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DEFINITION OF THE COMPLEX MONGE-AMPÈRE OPERATOR FOR ARBITRARY PLURISUBHARMONIC FUNCTIONS

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Abstract. In this paper we consider the problem of definition of the Monge-Ampère operator for an arbitrary plurisubharmonic function. The paper has a survey character; in it we discuss on the results related to this area. We give one construction of the definition of the Monge-Ampère operator, which will then be applied to maximal plurisubharmonic functions.

1 Introduction

Complex pluripotential theory, based on plurisubharmonic (Psh) functions and the Monge-Ampère operator $(dd^c u)^n$, is one of the important directions in potential theory and multi-dimensional complex analysis. Built in 1980s the theory has already found many applications in geometric questions of complex analysis and in the theory of Psh functions. We recall the following standard notation:

$$d = \partial + \bar{\partial}\,,\ d^{\,c} = i\left(\bar{\partial} - \partial\right),\text{where } \partial = \frac{\partial}{\partial z_1}dz_1 + ... + \frac{\partial}{\partial z_n}dz_n,\ \bar{\partial} = \frac{\partial}{\partial \bar{z}_1}d\bar{z}_1 + ... + \frac{\partial}{\partial \bar{z}_n}d\bar{z}_n,$$

so that

$$dd^{\,c}u = 2i\partial\bar{\partial}u,\ \ du \wedge d^{\,c}u = 2i\partial u \wedge \bar{\partial}u\ ,$$

$$(dd^{\,c}u)^n = dd^{\,c}u \wedge ... \wedge dd^{\,c}u = const\,\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)dV$$

In the classical case of $n = 1$ the operator $(dd^{\,c}u)^n$ is the linear Laplace operator, $dd^{\,c}u$ is easily defined and it is a positive Borel measure for every subharmonic function u .

Defining the Monge-Ampère operator $(dd^{\,c}u)^n$ in general case and solving the corresponding Dirichlet problem

$$(dd^{\,c}u)^n = \mu,\ u \in Psh\,(\Omega)\,,\ u|_{\partial\Omega} = \varphi, \tag{1.1}$$

have a long history and are more complicated.

Bremermann [1] noted, that if $u \in C^2(\Omega) \bigcap Psh(\Omega)$ is maximal (an analogue of harmonic functions, see. Ch 2), then $(dd^c u)^n = 0$. Later Kerzman [17] proved that if $(dd^c u)^n = 0$ then u is maximal. Chern, Levine and Nirenberg [12] proved that the operator $(dd^c u)^n$ is bounded in the mean for the class of uniformly bounded Psh function of class C^2. We note that this proposition easily follows also from the integral formula below, proved by author [20, 21]: if $G = \{\rho(z) < 0\}$ is a strictly pseudoconvex domain, $\rho \in C^2(G)$, $\sigma = \min_G \rho(z)$ and u is a C^2 Psh function in G, then for each r and k such, that $\sigma < r < 0$, $1 \leqslant k \leqslant n$

$$\int_\sigma^r dt \int_{\rho(z)\leqslant t} (dd^c\rho)^{n-k} \wedge (dd^c u)^k \leqslant (M-m) \int_{\rho(z)\leqslant r} (dd^c\rho)^{n-k+1} \wedge (dd^c u)^{k-1}, \qquad (1.2)$$

where $M = \max_G u(z)$, $m = \min_G u(z)$.

Chern - Levine - Nirenberg's result impies that if $u \in Psh(\Omega) \cap L^\infty_{loc}(\Omega)$ and $u_j \downarrow u$ is a C^∞ approximation of u, then $\{(dd^c u_j)^n\}$ is a compact family of (positive) Borel measures and the sequence $\{(dd^c u_j)^n\}$ has partial weak limits ($\mapsto$) for subsequences $\{(dd^c u_{j_k})^n\}$. The main problem here is to prove that the sequence $\{(dd^c u_j)^n\}$ has a limit, which can be considered to be $(dd^c u)^n$.

In this way Bedford and Taylor [6] have defined the Monge-Ampère operator for $u \in Psh(\Omega) \cap L^\infty_{loc}(\Omega)$ by the recurrent relation as *current*:

$$\int (dd^c u)^k \wedge \varphi = \int u\,(dd^c u)^{k-1} \wedge dd^c\varphi\ ,\ \ \varphi \in D^{(n-k,n-k)}, k = 1,2,...,n-1.$$

Here the $D^{(n-k,n-k)}$ is the space of test forms, i.e. space of all finite differential forms of bi-degree $(n-k, n-k)$, with C^∞ coefficients. Moreover, later in [7] it was proved that $(dd^c u)^k$ defined as above is *well-defined*, i.e. $(dd^c u_j)^k \mapsto (dd^c u)^k$ for any approximation $u_j \downarrow u$.

But the definition of $(dd^c u)^n$ for an arbitrary $u \in Psh(\Omega)$ is still a hard problem. This is also an obstacle for solving the Dirichlet problem (1.1) in general case and for providing answers to many other questions .

