bigr).$$

We multiply this by R/r and find

$$\sum_{i=1}^{s} \alpha_i \cdot \Bigl(p \cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_i) - R \cdot P\bigl(\widetilde{\mathscr{E}}_i^{\mathrm{total}}(l)\bigr)\Bigr) +$$
$$+ d \cdot \Bigl(\sum_{i=1}^{s} \alpha_i \cdot \bigl(\nu_i(\underline{\iota_0}) \cdot R - a \cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_i)\bigr)\Bigr) + R \cdot \sum_{i=1}^{s} \alpha_i \cdot \Bigl(\sum_{v \in V} \chi_v(l) \cdot \mathrm{rk}(\mathscr{E}_i^v)\Bigr).$$

This can be rewritten as

$$M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)(l) + \delta(l) \cdot \Bigl(\sum_{i=1}^{s} \alpha_i \cdot \bigl(\nu_i(\underline{\iota_0}) \cdot R - a \cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_i)\bigr)\Bigr).$$

As in [14], page 156, one verifies that

$$\mu\bigl(\mathscr{E}_\bullet, \underline{\alpha}, \varphi\bigr) \geq \sum_{i=1}^{s} \alpha_i \cdot \bigl(\nu_i(\underline{\iota_0}) \cdot R - a \cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_i)\bigr),$$

so that $\mu(\lambda, \Gamma(m))$ $(\geq)$ 0 implies Inequality (3.5). □

Semistability implies GIT-semistability

We now address the converse direction to Theorem 3.3, i.e., the following statement:

Theorem 3.4. *There is an $l_0 \in \mathbb{N}$, such that, for every $l \geq l_0$, the following statement holds true: Let $m = (q_v\colon W_v \otimes \mathscr{O}_X(-l) \longrightarrow \mathscr{E}_v, v \in V, \varphi)$ be a point in the parameter space $\mathfrak{M}$, such that $(\underline{\mathscr{E}}, \varphi)$ is a (semi)stable decorated V-split sheaf $(\underline{\mathscr{E}}, \varphi)$ of type $(\underline{P}, a, b, c, m)$. Then, the point $\Gamma(m)$ is (semi)stable with respect to the chosen linearization in $\mathscr{O}(\varrho, \vartheta_v, v \in V)$.*

Apart from some subtleties, we will follow the previous calculations. Let $\lambda\colon \mathbb{C}^\star \longrightarrow \widetilde{G}$ be a one parameter subgroup. We write λ as a tuple $(\lambda_v, v \in V)$ where λ_v is a one parameter subgroup of $\mathrm{GL}(W_v)$, $v \in V$. Then, λ_v induces a weighted filtration $(\widehat{W}^v_\bullet, \gamma^v_\bullet)$ of W_v, $v \in V$. The condition that λ be a one parameter subgroup of $\mathrm{SL}(\widetilde{M})$ translates into the condition that

$$\sum_{v \in V} \kappa_v(l) \cdot \sum_{i=1}^{s_v+1} \gamma_i^v \cdot \bigl(\dim(\widehat{W}_i^v) - \dim(\widehat{W}_{i-1}^v)\bigr) = 0.$$

We define $\widehat{\mathscr{E}}_i^v$ as the saturated subsheaf of $\mathscr{E}_v$ that is generically generated by $q_v(\widehat{W}_i^v \otimes \mathscr{O}_X(-l))$, $i = 1, ..., s_v$, $v \in V$.

For the point $\Gamma(m) = ([m'], [m_v], v \in V)$, we find that

$$\frac{\mu\big(\lambda, \Gamma(m)\big)}{\varrho} = \mu\big(\lambda, [m']\big) + \varepsilon \cdot A + B$$

with

$$\begin{aligned} A &:= \sum_{v \in V} \varepsilon_v \cdot \mu\big(\lambda, [m_v]\big) - \\ &\quad - \sum_{v \in V} \kappa_v(l) \cdot \left(\frac{r_v}{p_v} - \frac{r}{p}\right) \cdot \left(\sum_{i=1}^{s_v+1} \gamma_i^v \cdot \Big(\dim\big(\widehat{W}_i^v\big) - \dim\big(\widehat{W}_{i-1}^v\big)\Big)\right), \\ B &:= \sum_{v \in V} \left(x'_v \cdot \sum_{i=1}^{s_v+1} \gamma_i^v \cdot \Big(\dim\big(\widehat{W}_i^v\big) - \dim\big(\widehat{W}_{i-1}^v\big)\Big)\right). \end{aligned}$$

As before, we write

$$\sum_{i=1}^{s_v+1} \gamma_i^v \cdot \Big(\dim\big(\widehat{W}_i^v\big) - \dim\big(\widehat{W}_{i-1}^v\big)\Big) = \gamma_{s_v+1}^v \cdot p_v - \sum_{i=1}^{s_v} \big(\gamma_{i+1}^v - \gamma_i^v\big) \cdot \dim\big(\widehat{W}_i^v\big).$$

Moreover, the following holds true

$$\mu\big(\lambda, [m_v]\big) = \sum_{i=1}^{s_v} \frac{\gamma_{i+1}^v - \gamma_i^v}{p_v} \cdot \Big(p_v \cdot \mathrm{rk}(\widehat{\mathscr{E}}_i^v) - \dim\big(\widehat{W}_i^v\big) \cdot r_v\Big).$$

