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ON SELECTION OF INFINITELY DIFFERENTIABLE SOLUTIONS OF A CLASS OF PARTIALLY HYPOELLIPTIC EQUATIONS

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Abstract. In this paper the existence of a constant $\kappa_0 > 0$ is proved such that all solutions of a class of regular partially hypoelliptic (with respect to the hyperplane $x^{''} = (x_2, ..., x_n) = 0$ of the space E^n) equations $P(D)u = 0$ in the strip $\Omega_\kappa = \{(x_1, x^{''}) = (x_1, x_2, ..., x_n) \in E^n; |x_1| < \kappa\}$ are infinitely differentiable when $\kappa \geq \kappa_0$ and $D^\alpha u \in L_2(\Omega_\kappa)$ for all multi-indices $\alpha = (0, \alpha^{''}) = (0, \alpha_2, \cdots, \alpha_n)$ in the Newton polyhedron of the operator $P(D)\cdot$.

1 Introduction

After in 1950's in connection with a study of the regularity of a solution of the problem $P(D)u = 0$ in the space of generalized functions (distributions) L. Hörmander introduced the concept of a hypoelliptic differential equation all distributional solutions u of which are infinitely differentiable (see [13], [14]), a problem arose of finding additional assumptions on solutions u of more general, non-hypoelliptic equations ensuring that these solutions are infinitely differentiable.

In [8] L. Gårding and B. Malgrange, in [18] B. Malgrange, in [23] J. Peetre, in [6] L. Ehrenpreis, in [11] and [12] E.A. Gorin, in [7] J. Friberg and others introduced the concept of partially hypoelliptic equations $P(D)u = f$, all distributional solutions u of which with an infinitely differentiable right-hand side are infinitely differentiable under the a priori assumption that they are infinitely differentiable with respect to a certain group of the variables.

In [2] Ya.S. Bugrov constructed an example of a non-hypoelliptic equation, all solution of which in the half-space are infinitely differentiable provided they are square integrable in the half-space together with some of their derivatives.

In [3], [4] and [5] V.I. Burenkov considered the equation $P(D)u = f$ in the cylinder $\Omega = \Omega_l \times E^{n-l}$ where $0 \leq l < n$ and Ω_l is an open set in E^l (if $l = 0$ then $\Omega = E^n$) and f and all its derivatives are $l-$ locally square integrable on Ω, i.e. square integrable on $Q_l \times E^{n-l}$ for all compacts $Q_l \subset \Omega_l$ (if $l = 0$ square integrable on E^n). Necessary and sufficient conditions on P were found ensuring that all solutions u of this equation with any such f, which are $l-$ locally square integrable on Ω together with some of their derivatives, are of the same class as f (in particular they are infinitely differentiable).

The class of hypoelliptic by Burenkov operators is essentially wider than the class of hypoelliptic operators.

Since this paper directly adjoins the results in [2] and [3], we formulate these results in a suitable for us formulation. Let $E^n_+ = \{x \in E^n : x_n > 0\}$, and for $\delta > 0$ and $G \subset E^n_+$ $G_\delta = \{x \in G : \rho(x, \partial G) \geq \delta\}\cdot$ Let $m = (m_1, m_2, ..., m_n)$ be a vector with positive integer coordinate and $M = \{\alpha \neq 0 : 0 \leq \alpha_k \leq m_k\}$, $k = 1, ..., n-1$, and $P(D) = \sum\limits_{\alpha \in M} D^{2\alpha}\cdot$ Note that $P(D)$ is non - hypoelliptic differential operator.

Bugrov's Theorem (see [2]). *Let* $\sum\limits_{\alpha \in M} ||D^\alpha u||_{L_2(E^n_+)} < \infty$ *and* $P(D)u = 0\cdot$ *Then* $||D^\beta u||_{L_2((E^n_+)_\delta)} < \infty$ $\forall \delta > 0$, $\forall \beta \neq 0$. *In particular* $u \sim v \in C^\infty(E^n_+)$.

Let $0 \leq m \leq n$, $\Omega = \Omega_m \times E^{n-m}$, where Ω_m is any open set in $E^m\cdot$ Denote by Q_m the set of all paralleiepipeds $G = G_m \times E^{n-m}$, where $G_m = \{-\infty < a_k < x_k < b_k < \infty,\ k = 1, ..., m\}$; $\overline{G}_m \subset \Omega_m\cdot$ One say that $u \in [L_2]_m(\Omega)$, if $u \in L_2(G)$ for all $G \subset Q_m\cdot$

Denote by $[J^\infty_2]_m(\Omega)$ the set of all functions u such that $||D^\alpha u||_{[L_2]_m(G)} < \infty$ for all $G \subset Q_m$ and for all $\alpha \geq 0\cdot$ Note that if $u \in [J^\infty_2]_m(\Omega)$ then $u \sim v \in C^\infty(\Omega)\cdot$