Namely, firstly, in 1975 Shiffman and Taylor showed that there is $u \in Psh(\mathbb{C}^n)$ such that $\int (dd^c u_j)^n \to \infty$, where $u_j \downarrow u$. Kiselman [18] constructed a simple example of a function $u(z) = (-\ln|z_1|)^{1/n} \left(|z_2|^2 + ... + |z_n|^2 - 1\right)$, which is Psh near the origin, has this property and is such that its Monge-Ampère mass is unbounded near $z_1 = 0$.

Secondly, Cegrell [8] suggested the following example. For the Psh function $u(z) = \ln|z_1|^2 + ... + \ln|z_n|^2$ if we take the approximation $u_j(z) = \ln\left(|z_1...z_n|^2 + \frac{1}{j}\right) \downarrow u(z)$, then $(dd^c u_j)^n \mapsto 0$. On the other hand, if we take the approximation $v_j(z) = \ln\left(|z_1|^2 + \frac{1}{j}\right) + ... + \ln\left(|z_n|^2 + \frac{1}{j}\right) \downarrow u(z)$, then $(dd^c u_j)^n \mapsto n!4^n\delta_0$, where δ_0 is the Dirac measure. This example shows that for arbitrary Psh functions the operator$(dd^c u)^n$ cannot be well-defined by appoximations $u_j \downarrow u$.

There were some more attempts to define $(dd^c u)^n$ for arbitrary Psh functions.

1. Kiselman [18] proposed of following method for defining $(dd^c u)^n$: he considered a domain $\Omega \times \left\{w \in \mathbb{C} : |\mathrm{Im}\, w| < \frac{1}{2}\right\} \subset \mathbb{C}^{n+1}$ and the auxiliary function $F(z,w) =$

$(u(z) - \mathrm{Re}\, w)^+$. Then the measure $(dd^cF)^{n+1}$ is supported on the graph $\{(z,w) \in \mathbb{C}^{n+1} : \mathrm{Re}\, w = u(z)\}$ and its projection onto $\mathbb{C}^n_z$ is equal to $(dd^cu)^n$ for bounded functions. For an arbitrary Psh function u we denote this projection by $K(u)$. It is clear that $K(u)$ characterizes $(dd^cu)^n$ outside the singular set $S = \{u = -\infty\}$. For example, for $u(z) = \ln \|z\|^2$ we have $K(u) = 0$.

2. Another method of defining $(dd^cu)^n$ outside S was given by Błocki: for a bounded Psh function u it is clear that $e^{2u}dd^cu = 2e^u dd^ce^u - \frac{1}{2}dd^ce^{2u}$. Therefore, for unbounded functions we can define $B^p(U) = (dd^cu)^p$, $1 \leqslant p \leqslant n$, outside S using $e^{2pu}B^p(u) = (e^{2u}dd^cu)^p = (2e^u dd^ce^u - \frac{1}{2}dd^ce^{2u})^p$. Operators $(dd^cu)^p$ in this definition do not take into account the mass, supported on the singular set $S = \{u(z) = -\infty\}$.

These methods show that in $\Omega \setminus S$ the operator $(dd^cu)^p$ is always defined correctly.

However, for some class of unbounded Psh functions it is possible to well-define $(dd^cu)^n$.

3. In the paper [22] the author proved that for class

$$L^+ = \{u(z) \in Psh(\mathbb{C}^n) : \alpha + \ln\|z\| \leqslant u(z) \leqslant \beta + \ln\|z\|,\ \forall \|z\| \geqslant r\},$$

where r, α, β – const, the operators $(dd^cu)^k$ and $u(dd^cu)^{k-1}$, $0 < k \leqslant n$, are well-defined. It follows that if Psh function $u(z)$ is bounded in some sphere $S(0,r)$ (or some level of Psh function) then these operators are well-defined inside $B(0,r)$.

Similar results were also proved by Demailly [14], that if the singular set $S : \{u = -\infty\}$ of Psh function u is compact, then the Monge - Ampère operators $(dd^cu)^k$ are well-defined. Bedford [5] proved that if Psh function u has only "small degree" singularities, then $(dd^cu)^k$ are well-defined.

4. Cegrell [9] introduced several classes of Psh functions, where $(dd^cu)^n$ can be well-defined. Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex domain, i.e. there exists $v \in Psh(\Omega), v|_\Omega < 0$ and $\lim\limits_{z \to \partial\Omega} v(z) = 0$. We put

$$E_0(\Omega) = \left\{ u \in Psh(\Omega) \cap L^\infty(\Omega) : u|_\Omega \leqslant 0,\ \lim_{z \to \partial\Omega} u(z) = 0 \text{ and } \int_\Omega (\mathrm{dd^c}u)^n < \infty \right\},$$

$$F(\Omega) = \left\{ u \in Psh(\Omega) : \exists u_j \downarrow u, u_j \in E_0(\Omega) \text{ and } \sup \int_\Omega (dd^cu_j)^n < \infty, \right\} \tag{1.3}$$

$$E(\Omega) = \{u \in Psh(\Omega) : \text{the sequense } u_j \text{ exists locally}\}.$$

Cegrell prove that, for $u \in E(\Omega) \supset F(\Omega)$ it is possible to define $(dd^cu)^n$, using compactness of $(dd^cu_j)^n \mapsto \mu$.