For $v \in V$, we now compute

$$\begin{aligned} &\varepsilon \cdot \varepsilon_v \cdot \mu\big(\lambda, [m_v]\big) - x'_v \cdot \sum_{i=1}^{s_v} \big(\gamma_{i+1}^v - \gamma_i^v\big) \cdot \dim\big(\widehat{W}_i^v\big) + x'_v \cdot \gamma_{s_v+1}^v \cdot p_v \\ = &\sum_{i=1}^{s_v} \frac{\gamma_{i+1}^v - \gamma_i^v}{p_v} \cdot \Big(\varepsilon \cdot \varepsilon_v \cdot \Big(p_v \cdot \mathrm{rk}(\widehat{\mathscr{E}}_i^v) - \dim\big(\widehat{W}_i^v\big) \cdot r_v\Big) - x'_v \cdot p_v \cdot \dim\big(\widehat{W}_i^v\big)\Big) \\ &+ x'_v \cdot \gamma_{s_v+1}^v \cdot p_v \\ = &\sum_{i=1}^{s_v} \frac{\gamma_{i+1}^v - \gamma_i^v}{p_v} \cdot \Big(\varepsilon \cdot \varepsilon_v \cdot \Big(p_v \cdot \mathrm{rk}(\widehat{\mathscr{E}}_i^v) - \dim\big(\widehat{W}_i^v\big) \cdot r_v\Big) - x_v \cdot r_v \cdot \dim\big(\widehat{W}_i^v\big)\Big) \\ &+ x_v \cdot \gamma_{s_v+1}^v \cdot r_v \\ = &\sum_{i=1}^{s_v} \frac{\gamma_{i+1}^v - \gamma_i^v}{p_v} \cdot \Big(\varepsilon \cdot \kappa_v(l) \cdot \Big(p_v \cdot \mathrm{rk}(\widehat{\mathscr{E}}_i^v) - \dim\big(\widehat{W}_i^v\big) \cdot r_v\Big) - x_v \cdot p_v \cdot \mathrm{rk}(\widehat{\mathscr{E}}_i^v)\Big) \\ &+ x_v \cdot \gamma_{s_v+1}^v \cdot r_v \\ = &\ \varepsilon \cdot \kappa_v(l) \cdot \sum_{i=1}^{s_v} \left(\frac{\gamma_{i+1}^v - \gamma_i^v}{p_v} \cdot \Big(p_v \cdot \mathrm{rk}(\widehat{\mathscr{E}}_i^v) - \dim\big(\widehat{W}_i^v\big) \cdot r_v\Big)\right) + \\ &+ \sum_{i=1}^{s_v+1} x_v \cdot \gamma_i^v \cdot \big(\mathrm{rk}(\widehat{\mathscr{E}}_i^v) - \mathrm{rk}(\widehat{\mathscr{E}}_{i-1}^v)\big). \end{aligned}$$

Via the isomorphism

$$H^0\big(q_v(l)\big)\colon W_v \longrightarrow H^0\big(\mathscr{E}_v(l)\big),$$

we may view $\widehat{W}_i^v$ as a subspace of $H^0(\widehat{\mathscr{E}}_i^v(l))$, $i = 1, ..., s_v$, $v \in V$. In particular, the above computations show that, for $v \in V$,

$$\begin{aligned} & \varepsilon \cdot \varepsilon_v \cdot \mu\big(\lambda, [m_v]\big) + \sum_{i=1}^{s_v+1} x_v \cdot \gamma_i^v \cdot \big(\mathrm{rk}(\widehat{\mathscr{E}}_i^v) - \mathrm{rk}(\widehat{\mathscr{E}}_{i-1}^v)\big) \\ \geq\ & \varepsilon \cdot \kappa_v(l) \cdot \sum_{i=1}^{s_v} \bigg(\frac{\gamma_{i+1}^v - \gamma_i^v}{p_v} \cdot \Big(p_v \cdot \mathrm{rk}(\widehat{\mathscr{E}}_i^v) - h^0\big(\widehat{\mathscr{E}}_i^v(l)\big) \cdot r_v\Big)\bigg) + \\ & + \sum_{i=1}^{s_v+1} x_v \cdot \gamma_i^v \cdot \big(\mathrm{rk}(\widehat{\mathscr{E}}_i^v) - \mathrm{rk}(\widehat{\mathscr{E}}_{i-1}^v)\big). \end{aligned}$$

Let the filtration

$$\widetilde{\mathscr{E}}_\bullet^{\mathrm{total}} : 0 \subseteq \widetilde{\mathscr{E}}_1^{\mathrm{total}} \subseteq \cdots \subseteq \widetilde{\mathscr{E}}_s^{\mathrm{total}} \subseteq \widetilde{\mathscr{E}}^{\mathrm{total}}$$

be constructed as described in the footnote on page 119 and set

$$\widetilde{\mathscr{E}}_i^v := \widehat{\mathscr{E}}_{s_v(i)}^v, \quad v \in V,$$

so that

$$\widetilde{\mathscr{E}}_i^{\mathrm{total}} = \bigoplus_{v \in V} \widetilde{\mathscr{E}}_i^{v, \oplus \kappa_v(l)}, \quad i = 1, ..., s.$$

With Proposition 3.1, $\mathrm{rk}(\widehat{\mathscr{E}}_{s_v+1}^v) = r_v$, $v \in V$, and the equation $\sum_{v \in V} x_v \cdot r_v = 0$ (Remark 3.1, ii), we find that

$$\begin{aligned} \varepsilon \cdot A + B \ \geq\ & \varepsilon \cdot \sum_{i=1}^{s} \frac{\gamma_{i+1} - \gamma_i}{p} \cdot \Big(p \cdot \mathrm{rk}(\widetilde{\mathscr{E}}_i^{\mathrm{total}}) - h^0\big(\widetilde{\mathscr{E}}_i^{\mathrm{total}}(l)\big) \cdot r\Big) \\ & - \sum_{v \in V} x_v \cdot \sum_{i=1}^{s_v} (\gamma_{i+1}^v - \gamma_i^v) \cdot \mathrm{rk}(\widehat{\mathscr{E}}_i^v) \\ =\ & \varepsilon \cdot \sum_{i=1}^{s} \frac{\gamma_{i+1} - \gamma_i}{p} \cdot \Big(p \cdot \mathrm{rk}(\widetilde{\mathscr{E}}_i^{\mathrm{total}}) - h^0\big(\widetilde{\mathscr{E}}_i^{\mathrm{total}}(l)\big) \cdot r\Big) \\ & - \sum_{i=1}^{s} (\gamma_{i+1} - \gamma_i) \cdot \sum_{v \in V} x_v \cdot \mathrm{rk}(\widetilde{\mathscr{E}}_i^v). \end{aligned}$$