In this way Błocki [3] introduced a class $\mathcal{D}$, where the Monge-Ampère operator can be well-defined.

Definition 1. We say that the operator $(dd^cu)^n$ is *well-defined* for a given $u \in Psh(\Omega)$ if in Ω there exist a Borel measure μ such that for each open set $U \subset\subset \Omega$ and $u_j \in Psh(U) \cap C^\infty(U), u_j \downarrow u$, we have $(dd^cu_j)^n \mapsto \mu$. In this case we put $(dd^cu)^n = \mu$ and the set of all such u we denote by $\mathcal{D}$.

Błocki proved a series of properties of $\mathcal{D}$. In particular, $u \in \mathcal{D}(\Omega) \Leftrightarrow (\mathrm{dd^c}u_j)^n$ is weakly bounded for each sequence $u_j \downarrow u$. Moreover, the class $\mathcal{D}$ is characterized by boundedness of currents $|u_j|^{n-p-2}\, du_j \wedge d^c u_j \wedge (dd^c u_j)^p \wedge \left(dd^c \|z\|^2\right)^{n-p-1}$, $p = 0, 1, ..., n-2,$ for a single sequence $u_j \downarrow u$. Then it also follows that these currents are bounded for all sequences $u_j \downarrow u$.

For hyperconvex domain $\Omega \subset\subset \mathbb{C}^n$ Cegrell's class $E(\Omega)$ is equal $\mathcal{D}(\Omega)$ and for each $u \in E(\Omega) = \mathcal{D}(\Omega)$, all currents $(dd^c u)^k$, $1 \leqslant k \leqslant n$, are also well-defined.

Other properties of the Cegrell's classes, in particular, the solution of the Dirichlet problem in these classes are given in a series of works of Cegrell, Coman, Guedj, Zeriahi, Kolodziej and others (see [11, 13, 16, 19]). The aim of this paper is to give a survey in this direction and continue the definition of currents $(dd^c u)^k, 1 \leqslant k \leqslant n,$ from Cegrell's class to a wider class of $Psh(\Omega)$.

2 Maximal Psh functions

Maximal Psh function is an analogue of harmonic functions. They possess the maximality condition of harmonic functions.

Definition 2 ([21]). We say that a function $u \in Psh(\Omega)$ is maximal in the domain Ω if the maximality principle holds, i.e. if $v \in psh(\Omega) : \varliminf_{z \to \partial\Omega} (u(z) - v(z)) \geqslant 0$, then $u(z) \geqslant v(z), \quad \forall z \in \Omega.$

The condition $\varliminf_{z \to \partial D} (u(z) - v(z)) \geqslant 0$ here means that for arbitrary fixed $\varepsilon > 0$ there exist a compact $K \subset \Omega$ such that $v(z) \leqslant u(z) + \varepsilon$ outside K. In particular, $v(z) = -\infty$ if $u(z) = -\infty$. Next proposition is convenient in applications

Proposition. The following statements are equivalent:

1) u - is maximal in Ω;

2) for any domain $G \subset\subset \Omega$ and for any function $v \in psh(G) : \varliminf_{z \to \partial G} (u(z) - v(z)) \geqslant 0$ implies $u(z) \geqslant v(z), \quad \forall z \in G$;

3) for any domain $G \subset\subset \Omega$ and for any function $v \in psh(\Omega) : u|_{\partial G} \geqslant v|_{\partial G}$ implies $u(z) \geqslant v(z), \quad \forall z \in G.$

Proof. The implication 3) $\Rightarrow$ 1) is clear. For implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) we note that the function

$$w(z) = \begin{cases} \max\{u(z), v(z)\}, & \text{if } z \in G \\ u(z), & \text{if } z \in \Omega \backslash G \end{cases} \tag{2.1}$$

is Psh in Ω and $\varliminf_{z \to \partial\Omega} (u(z) - w(z)) = 0$. Hence $u(z) \geqslant w(z), \quad \forall z \in \Omega$ and $u(z) \geqslant v(z), \quad \forall z \in G.$ □

In the class of bounded Psh functions, a function $u(z) \in Psh(\Omega) \cap L^\infty_{loc}(\Omega)$ is maximal if and only if $(dd^c u)^n = 0$. Moreover, the following comparison principle of Bedford-Taylor (see [6, 7]) is true: if $u, v \in Psh(\Omega) \cap L^\infty_{loc}(\Omega)$ and the set $F = \{z \in \Omega : \ u(z) < v(z)\} \subset\subset \Omega$, then

$$\int_F (dd^c u)^n \geqslant \int_F (dd^c v)^n \tag{2.2}$$

An example of unbounded maximal Psh function in complex space $\mathbb{C}^2$ is $u(z) = \ln|z_1|$

Theorem 1. *If for $u(z) \in Psh(\Omega)$ there exists a sequence $u_j(z) \in Psh(\Omega) \cap L^{\infty}_{loc}(\Omega)$ such that $u_j \downarrow u, \ (dd^c u_j)^n \mapsto 0$, then u is maximal. Conversely, if u is maximal, then there exists an approximation $\{u_j(z)\}$ such that*

$$u_j \in Psh(\Omega_j) \cap L^{\infty}_{loc}(\Omega_j), \ (dd^c u_j)^n = 0 \ , \ \Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega,$$

$$\Omega = \bigcup_{j=1}^{\infty} \Omega_j, \ \ u_j(z) \downarrow u(z) \ \ \forall z \in \Omega. \tag{2.3}$$

Proof. Let $u(z) \in Psh(\Omega)$ and let there exists a sequence $u_j(z) \in Psh(\Omega) \cap L^{\infty}_{loc}(\Omega)$ such that $u_j \downarrow u, \ (dd^c u_j)^n \mapsto 0$. Suppose to the contrary that u is not maximal . Then there exist a domain $G \subset\subset \Omega$ and a function $v \in psh(\Omega)$ such that $u(z) \geqslant v(z)$ in a neighborhood of ∂G, but $u(z^0) < v(z^0), \ z^0 \in G$.

We fix a small $\varepsilon > 0$ such that $u(z^0) + \varepsilon < v(z^0)$ and put

$$\delta = \frac{\varepsilon}{2 \cdot \max\left\{|z|^2 : \ z \in \overline{G}\right\}}.$$

Then the plurisubharmonic in Ω function $\tilde{v} = v + \delta|z|^2$ satisfies the condutions

$$u\left(z^0\right) + \varepsilon < \tilde{v}\left(z^0\right), \ u|_{\partial G} + \varepsilon \geqslant \tilde{v}|_{\partial G}. \tag{2.4}$$

We can choose $j_0 \in N$ so large that $u_{j_0}(z^0) + \varepsilon < \tilde{v}(z^0), \ j \geq j_0$. Since $u_j|_{\partial G} + \varepsilon \geqslant \tilde{v}|_{\partial G}$, then approximating u_j, v in a neighborhood of $\overline{G}$ by standard sequences $u_{k,j} \downarrow u_j, \ v_k \downarrow v, \ u_{k,j}, v_k \in C^{\infty}, \ k = 1, 2, ...$, and putting $\tilde{v}_k = v_k + \delta|z|^2$, by comparison princple we have

$$\int_F (dd^c u_{k,j})^n \geqslant \int_F (dd^c \tilde{v}_k)^n, \ \ F = \{z \in G : \ u_{k,j} + \varepsilon < \tilde{v}_k\} \subset\subset G. \tag{2.5}$$

We note that the set $E = \{u(z) + \varepsilon < \tilde{v}(z)\} \neq \oslash$ and the Lebesgue measure meas$E > 0$ by (2.4). Since $E = \bigcup_j E_j$, where $E_j = \{u_j(z) + \varepsilon < \tilde{v}(z)\}, \ E_j \subset E_{j+1}$, then $\lim_{j\to\infty} \text{meas} E_j = \text{meas} E$. By (2.5) we have

$$\int_{\overline{G}} (dd^c u_j)^n \geqslant \overline{\lim_{k\to\infty}} \int_{\overline{G}} (dd^c u_{k,j})^n \geqslant \overline{\lim_{k\to\infty}} \int_F (dd^c u_{k,j})^n \geqslant \overline{\lim_{k\to\infty}} \int_F (dd^c \tilde{v}_k)^n \geqslant$$

$$\geqslant \overline{\lim_{k\to\infty}} \int_{\{u_{k,j} < \tilde{v}\}} (dd^c \tilde{v}_k)^n \geqslant \delta^n \overline{\lim_{k\to\infty}} \int_{\{u_{k,j} < \tilde{v}\}} \left(dd^c |z|^2\right)^n \geqslant$$

$$\geqslant \delta^n \overline{\lim_{k\to\infty}} \text{meas}\{u_{k,j} < \tilde{v}\} = \delta^n \text{meas} E_j, \tag{2.6}$$

which if we let $j \to \infty$ gives

$$\underline{\lim}_{j\to\infty} \int\limits_{\overline{G}} (dd^c u_j)^n \geqslant \delta^n \text{meas} E > 0.$$

This is a contradiction to the claim $(dd^c u_j)^n \mapsto 0$.