Next, we point out that there might occur equalities in the filtration

$$0 \subseteq \widetilde{\mathscr{E}}_1^{\mathrm{total}} \subseteq \cdots \subseteq \widetilde{\mathscr{E}}_s^{\mathrm{total}} \subseteq \widetilde{\mathscr{E}}^{\mathrm{total}}.$$

We clear these and obtain a filtration

$$0 \subseteq \overline{\mathscr{E}}_1^{\mathrm{total}} \subsetneq \cdots \subsetneq \overline{\mathscr{E}}_{\overline{s}}^{\mathrm{total}} \subsetneq \overline{\mathscr{E}}^{\mathrm{total}} = \widetilde{\mathscr{E}}^{\mathrm{total}}.$$

in which all inclusions are strict. For $i \in \{1, ..., \overline{s}+1\}$, we set

$$\begin{aligned} \overline{\jmath}(i) &:= \min\Big\{ j \in \{1, ..., s\} \,\Big|\, \widetilde{\mathscr{E}}_j^{\text{total}} = \overline{\mathscr{E}}_i^{\text{total}} \Big\}, \\ \text{and} \quad \overline{\gamma}_i &:= \gamma_{\overline{\jmath}(i)}. \end{aligned}$$

Writing

$$\overline{\mathscr{E}}_i^{\text{total}} = \bigoplus_{v \in V} \overline{\mathscr{E}}_i^v, \quad i = 1, ..., s,$$

we find

$$\begin{aligned} & \varepsilon \cdot \sum_{i=1}^{s} \frac{\gamma_{i+1} - \gamma_i}{p} \cdot \Big(p \cdot \operatorname{rk}(\widetilde{\mathscr{E}}_i^{\text{total}}) - h^0\big(\widetilde{\mathscr{E}}_i^{\text{total}}(l)\big) \cdot r \Big) \\ & - \sum_{i=1}^{s} (\gamma_{i+1} - \gamma_i) \cdot \sum_{v \in V} x_v \cdot \operatorname{rk}(\widetilde{\mathscr{E}}_i^v) \\ = \; & \varepsilon \cdot \sum_{i=1}^{\overline{s}} \frac{\overline{\gamma}_{i+1} - \overline{\gamma}_i}{p} \cdot \Big(p \cdot \operatorname{rk}(\overline{\mathscr{E}}_i^{\text{total}}) - h^0\big(\overline{\mathscr{E}}_i^{\text{total}}(l)\big) \cdot r \Big) \\ & - \sum_{i=1}^{\overline{s}} (\overline{\gamma}_{i+1} - \overline{\gamma}_i) \cdot \sum_{v \in V} x_v \cdot \operatorname{rk}(\overline{\mathscr{E}}_i^v). \end{aligned} \tag{3.7}$$

We define $\alpha_\bullet = (\alpha_1, ..., \alpha_s)$ via

$$\alpha_i := \frac{\overline{\gamma}_{i+1} - \overline{\gamma}_i}{p}, \quad i = 1, ..., \overline{s}.$$

Then, the expression in (3.7) takes the form

$$\begin{aligned} \sum_{i=1}^{s} \alpha_i \cdot \Bigg(& \frac{p^2 \cdot \operatorname{rk}(\overline{\mathscr{E}}_i^{\text{total}})}{r \cdot d} - \frac{p \cdot a \cdot \operatorname{rk}(\overline{\mathscr{E}}_i^{\text{total}})}{r} - \frac{p \cdot h^0\big(\overline{\mathscr{E}}_i^{\text{total}}(l)\big)}{d} + \\ & + a \cdot \sum_{v \in V} \kappa_v(l) \cdot h^0\big(\overline{\mathscr{E}}_i^v(l)\big) \Bigg) - p \cdot \sum_{i=1}^{s} \alpha_i \cdot \Bigg(\sum_{v \in V} x_v \cdot \operatorname{rk}(\overline{\mathscr{E}}_i^v) \Bigg). \end{aligned} \tag{3.8}$$

Let $(\widetilde{M}^{\text{total}}, \gamma_\bullet)$, $\gamma_\bullet = (\gamma_1, ..., \gamma_{s+1})$, be the weighted filtration of $\widetilde{M}$ that comes from the weighted filtrations $(\widehat{W}_\bullet^v, \gamma_\bullet^v)$ of the W_v, $v \in V$. Then, for

$$m' \colon \big(\widetilde{M}^{\otimes a}\big)^{\oplus b} \longrightarrow H^0\big(\mathscr{L}^{\otimes c}(a \cdot l)\big),$$

we compute

$$\begin{aligned} & \mu\big(\lambda, [m']\big) \\ & = -\min\Big\{ \gamma_{\iota_1} + \cdots + \gamma_{\iota_a} \,\Big|\, (\iota_1, ..., \iota_a) \in \{1, ..., s+1\}^{\times a} : m'_{|(\widetilde{M}_{\iota_1}^{\text{total}} \otimes \cdots \otimes \widetilde{M}_{\iota_a}^{\text{total}})^{\oplus b}} \neq 0 \Big\}. \end{aligned}$$

Since $q_v(\widetilde{W}_v \otimes \mathscr{O}_X)$ generically generates $\widetilde{\mathscr{E}}_i^v(l)$, $i = 1, ..., s_v$, $v \in V$, we infer

$$\forall \underline{\iota} = (\iota_1, ..., \iota_a) \in \{1, ..., s\}^{\times a} : \; m'_{|(\widetilde{M}_{\iota_1}^{\text{total}} \otimes \cdots \otimes \widetilde{M}_{\iota_a}^{\text{total}})^{\oplus b}} \neq 0 \iff \varphi_{|(\widetilde{\mathscr{E}}_{\iota_1}^{\text{total}} \otimes \cdots \otimes \widetilde{\mathscr{E}}_{\iota_a}^{\text{total}})^{\oplus b}} \neq 0.$$