Let now $u(z)$ be maximal. For a fixed domain $G \subset\subset \Omega$ with smooth boundary ∂G, we fix an approximation $w_j \downarrow u$, $w_j \in Psh(G') \cap C^\infty(G')$, $j = 1, 2, ...$, where $G \subset\subset G' \subset\subset \Omega$. It is well-known that the regularization $v_j^*(z) = \overline{\lim}_{\lambda\to z} v_j^*(\lambda)$ of

$$v_j(z) = \sup\left\{v(z) \in Psh(G) \cap C\left(\overline{G}\right) : v|_{\partial G} \leqslant w_j|_{\partial G}\right\} \tag{2.7}$$

is a bounded Psh function in G with vanishing Monge-Ampère operator, $(dd^c v_j)^n = 0$. Moreover, since u is maximal, we have $v_j(z) \downarrow u(z)$, $z \in G$.

Now it is not difficult to construct, by applying this process to an arbitrary compact $G \subset\subset \Omega$, a sequence of domains $\Omega_j \subset \Omega$ and approximations $u_j(z) \downarrow u(z)$ such that $u_j \in Psh(\Omega_j) \cap L^\infty_{loc}(\Omega_j)$, $(dd^c u_j)^n = 0$, where $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$, $\Omega = \bigcup\limits_{j=1}^{\infty} \Omega_j$. □

Remark 1. If the domain $G \subset\subset \Omega$ above is strongly pseudoconvex , then the upper envelop (2.7) is continuous in $\overline{G}$ by Bremermann-Walsh theorem. Since for an arbitrary domain $G \subset \Omega$ with smooth boundary ∂G the functions u_j are Psh in the neighborhood $G' \supset \overline{G}$, then the technique of Walsh allows also to prove continuity of $v_j^* :\ v_j^* \in Psh(G) \cap C(\overline{G})$. Therefore, the sequence $\{u_j(z)\}$ in (2.3) we can choose to be continuous: $u_j \in Psh(\Omega_j) \cap C(\Omega_j)$.

Remark 2. Theorem 1 was proved in [21] under the assumption, that the sequence $\{u_j(z)\}$ is continuous. Similar properties of maximal Psh functions considered also by Błocki [2] and Cegrell [10].

Remark 3. Theorem 1 shows that for given maximal function $u \in Psh(\Omega)$, locally, in a fixed ball $B \subset\subset \Omega$ there exist at least one sequence $u_j \downarrow u$, $u_j \in Psh(B) \cap C(B)$ such that $(dd^c u_j)^n \mapsto 0$. On the other hand, Błocki [4] showed that there is maximal function $u \in B$, for which $(dd^c \max\{u, j\})^n$ does not tends 0 (see Section 3).

3 One way of defining the Monge-Ampère operator

The Cegrell class $\mathcal{D} \subset Psh(\Omega)$ is the biggest class, where $(dd^c u)^n$ is well-defined, i.e. for any sequence $\{u_j\} \subset Psh(\Omega) \cap L^\infty_{loc}(\Omega)$, $u_j \downarrow u$ $(dd^c u_j)^n$ weekly tends to the unique limit T. Let now $u \in Psh(\Omega) \setminus D$. Then the set $\{T\}$ of all Borel measures $T = \lim\limits_{j\to\infty} (dd^c u_j)^n$, where $\{u_j\} \subset Psh(\Omega) \cap L^\infty_{loc}(\Omega)$, $u_j \downarrow u$ may contain many elements or may be empty (when $(dd^c u)^n$ is not compact). The problem of definition of Monge-Ampère operator is, generally speaking, consists of a reasonable selection of $T_u \in \{T\}$. That is,

a) the comparison principle (2.2) holds for T_u;

b) u is maximal if and only if $T_u = 0$;

c) T_u is minimal among $\{T\}$, i.e., if $T \in \{T\}$ is another measure, then $T_u - T$ is not positive.

1. We consider here one interesting example given by Błocki [4]. The function $u(z,w) = -2\sqrt{\ln|z|\ln|w|}$ is Psh in the unit disk U^2. Then u is maximal in $U^2 \setminus \{0,0\}$, but it is not maximal in all U^2. In $U^2 \setminus \{0,0\}$

$$\lim_{j\to\infty} (dd^c \max\{u,-j\})^n = T$$

exists and

$$T = \begin{cases} 0 & \text{if } zw \neq 0 \\ -\frac{2\pi dV_w}{|w|^2 \ln|w|} & \text{if } z = 0 \\ -\frac{2\pi dV_z}{|z|^2 \ln|w|} & \text{if } w = 0 \end{cases} \tag{3.1}$$

This example shows that $\lim\limits_{j\to\infty} (dd^c \max\{u,-j\})^n$ is not good for defining the $(dd^c u)^n$ (condition b) does not hold). We note that if $B \subset U^2$, $0 \in B$, is an open set , then $\lim\limits_{j\to\infty} \int\limits_B (dd^c \max\{u,-j\})^n = \infty$.