We thus find the estimate

$$\begin{aligned}&\mu\big(\lambda,[m']\big)\\&=-\min\Big\{\,\gamma_{\iota_1}+\cdots+\gamma_{\iota_a}\,\Big|\,(\iota_1,...,\iota_a)\in\{\,1,...,s+1\,\}^{\times a}:\varphi_{|(\widetilde{\mathscr{E}}^{\text{total}}_{\iota_1}\otimes\cdots\otimes\widetilde{\mathscr{E}}^{\text{total}}_{\iota_a})^{\oplus b}}\neq 0\,\Big\}\\&\geq-\min\Big\{\,\overline{\gamma}_{\iota_1}+\cdots+\overline{\gamma}_{\iota_a}\,\Big|\,(\iota_1,...,\iota_a)\in\{\,1,...,\overline{s}+1\,\}^{\times a}:\varphi_{|(\overline{\mathscr{E}}^{\text{total}}_{\iota_1}\otimes\cdots\otimes\overline{\mathscr{E}}^{\text{total}}_{\iota_a})^{\oplus b}}\neq 0\,\Big\}.\end{aligned}$$

By definition, the entries of $\overline{\gamma}_\bullet=(\overline{\gamma}_1,...,\overline{\gamma}_{\overline{s}+1})$ are the entries of the vector

$$\sum_{i=1}^{\overline{s}}\alpha_i\cdot\big(\underbrace{m_i-p,..,m_i-p}_{m_i\times},\underbrace{m_i,...,m_i}_{(p-m_i)\times}\big),\quad m_i:=h^0\big(\overline{\mathscr{E}}^{\text{total}}_i(l)\big)\quad i=1,..,\overline{s}.$$

Let $\underline{\iota}\in\{\,1,...,\overline{s}+1\,\}^{\times a}$ be such that $\varphi_{|(\overline{\mathscr{E}}^{\text{total}}_{\iota_1}\otimes\cdots\otimes\overline{\mathscr{E}}^{\text{total}}_{\iota_a})^{\oplus b}}\neq 0$. With (3.6), we thus find

$$\mu\big(\lambda,[m']\big)\geq\sum_{i=1}^{\overline{s}}\alpha_i\cdot\big(\nu_i(\underline{\iota})\cdot p-a\cdot m_i\big).$$

By now, it has become our task to show that, for some $\underline{\iota}\in\{1,...,\overline{s}+1\}^{\times a}$ with $\varphi_{|(\overline{\mathscr{E}}^{\text{total}}_{\iota_1}\otimes\cdots\otimes\overline{\mathscr{E}}^{\text{total}}_{\iota_a})^{\oplus b}}\neq 0$, we have

$$\begin{aligned}\sum_{i=1}^{s}\alpha_i\cdot\left(\frac{p^2\cdot\mathrm{rk}(\overline{\mathscr{E}}^{\text{total}}_i)}{r\cdot d}-\frac{p\cdot a\cdot\mathrm{rk}(\overline{\mathscr{E}}^{\text{total}}_i)}{r}-\frac{p\cdot h^0\big(\overline{\mathscr{E}}^{\text{total}}_i(l)\big)}{d}\right)+\\+p\cdot\sum_{i=1}^{s}\alpha_i\cdot\nu_i(\underline{\iota})-p\cdot\sum_{i=1}^{s}\alpha_i\cdot\left(\sum_{v\in V}x_v\cdot\mathrm{rk}(\overline{\mathscr{E}}^{v}_i)\right)\quad(\geq)\quad 0.\end{aligned}\tag{3.9}$$

We introduce the weighted filtration $(\underline{\mathscr{E}}_\bullet,\alpha_\bullet)$ of $\underline{\mathscr{E}}=(\mathscr{E}_v,v\in V)$ with $\underline{\mathscr{E}}_i=(\overline{\mathscr{E}}^{v}_i,v\in V)$, $i=1,...,s$, and $\alpha_\bullet=(\alpha_1,...,\alpha_s)$. By computations analogous to those on page 126, inequality (3.9) amounts to the inequality

$$\begin{aligned}&\sum_{i=1}^{\overline{s}}\alpha_i\cdot\left(p\cdot\mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_i)-R\cdot h^0\big(\overline{\mathscr{E}}^{\text{total}}_i(l)\big)\right)+R\cdot\sum_{i=1}^{\overline{s}}\alpha_i\cdot\left(\sum_{v\in V}\chi_v(l)\cdot\mathrm{rk}(\overline{\mathscr{E}}^{v}_i)\right)\\&+d\cdot\left(\sum_{i=1}^{\overline{s}}\alpha_i\cdot\big(\nu_i(\underline{\iota_0})\cdot R-a\cdot\mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_i)\big)\right)\quad(\geq)\quad 0.\end{aligned}$$

Let $\underline{\gamma}'_\bullet=(\gamma'_1,...,\gamma'_{\overline{s}})$ be the entries of the vector

$$\sum_{i=1}^{\overline{s}}\alpha_i\cdot\big(\underbrace{R_i-R,..,R_i-R}_{R_i\times},\underbrace{R_i,...,R_i}_{(R-R_i)\times}\big).$$

Choosing $\underline{\iota}$ with $\varphi_{|(\overline{\mathscr{E}}^{\text{total}}_{\iota_1}\otimes\cdots\otimes\overline{\mathscr{E}}^{\text{total}}_{\iota_a})^{\oplus b}}\neq 0$ in such a way that

$$\gamma'_{\iota_1}+\cdots+\gamma'_{\iota_a}$$

becomes minimal, we see that

$$\sum_{i=1}^{s}\alpha_i\cdot\bigl(\nu_i(\underline{\iota})\cdot R-a\cdot \mathrm{rk}(\overline{\mathscr{E}}{}_i^{\mathrm{total}})\bigr)=\mu(\underline{\mathscr{E}}_\bullet,\alpha_\bullet,\varphi).$$

As in the proof of Lemma 3.2, we write

$$\{\,1,...,\overline{s}\,\}=\{\,i_1,...,i_{s_1}\,\}\sqcup\{\,j_1,...,j_{s_2}\,\}.$$

Here, $i_1,...,i_{s_1}$ are those i in $\{\,1,...,\overline{s}\,\}$ with $h^j(\overline{\mathscr{E}}{}_i^{\mathrm{total}}(l))=0$, $j>0$. We have

$$\begin{aligned}
&\sum_{i=1}^{\overline{s}}\alpha_i\cdot\Bigl(p\cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_i)-R\cdot h^0\bigl(\overline{\mathscr{E}}{}_i^{\mathrm{total}}(l)\bigr)\Bigr)+R\cdot\sum_{i=1}^{s}\alpha_i\cdot\Bigl(\sum_{v\in V}\chi_v(l)\cdot \mathrm{rk}(\overline{\mathscr{E}}{}_i^{v})\Bigr)\\
=\;&\sum_{f=1}^{s_1}\alpha_{i_f}\cdot\Bigl(p\cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_{i_f})-R\cdot h^0\bigl(\overline{\mathscr{E}}{}_{i_f}^{\mathrm{total}}(l)\bigr)\Bigr)+R\cdot\sum_{f=1}^{s_1}\alpha_{i_f}\cdot\Bigl(\sum_{v\in V}\chi_v(l)\cdot \mathrm{rk}(\overline{\mathscr{E}}{}_{i_f}^{v})\Bigr)+\\
&+\sum_{g=1}^{s_2}\alpha_{j_g}\cdot\Bigl(p\cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_{j_g})-R\cdot h^0\bigl(\overline{\mathscr{E}}{}_{j_g}^{\mathrm{total}}(l)\bigr)\Bigr)+R\cdot\sum_{g=1}^{s_2}\alpha_{j_g}\cdot\Bigl(\sum_{v\in V}\chi_v(l)\cdot \mathrm{rk}(\overline{\mathscr{E}}{}_{j_g}^{v})\Bigr)
\end{aligned}$$

and

$$\mu(\underline{\mathscr{E}}_\bullet,\alpha_\bullet,\varphi)\ge\mu(\underline{\mathscr{E}}^1_\bullet,\alpha^1_\bullet,\varphi)-a\cdot(R-1)\cdot\sum_{g=1}^{s_2}\alpha_{j_g}.$$

Now, we let $(\underline{\mathscr{E}}^1,\alpha^1_\bullet)$ be the weighted filtration of $\underline{\mathscr{E}}$ containing the V-split sheaves $\underline{\mathscr{E}}_{i_f}$, $f=1,...,s_1$, and the vector $\alpha_\bullet=(\alpha^1_1,...,\alpha^1_s):=(\alpha_{i_1},...,\alpha_{i_{s_1}})$. Then,

$$\begin{aligned}
&\sum_{f=1}^{s_1}\alpha_{i_f}\cdot\Bigl(p\cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_{i_f})-R\cdot h^0\bigl(\overline{\mathscr{E}}{}_{i_f}^{\mathrm{total}}(l)\bigr)\Bigr)+\\
&+R\cdot\sum_{f=1}^{s_1}\alpha_{i_f}\cdot\Bigl(\sum_{v\in V}\chi_v(l)\cdot \mathrm{rk}(\overline{\mathscr{E}}{}_{i_f}^{v})\Bigr)+d\cdot\mu(\underline{\mathscr{E}}^1_\bullet,\alpha^1_\bullet,\varphi)\\
=\;&M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}^1_\bullet,\alpha^1_\bullet)(l)+\delta(l)\cdot\mu(\underline{\mathscr{E}}^1_\bullet,\alpha^1_\bullet,\varphi)\quad(\ge)\quad 0
\end{aligned}$$

follows from the (semi-)stability of $(\underline{\mathscr{E}},\varphi)$. By the boundedness of semistable decorated V-split sheaves (Proposition 2.1), we may prescribe a constant $C>0$ and assume

$$\mu(\overline{\mathscr{E}}{}_i^{\mathrm{total}})<\mu(\mathscr{E})-C,\quad i=j_1,...,j_{s_2}.$$

Then, we use the LePotier–Simpson estimate ([10], Corollary 3.3.8) to obtain an inequality of the form

$$p\cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_i)-R\cdot h^0\bigl(\overline{\mathscr{E}}{}_i^{\mathrm{total}}(l)\bigr)>C'\cdot m^{\dim(X)-1}+\text{lower order terms},\quad i=j_1,...,j_{s_2}.$$

The lower order terms depend only on $\underline{P}$ and $(X,\mathscr{O}_X(1))$, and, given $C'>0$, we can find an appropriate constant C which yields C'. It is, therefore, possible to find C' and l_0, such that

$$p\cdot \mathrm{rk}_{\underline{\sigma}}(\underline{\mathscr{E}}_i)-R\cdot h^0\bigl(\overline{\mathscr{E}}{}_i^{\mathrm{total}}(l)\bigr)+R\cdot\Bigl(\sum_{v\in V}\chi_v(l)\cdot \mathrm{rk}(\overline{\mathscr{E}}{}_i^{v})\Bigr)-\delta(l)\cdot a\cdot(R-1)>0$$

for $i=j_1,...,j_{s_2}$ and all $l\ge l_0$. □

Properness of the Gieseker morphism

For our construction, we work with the Gieseker morphism

$$\Gamma\colon \mathfrak{M} \longrightarrow \widetilde{\mathfrak{P}}' \times \bigtimes_{v\in V} \mathfrak{G}_v.$$

There is the $\widetilde{G}$-invariant open subset

$$U := \Big(\widetilde{\mathfrak{P}}' \times \bigtimes_{v\in V} \mathfrak{G}_v\Big)^{\mathrm{ss}}$$

of those points that are semistable with respect to the fixed linearization. By Geometric Invariant Theory, the categorical quotient $U/\!\!/\widetilde{G}$ exists as a **projective** scheme. Theorem 3.3 and 3.4 show that

$$\mathfrak{M}^{\mathrm{ss}} := \Gamma^{-1}(U)$$

consists exactly of those points $m = (q_v\colon W_v \otimes \mathscr{O}_X(-l) \longrightarrow \mathscr{E}_v, v \in V, \varphi)$, such that $(\underline{\mathscr{E}}, \varphi)$ is a semistable decorated V-split sheaf of type $(\underline{P}, a, b, c, m)$. The existence of the moduli space, claimed in Theorem 2.1, reduces to the following statement:

Proposition 3.2. *The induced morphism*

$$\Gamma_{|\mathfrak{M}^{\mathrm{ss}}}\colon \mathfrak{M}^{\mathrm{ss}} \longrightarrow U$$

is proper and therefore affine.