2. Let $u(z) \in Psh(\Omega)$ be an arbitrary Psh function in a domain $\Omega \subset^n$. We put $v = e^u$ and $u_a = \ln(v+a) = \ln(e^u + a)$, $a > 0$. Then $u_a \downarrow u$ as $a \downarrow 0$ and $v \in Psh(\Omega) \cap L^\infty_{loc}(\Omega)$. So the operators $(dd^c v)^p$ and $v(dd^c v)^p$, $dv \wedge d^c v \wedge (dd^c v)^{p-1}$ are correctly defined. We have

$$dd^c u_a = (v+a)^{-1}\left[dd^c v - (v+a)^{-1} dv \wedge d^c v\right],$$

$$(dd^c u_a)^p = (v+a)^{-p}\left[(dd^c v)^p - p(v+a)^{-1} dv \wedge d^c v \wedge (dd^c v)^{p-1}\right] =$$

$$= \frac{1}{(v+a)^{p+1}}\left[v(dd^c v)^p - p dv \wedge d^c v \wedge (dd^c v)^{p-1}\right] + \frac{a}{(v+a)^{p+1}}(dd^c v)^p = \omega_{1,a} + \omega_{2,a}.$$

The currents $\omega_{1,a}$ and $\omega_{2,a}$ are positive. This is clear for $\omega_{2,a}$. To prove it for $\omega_{1,a}$, we show that the current $\phi^p = v(dd^c v)^p - p dv \wedge d^c v \wedge (dd^c v)^{p-1}$ is positive. We take the standard approximation $u_j \downarrow u$ and put $v_j = e^{u_j}$. Then we have

$$\phi_j^p = v_j (dd^c v_j)^p - p dv_j \wedge d^c v_j \wedge (dd^c v_j)^{p-1} = e^{(p+1)u_j}(dd^c u_j + du_j \wedge d^c u_j)^p -$$

$$-pe^{(p+1)u_j} du_j \wedge d^c u_j \wedge (dd^c u_j + du_j \wedge d^c u_j)^{p-1} = e^{(p+1)u_j}(dd^c u_j)^p \geq 0.$$

It is clear that $\phi_j^p \mapsto \phi^p$ as $j \to \infty$. Thus $\phi^p \geqslant 0$. We put formally

$$\omega_1^p = \lim_{a\to 0} \omega_{1,a}^p = \frac{\phi^p}{v^{p+1}}. \tag{3.2}$$

The current ω_1^p characterizes $(dd^c u)^p$ completely outside a singular set S : $\{u(z) = -\infty\}$. If the current ϕ^p / v^{p+1} is locally bounded in Ω, i.e,

$$\int\limits_{K\setminus S} \frac{\phi^p \wedge \left(dd^c |z|^2\right)^{n-p}}{v^{p+1}} < \infty, \ \forall K \subset\subset \Omega,$$

then $\omega_1^p = \ \phi^p/v^{p+1}$ represents a current in Ω and we call it the *regular part* (the part outside S) of $(dd^c u)^p$. However, $\omega_1^n = \ \phi^n/v^{n+1}$ is not bounded near $z_1 = 0$ for Kiselman's example $u(z) = (-\ln|z_1|)^{1/n} \cdot \cdot \left(|z_2|^2 + ... + |z_n|^2 - 1\right)$. It follows that for some Psh functions ω_1^p may be unbounded near a singular set S. In this case it is not possible to define of $(dd^c u)^p$ as a current, i.e., $(dd^c u)^p$ is undefinable.

Definition 3. We say that $(dd^c u)^p$ is definable at a point $o \in \Omega$, if there exist a neighborhood U of o such that ω_1 bounded in U (then it is a current) and $\omega_{2,a}$ weekly tends to some current ω_2 as $a \to 0$.

We note that if $(dd^c u)^p$ is definable at a point $o \in \Omega$, then supp $\omega_2 \subset S$. ω_2 is called to be *singular part* (the part on S) of $(dd^c u)^p$.

We shall study the current $\omega_{2,a}$ and its limit. Since $v = e^u \in Psh(\Omega) \cap L^\infty_{loc}(\Omega)$ then $dd^c v = v(du \wedge d^c u + dd^c u)$ and $(dd^c v)^p = v^p \left[du \wedge d^c u \wedge (dd^c u)^{p-1} + (dd^c u)^p\right]$. Therefore we have

$$\omega_{2,a} = \frac{a}{(v+a)^{p+1}}(dd^c v)^p = \frac{av^{p-1}}{(v+a)^{p+1}} \ e^u \left[du \wedge d^c u + dd^c u\right] \wedge (dd^c u)^{p-1} =$$

$$= \frac{av^{p-1}}{(v+a)^{p+1}} d\left(e^u d^c u\right) \wedge (dd^c u)^{p-1}. \tag{3.3}$$

It is clear that

$$\left(dd^c e^{u/p}\right)^p = \frac{1}{p^{p+1}} e^u \left[du \wedge d^c u \wedge (dd^c u)^{p-1}\right] + \frac{1}{p^p} e^u (dd^c u)^p$$

$$e^{p/2}\left(dd^c e^{u/2p}\right)^p = \frac{1}{(2p)^{p+1}} e^u \left[du \wedge d^c u \wedge (dd^c u)^{p-1}\right] + \frac{1}{(2p)^{2p}} e^u (dd^c u)^p.$$