As usual, one uses the valuative criterion of properness to see this. So, let R be a discrete valuation ring with quotient field K, $S := \mathrm{Spec}(R)$ and $S^\star = \mathrm{Spec}(K)$. Suppose we have a family $m_{S^\star} = (q_v^{S^\star}\colon W_v \otimes \pi_{S^\star}^\star(\mathscr{O}_X(-l)) \longrightarrow \mathscr{E}_v^{S^\star}, v \in V, \varphi_{S^\star})$ of semistable decorated V-split sheaves of type $(\underline{P}, a, b, c, m)$ over $S^\star \times X$ and let

$$\kappa^\star\colon S^\star \longrightarrow \mathfrak{M}$$

be the classifying morphism. Suppose further that $\lambda^\star := \Gamma \circ \kappa^\star$ extends to a morphism

$$\overline{\lambda}\colon S \longrightarrow U.$$

The family $m_{S^\star}$ extends to a family $\widetilde{m}_S = (\widetilde{q}_v^S\colon W_v \otimes \pi_S^\star(\mathscr{O}_X(-l)) \longrightarrow \widetilde{\mathscr{E}}_v^S, v \in V, \widetilde{\varphi}_S)$ over $S \times X$. The sheaf $\widetilde{\mathscr{E}}_v^S$ is flat over S, but $\widetilde{\mathscr{E}}_{v|\{0\}\times X}^S$ may have torsion, $v \in V$. There are a family $\mathscr{E}_v^S$ of **torsion free** sheaves on X parameterized by S and a homomorphism $\tau_v^S\colon \widetilde{\mathscr{E}}_v^S \longrightarrow \mathscr{E}_v^S$ which is an isomorphism on $S^\star \times X$, such that the kernel of $\tau_{v|\{0\}\times X}^S$ is just the torsion subsheaf $\mathscr{T}_v$ of $\widetilde{\mathscr{E}}_{v|\{0\}\times X}^S$, $v \in V$ (see [10], Proposition 4.4.2). Let $T \subset \{0\} \times X$ be the union of the supports of the torsion subsheaves $\mathscr{T}_v$, $v \in V$, and $W := (S \times X) \setminus (\{0\} \times T)$. Finally, let $\iota\colon W \longrightarrow S \times X$ be the inclusion. Define

$$\varphi_S\colon \underline{\mathscr{E}}_{a,b,c}^S \longrightarrow \iota_\star\big(\iota^\star(\underline{\widetilde{\mathscr{E}}}_{a,b,c}^S)\big) \stackrel{\iota_\star(\iota^\star(\widetilde{\varphi}_S))}{\longrightarrow} \iota_\star\bigg(\iota^\star\Big(\pi_X^\star\big(\mathscr{O}_X(-l)\big)\Big)\bigg) = \pi_X^\star\big(\mathscr{O}_X(-l)\big).$$

The family $m_S = (q_v^S\colon W_v \otimes \pi_{S^\star}^\star(\mathscr{O}_X(-l)) \longrightarrow \mathscr{E}_v^S, v \in V, \varphi_S)$ defines a morphism

$$S \longrightarrow \widetilde{\mathfrak{P}}' \times \bigtimes_{v\in V} \mathfrak{G}_v$$

which agrees with $\lambda^\star$ on $S^\star$ and consequently identifies with $\overline{\lambda}$. It suffices now to check the following:

Theorem 3.5. *The set of isomorphism classes of torsion free sheaves $\mathscr{E}'$, such that there exist an $l \in \mathbb{N}$, a tuple $m = (q_v \colon W_v \otimes \mathscr{O}_X(-l) \longrightarrow \mathscr{E}_v, v \in V, \varphi)$ in which q_v is a* ***generically surjective*** *homomorphism, $v \in V$, whose associated point $\Gamma(m)$ is semistable with respect to the chosen linearization in $\mathscr{O}(\varrho, \vartheta_v, v \in V)$, and an index $v_0 \in V$ with $\mathscr{E}' \cong \mathscr{E}_{v_0}$ is bounded.*

One checks that the semistability of $\Gamma(m)$ implies that $H^0(q_v(l)) \colon W_v \longrightarrow H^0(\mathscr{E}_v(l))$ is injective, $v \in V$. With this, the proof of this result is an easy adaptation of the proof of Theorem 3.1. In fact, in (3.4), we replace $H^0(\widetilde{\mathscr{E}}^{\text{total}}(l))$ by $\widetilde{M}$ and $H^0(\mathscr{F}^{\text{total}}(l))$ by $\widetilde{M} \cap H^0(\mathscr{F}^{\text{total}}(l))$. □

4 Proof of the main theorem

In this final section, we explain how the main theorem may be reduced to Theorem 2.1. This is again very similar to the corresponding step in [13], so that we may be rather sketchy.

We return to the setting of quiver representations. So, let $Q = (V, A, t, h)$ be a quiver with vertices $V = \{v_1, ..., v_t\}$, arrows $A = \{a_1, ..., a_u\}$, the tail map $t \colon A \longrightarrow V$, and the head map $h \colon A \longrightarrow V$. Since we allow twisting by vector bundles in the representations, we may assume that no multiple arrows occur. Fix a tuple of locally free coherent sheaves $\underline{\mathscr{G}} = (\mathscr{G}_a, a \in A)$. An *augmented representation of Q of type $(\underline{P}, \underline{\mathscr{G}})$* is a tuple $(\mathscr{E}_v, v \in V, \varphi_a, a \in A, \varepsilon)$, consisting of

- a V-split sheaf $(\mathscr{E}_v, v \in V)$ of type P,
- homomorphisms $\varphi_a \colon \mathscr{G}_a \otimes \mathscr{E}_{t(a)} \longrightarrow \mathscr{E}_{h(a)}$, $a \in A$,
- a complex number ε,

such that either $\varepsilon \neq 0$ or one of the φ_a, $a \in A$, is non-trivial. Two augmented representations $(\mathscr{E}_v, v \in V, \varphi_a, a \in A, \varepsilon)$ and $(\mathscr{E}'_v, v \in V, \varphi'_a, a \in A, \varepsilon')$ are called *equivalent*, if there are isomorphisms $\psi_v \colon \mathscr{E}_v \longrightarrow \mathscr{E}'_v$, $v \in V$, and $z \in \mathbb{C}^\star$, such that

$$z \cdot (\psi_{h(a)} \circ \varphi_a \circ (\mathrm{id}_{\mathscr{G}_a} \otimes \psi_{t(a)})^{-1}) = \varphi'_a, \quad a \in A, \quad \text{and} \quad z \cdot \varepsilon = \varepsilon'.$$

In [13], p. 32, it is described how one may associate with an augmented representation $(\mathscr{E}_v, v \in V, \varphi_a, a \in A, \varepsilon)$ of Q of type $(\underline{\mathscr{G}}, \underline{P})$ a decorated V-split sheaf $(\underline{\mathscr{E}}, \psi)$, $\underline{\mathscr{E}} = (\mathscr{E}_v, v \in V)$, of type $(\underline{P}, a, b, c, m)$ for suitable positive integers a, b, c, m. We fix stability parameters $\underline{\kappa}$, $\underline{\eta}$ and δ and say that an augmented representation $(\mathscr{E}_v, v \in V, \varphi_a, a \in A, \varepsilon)$ of Q of type $(\underline{\mathscr{G}}, \underline{P})$ is $(\underline{\kappa}, \underline{\eta}, \delta)$-*(semi)stable*, if the associated decorated V-split sheaf $(\underline{\mathscr{E}}, \psi)$ of type $(\underline{P}, a, b, c, m)$ is.

We remind the reader that associating with an augmented quiver representation a decorated V-split sheaf involves choosing suitable positive integers b and m as well as surjections $\mathscr{O}_X(-m)^{\oplus b} \longrightarrow \mathscr{G}_a$, $a \in A$ ([13], p. 32). Using these surjections, we may assign to every quiver representation $(\mathscr{E}_v, v \in V, \varphi_a, a \in A)$ a homomorphism

$$\varphi \colon \mathscr{E}^{\text{total}} \longrightarrow \mathscr{E}^{\text{total}} \otimes \mathscr{O}_X(m)^{\oplus b}.$$

For $s \geq 1$, we set

$$\varphi^{(s)} := \varphi \otimes \mathrm{id}_{\mathscr{O}_X(s\cdot m)^{\oplus b^s}} : \mathscr{E}^{\mathrm{total}} \otimes \mathscr{O}_X(s\cdot m)^{\oplus b^s} \longrightarrow \mathscr{E}^{\mathrm{total}} \otimes \mathscr{O}_X\big((s+1)\cdot m\big)^{\oplus b^{s+1}}$$

and say that φ is *nilpotent*, if there is an $s \geq 1$ with

$$\varphi^{(s)} \circ \varphi^{(s-1)} \circ \cdots \circ \varphi^{(2)} \circ \varphi^{(1)} = 0.$$

Lemma 4.1. *Let $(\mathscr{E}_v, v \in V, \varphi_a, a \in A, \varepsilon)$ be an augmented quiver representation of type $(\underline{\mathscr{G}}, \underline{P})$, $(\underline{\mathscr{E}}, \psi)$ the associated decorated V-split sheaf and $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ a weighted filtration of $\underline{\mathscr{E}}$.*

i) *One finds*

$$\mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \psi) > 0,$$

if and only if there is an index $i \in \{1, ..., s\}$, such that

$$\varphi\big(\mathscr{E}_i^{\mathrm{total}}\big) \not\subset \mathscr{E}_i^{\mathrm{total}} \otimes \mathscr{O}_X(m)^{\oplus b}.$$

ii) *The condition*

$$\mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \psi) < 0$$

is equivalent to the fact that $\varepsilon = 0$ and

$$\varphi\big(\mathscr{E}_i^{\mathrm{total}}\big) \subset \mathscr{E}_{i-1}^{\mathrm{total}} \otimes \mathscr{O}_X(m)^{\oplus b}, \quad i = 1, ..., s+1.$$

Proof. Apply [14], Proposition 1.5.1.22, to the restriction of the objects involved to the generic point of X. □

Proposition 4.1. *Suppose the following data are fixed:*

- *a tuple $\underline{P} = (P_v, v \in V)$ of Hilbert polynomials,*
- *an integer $t \geq 0$,*
- *a tuple $\underline{\kappa} = (\kappa_v, v \in V)$ of positive integral polynomials of degree exactly t,*
- *a positive rational polynomial δ of degree at most $t + \dim(X) - 1$,*
- *and a tuple $\underline{\eta} = (\eta_v, v \in V)$ of rational numbers, subject to the condition $\sum_{v\in V} \eta_v \cdot r_v = 0$,*

and set

- *$\chi_v := \eta_v \cdot \delta$, $v \in V$, $\underline{\chi} = (\chi_v, v \in V)$,*
- *$\underline{\sigma} = (\sigma_v, v \in V)$, σ_v the leading coefficient of κ_v, $v \in V$.*

Then, the family of torsion free sheaves $\mathscr{E}'$ for which there exist an index v_0 and an augmented representation $(\mathscr{E}_v, v \in V, \varphi_a, a \in A, \varepsilon)$ of Q of type $(\underline{\mathscr{G}}, \underline{P})$, such that

a) *$\varepsilon \neq 0$ or φ is not nilpotent,*

b) *for any collection of saturated subsheaves* $\mathscr{F}_v \subset \mathscr{E}_v$, $v \in V$, *not all trivial and not all equal to* $\mathscr{E}_v$, *such that* $\varphi_a(\mathscr{G}_a \otimes \mathscr{F}_{t(a)}) \subset \mathscr{F}_{h(a)}$ *for all arrows* $a \in A$, *one has*

$$\frac{P_{\underline{\kappa},\underline{\chi}}(\mathscr{F}_v, v \in V)}{\mathrm{rk}_{\underline{\sigma}}(\mathscr{F}_v, v \in V)} \preceq \frac{P_{\underline{\kappa},\underline{\chi}}(\mathscr{E}_v, v \in V)}{\mathrm{rk}_{\underline{\sigma}}(\mathscr{E}_v, v \in V)},$$

and

c) $\mathscr{E}' \cong \mathscr{E}_{v_0}$

is bounded.