It follows that the $d(e^u d^c u) \wedge (dd^c u)^{p-1}$ represents a current of bi-degree (p,p). Moreover, it is well-defined, i.e. if $\{u_j\} \subset Psh(\Omega) \cap C^2(\Omega)$, $u_j \downarrow u$ then

$$d\left(e^{u_j} d^c u_j\right) \wedge (dd^c u_j)^{p-1} \mapsto d\left(e^u d^c u\right) \wedge (dd^c u)^{p-1}. \tag{3.4}$$

Now fix a $\alpha(z) \in C^\infty(\Omega)$, $B = \mathrm{supp}\alpha \subset\subset \Omega$. We can assume that $u < 0$ in B. Let $B_t = \{v < t\} \cap B$ and $\mu_\alpha(t) = \int\limits_{B_t} d\left(e^u d^c u\right) \wedge (dd^c u)^{p-1} \wedge \alpha(z) \left(dd^c |z|^2\right)^{n-p}$, $t > 0$.

We want to find

$$\lim_{a \to 0} \int\limits_{B_1} \frac{av^{p-1}}{(v+a)^{p+1}} d\left(e^u d^c u\right) \wedge (dd^c u)^{p-1} \wedge \alpha \left(dd^c |z|^2\right)^{n-p}. \tag{3.5}$$

For a C^2 to function u the integral in (3.5) is equal (see [15]) to

$$\int\limits_{B_1} \frac{av^{p-1}}{(v+a)^{p+1}} d\left(e^u d^c u\right) \wedge (dd^c u)^{p-1} \wedge \alpha \left(dd^c |z|^2\right)^{n-p} = \int\limits_0^1 \frac{at^{p-1}}{(t+a)^{p+1}} d\mu_\alpha(t). \tag{3.6}$$

Integrating by parts we have

$$\int_0^1 \frac{at^{p-1}}{(t+a)^{p+1}} d\mu_\alpha(t) = \frac{a\mu_\alpha(1)}{(1+a)^{p+1}} + a\int_0^1 \left[2 - \frac{a}{t+a}\right] \frac{t^{p-2}}{(t+a)^{p+1}} \mu_\alpha(t)dt$$

and

$$\int_{B_1} \frac{av^{p-1}}{(v+a)^{p+1}} d\left(e^u d^c u\right) \wedge \left(dd^c u\right)^{p-1} \wedge \alpha \left(dd^c |z|^2\right)^{n-p} =$$

$$\frac{a\mu_\alpha(1)}{(1+a)^{p+1}} + a\int_0^1 \left[2 - \frac{a}{t+a}\right] \frac{t^{p-2}}{(t+a)^{p+1}} \mu_\alpha(t)dt.$$

By (3.4) this relation is true for arbitrary $u \in Psh(\Omega)$.

Now we need the following

Lemma. *If the limit*

$$\lim_{t \to 0} \frac{\mu_\alpha(t)}{t} = A \tag{3.7}$$

exists, then the limit

$$\lim_{a \to 0} a \int_0^1 \frac{t^{p-2}}{(t+a)^{p+1}} \mu_\alpha(t)\, dt,$$

and consequently, limit (3.5) *also exist.*

Proof. It is clear that

$$\lim_{a \to 0} a \int_0^1 \frac{t^{p-1}}{(t+a)^{p+1}}\, dt = C = const,$$

so that exists

$$\lim_{a \to 0} a \int_0^1 \frac{t^{p-2}}{(t+a)^{p+1}} \mu_\alpha(t)\, dt = \lim_{a \to 0} a \int_0^1 \frac{t^{p-1}}{(t+a)^{p+1}} \left(A + O(t)\right) dt = AC.$$

□

We note that if the limit

$$\lim_{t \to 0} \frac{\int_{B_t} d\left(e^u d^c u\right) \wedge \left(dd^c u\right)^{p-1} \wedge \left(dd^c |z|^2\right)^{n-p}}{t} = A \tag{3.8}$$

exists for any $B \subset\subset \Omega$, then (3.7) exists for an arbitrary function $\alpha \in C^\infty(\Omega)$ such that supp $\alpha \subset\subset \Omega$. We arrive at the following statement.

Theorem 2. *If the Psh function u satisfies the condition* (3.8) *and ω_1 is a locally bounded current in Ω, then $(dd^c u)^p$ definable.*

4 Examples

1. $u=\ln|z|^2=\ln\left(|z_1|^2+...+|z_n|^2\right)$. Then $v=|z_1|^2+...+|z_n|^2$, $dd^cv=dz_1\wedge d\bar{z}_1+...+dz_n\wedge d\bar{z}_n$ and $dv\wedge d^cv=|z_1|^2dz_1\wedge d\bar{z}_1+...+|z_n|^2dz_n\wedge d\bar{z}_n$. It follows that for $p\leqslant n-1$

$$\omega_1^p=\frac{v\,(dd^cv)^p-pdv\wedge d^cv\wedge(dd^cv)^{p-1}}{|z|^{2(p+1)}}$$

and $\int\limits_{B(o,r)}\omega_1^p\wedge\left(dd^c|z|^2\right)^{n-p}\to 0$ as $r\to 0$. We note that

$$\omega_{2,a}^p=\frac{a}{(v+a)^{p+1}}(dd^cv)^p\mapsto 0$$

as $a\to 0$, so that $\omega_2^p=0$.