Proof. To the stated semistability condition of augmented quiver representations, there belongs a corresponding notion of slope semistability. As in Remark 2.2, one sees that this notion of slope semistability is already covered by the formalism in [13]. For this reason, the proposition follows from [13], Proposition 4.3.2. □

Proposition 4.2. *Fix the same data as in the previous proposition. Any* $(\underline{\kappa}, \underline{\eta}, \underline{\delta})$-*(semi)stable augmented representation* $(\mathscr{E}_v, v \in V, \varphi_a, a \in A, \varepsilon)$ *of* Q *of type* $(\underline{\mathscr{G}}, \underline{P})$ *satisfies Condition* a) *and* b) *from Proposition* 4.1.

Proof. This follows immediately from Lemma 4.1. □

Theorem 4.1. *Fix the same data as in the two previous propositions and assume that* δ *has degree* ***exactly*** $t + \dim(X) - 1$. *Then, there is a natural number* n_∞, *such that for any* $n \geq n_\infty$ *and any augmented representation* $(\mathscr{E}_v, v \in V, \varphi_a, a \in A, \varepsilon)$ *of* Q *of type* $(\underline{\mathscr{G}}, \underline{P})$, *the following conditions are equivalent:*

1. $(\mathscr{E}_v, v \in V, \varphi_a, a \in A, \varepsilon)$ *is* $(\underline{\kappa}, \underline{\eta}/n, n \cdot \delta)$-*(semi)stable,* $\underline{\eta}/n := (\eta_v/n, v \in V)$.

2. $(\mathscr{E}_v, v \in V, \varphi_a, a \in A, \varepsilon)$ *satisfies Semistability Condition* a) *and* b) *from Proposition* 4.1.

Proof. The implication "1.$\Longrightarrow$2." has been verified in greater generality in the last proposition. For the converse direction, note that, by Lemma 4.1, Condition 2. is equivalent to requiring that $\mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \psi) \geq 0$ holds for every weighted filtration $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ of $\underline{\mathscr{E}}$ and that the (semi)stability condition is verified for every weighted filtration $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ with $\mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \psi) = 0$.

Now, we use Lemma 3.1. By the boundedness result in Proposition 4.1, there is a constant $C > 0$, such that

$$M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}_\bullet, \alpha_\bullet) \succ -C \cdot x^{t+\dim(X)-1}$$

holds for every weighted filtration $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ of $\underline{\mathscr{E}}$ in which $\alpha_\bullet = (\alpha_1, ..., \alpha_s)$ consists of integers in the set $\{1, ..., A\}$. Assume that

$$n \geq n_\infty := \left\lceil \frac{C+1}{\text{coefficient of } x^{t+\dim(X)-1} \text{ in } \delta} \right\rceil$$

and that $(\underline{\mathscr{E}}_\bullet, \alpha_\bullet)$ is weighted filtration of $\underline{\mathscr{E}}$ in which $\alpha_\bullet = (\alpha_1, ..., \alpha_s)$ is a vector of integers from the set $\{1, ..., A\}$, such that

$$\mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \psi) > 0.$$

Since $\mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \psi)$ is an integer,

$$M_{\underline{\kappa},\underline{\chi}}(\underline{\mathscr{E}}_\bullet, \alpha_\bullet) + n \cdot \delta \cdot \mu(\underline{\mathscr{E}}_\bullet, \alpha_\bullet, \psi) \succ -C \cdot x^{t+\dim(X)-1} + n \cdot \delta \succ 0.$$

Thus, for $n \geq n_\infty$, an augmented representation $(\mathscr{E}_v, v \in V, \varphi_a, a \in A, \varepsilon)$ of Q of type $(\underline{\mathscr{G}}, \underline{P})$ which satisfies 2. is $(\underline{\kappa}, \underline{\eta}/n, n \cdot \delta)$-(semi)stable. □

Fix $n \geq n_\infty$ By Theorem 2.1, we get a projective moduli space

$$\mathscr{R}(Q, \underline{\mathscr{G}})^{\mathrm{ss}}_{\underline{P}/\underline{\kappa}/\underline{\eta}/\delta}$$

for $(\underline{\kappa}, \underline{\eta}/n, n \cdot \delta)$-semistable augmented representations of Q of type $(\underline{\mathscr{G}}, \underline{P})$. As in [13], p. 32ff, one constructs a projective space

$$\mathbb{H} = \mathbb{H}(Q, \underline{\mathscr{G}}, \underline{P})$$

and a morphism

$$\mathrm{Hit}(Q, \underline{\mathscr{G}}, \underline{P}) \colon \mathscr{R}(Q, \underline{\mathscr{G}})^{\mathrm{ss}}_{\underline{P}/\underline{\kappa}/\underline{\eta}/\delta} \longrightarrow \mathbb{H}.$$

We define the affine space $\mathbb{D}$ as the open set $\varepsilon = 1$ in $\mathbb{H}$ and

$$\mathscr{D}(Q, \underline{\mathscr{G}})^{\mathrm{ss}}_{\underline{P}/\underline{\kappa}/\underline{\eta}/\delta}$$

as the preimage of $\mathbb{D}$ under $\mathrm{Hit}(Q, \underline{\mathscr{G}}, \underline{P})$. This is the moduli space we wanted to construct. □

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