It is clear that $\omega_{1,a}^n=0$ and

$$\omega_{2,a}^n=const\frac{adV}{\left(|z|^2+a\right)^{n+1}},$$

where

$$\int\limits_{B(0,r)}\omega_{2,a}^n\mapsto 1$$

as $a\to 0$, i.e. $\omega_2^n=\delta_0$ (Dirac measure).

2. In $\mathbb{C}^n\{z_1,z_2,...,z_n\}$ let $u=\ln\left(|z_1|^2+|z_2|^2\right)$, $v=|z_1|^2+|z_2|^2$. Then $dd^cv=dz_1\wedge d\bar{z}_1+dz_2\wedge d\bar{z}_2$, $dv\wedge d^cv=|z_1|^2dz_1\wedge d\bar{z}_1+|z_2|^2dz_2\wedge d\bar{z}_2$.

For $p\geqslant 3$ we have $\omega_{1,a}^p=\omega_{2,a}^p=0$,i.e. $\omega_1^p=\omega_2^p=0$.

For $p=1$

$$\omega_{1,a}^1\mapsto\omega_1^1=\frac{|z_2|^2dz_1\wedge d\overline{z_1}+|z_1|^2dz_2\wedge d\overline{z_2}}{\left(|z_1|^2+|z_2|^2\right)^2}.$$

Therefore

$$\omega_1^1\wedge\left(dd^c|z|^2\right)^{n-1}=\frac{dV}{|z_1|^2+|z_2|^2}.$$

It is not difficult to see that

$$\omega_{2,a}^1\wedge\left(dd^c|z|^2\right)^{n-1}=\frac{adV}{\left(|z_1|^2+|z_2|^2\right)^2}\mapsto 0$$

as $a\to 0$ and $\omega_2^1=0$.

Finally, $\omega_1^2=0$ and

$$\omega_{2,a}^2\wedge\left(dd^c|z|^2\right)^{n-2}=\frac{adV}{\left(|z_1|^2+|z_2|^2+a\right)^3}\mapsto\delta_{\{z_1=0,z_2=0\}},$$

where $\delta_{\{z_1=0,z_2=0\}}$ is the Dirac measure supported on the plane $\{z_1=0,z_2=0\}\subset\mathbb{C}^n$.

3. $u = \ln |f(z)|^2$, where $f \in O(\mathbb{C}^n)$. Then $v = f\bar{f}$, $dv \wedge d^c v = |f|^2 df \wedge d\bar{f}$, $dd^c v = df \wedge d\bar{f}$. It is clear that $\omega_{1.a}^p + \omega_{2,a}^p = 0$ for $p \geqslant 2$. We have $\omega_{1,a}^1 = 0$ and

$$\omega_{2,a}^1 = \frac{a}{\left(|f|^2 + a\right)^2} df \wedge d\bar{f} \mapsto [Z_f],$$

where $[Z_f]$ is $(1,1)$ current corresponding to the analytic set $Z_f = \{f = 0\}$.

4. Let $u \in Psh(\Omega)$, $u < 0$ and let there exist $M < 0$ such that the set $\{u < M\} \subset\subset \Omega$, i.e. u has compact singularities. For $u \in Psh(\Omega) \cap C^2(\Omega)$,

$$\int\limits_{B_t} d\left(e^u d^c u\right) \wedge (dd^c u)^{p-1} \wedge \left(dd^c |z|^2\right)^{n-p} = \int\limits_{\partial B_t} e^u d^c u \wedge (dd^c u)^{p-1} \wedge \left(dd^c |z|^2\right)^{n-p} =$$

$$= t \int\limits_{\partial B_t} d^c u \wedge (dd^c u)^{p-1} \wedge \left(dd^c |z|^2\right)^{n-p}.$$

Therefore, for $s > t > 0$ we have

$$\int\limits_{\partial B_s} d^c u \wedge (dd^c u)^{p-1} \wedge \left(dd^c |z|^2\right)^{n-p} - \int\limits_{\partial B_t} d^c u \wedge (dd^c u)^{p-1} \wedge \left(dd^c |z|^2\right)^{n-p} =$$

$$= \int\limits_{B_s \setminus B_t} (dd^c u)^p \wedge \left(dd^c |z|^2\right)^{n-p}.$$

This equality shows that for our given function $u \in Psh(\Omega)$, $u < 0$ integral $\int\limits_{B_s \setminus S} (dd^c u)^p \wedge \left(dd^c |z|^2\right)^{n-p}$ is bounded and limit (2.8) exists. We note, that the current $(dd^c u)^p$ is definable.

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