Let $P(D)$ be an arbitrary linear differential operator with constant coefficients, $\mathcal{E}_m = \{\alpha; \alpha = (\alpha_1, ..., \alpha_m, 0, ..., 0) \geq 0\}$, $\mathcal{E}'_m = \{\alpha \in \mathcal{E}_m : \alpha \neq 0\}$ and let $[U_2]_m(\Omega)$ be the set of all functions u measurable on Ω and such that

$$||u||_{[U_2]_m(G)} = ||u||_{L_2(G)} + \sum_{\alpha \in \mathcal{E}'_m} ||P^{(\alpha)} u||_{L_2(G)} < \infty$$

for all $G \subset Q_m\cdot$

Burenkov's Theorem (see [3]). *The conditions* $u \in [U_2]_m(\Omega)$ *and* $P(D)u \in [J^\infty_2]_m(\Omega)$ *imply that* $u \in [J^\infty_2]_m(\Omega)$ *if and only if*

1) $P(\xi) \neq 0$ *for sufficiently large* $\xi \in R^n$

and

2)

$$\lim_{\xi \to \infty} \frac{P^{(\beta)}(\xi)}{P(\xi)} = 0 \ \ \forall \beta \in \mathcal{E}'_m\cdot$$

In this paper we consider a class of partially hypoelliptic (with respect to hyperplane $x'' = (x_2, ..., x_n) = 0$ of the space E^n) regular equations $P(D)u = 0$ and prove that all distributional solutions of such equations which belong to a certain weighted Sobolev space in a certain strip in E^n are infinitely differentiable. Namely we prove that there exists a number $\kappa > 0$ such that all solutions of equation $P(D)u = 0$ on $\Omega_\kappa = \{x \in E^n : |x_1| < \kappa\}$, satisfying conditions $D^{0, \alpha_2, ..., \alpha_n} u \in L_2(\Omega_\kappa)$ for all α; $\alpha_2 + ..., +\alpha_n \leq m = ordP$ are infinitely differentiable.

To state the problem and formulate the results we need some notation and definitions. We use the following standard notation: N denotes the set of all natural numbers, $N_0 = N \cup \{0\}$, $N^n_0 = N_0 \times ... \times N_0$ is the set of all $n-$ dimensional multi-indices, E^n and R^n are the n-dimensional Euclidean spaces of points (vectors)

$x = (x_1, ..., x_n)$ and $\xi = (\xi_1, ..., \xi_n)$ respectively. For $\xi \in R^n, x \in E^n$ and $\alpha \in N_0^n$ we put $|\xi| = \sqrt{\xi_1^2 + ... + \xi_n^2}$, $|x| = \sqrt{x_1^2 + ... + x_n^2}$, $|\alpha| = \alpha_1 + ... + \alpha_n$, $\xi^\alpha = \xi_1^{\alpha_1} ... \xi_n^{\alpha_n}$, $D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$, where $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$ $(j = 1, ..., n)$.

Let $A = \{\alpha^j = (\alpha_1^j, ..., \alpha_n^j)\}_1^M$ be a finite set of multi-indices in N_0^n. By the Newton polyhedron of the set A we mean the minimal convex polyhedron $\Re = \Re(A)$ in $R_+^n = \{\xi \in R^n; \xi_j \geq 0 \ (j = 1, ..., n)\}$ containing all points of A.

A polyhedron $\Re \subset R_+^n$ with vertices in N_0^n is said to be complete (see [20] or [21]) if $\Re$ has a vertex at the origin and one vertex (distinct from the origin) on each coordinate axis of N_0^n. A complete polyhedron $\Re$ is called regular (completely regular), if all coordinates of the outward normals of its noncoordinate $(n-1)$-dimensional faces are non-negative (positive) (see [24] and [16]).

Let $P(D) = P(D_1, ..., D_n) = \sum_\alpha \gamma_\alpha D^\alpha$ be a linear differential operator with constant coefficients and let $P(\xi) = P(\xi_1, ..., \xi_n) = \sum_\alpha \gamma_\alpha \xi^\alpha$ be its characteristic polynomial (the complete symbol). Here the sum goes over a finite set of multi-indices $(P) = \{\alpha \in N_0^n; \gamma_\alpha \neq 0\}$.

The Newton polyhedron $\Re = \Re(P)$ of the set $(P) \cup \{0\}$ is called the Newton or characteristic polyhedron of the operator $P(D)$ (the polynomial $P(\xi)$) (see [21] or [24]) and is denoted by $\Re(P)$.

An operator $P(D)$ (a polynomial $P(\xi)$) is called hypoelliptic (see [13] or [14], Definition 11.1.2 and Theorem 11.1.1) if the following equivalent conditions are satisfied:

1) if $u \in D'(\Omega)$ (Ω is an open set in E^n, $D'(\Omega)$ is the set of distributions defined in Ω) is a solution of the equation $P(D)u = 0$ then $u \in C^\infty(\Omega)$,

2) all solutions $u \in D' = D'(E^n)$ of the equation $P(D)u = f$ are infinitely differentiable (belong to $C^\infty = C^\infty(E^n)$) for all $f \in C^\infty$.

3) if $|\xi| \to \infty$, and $0 \neq \alpha \in N_0^n$ then

$$P^{(\alpha)}(\xi)/P(\xi) \equiv D^\alpha P(\xi)/P(\xi) \to 0.$$

An operator $P(D)$ is called partially hypoelliptic with respect to hyperplane $x'' = (x_2, ..., x_n) = 0$ of the space E^n (a polynomial $P(\xi)$ is called partially hypoelliptic with respect to $\xi'' = (\xi_2, ..., \xi_n)$ (see [8], or [14] Definition 11.2.4 and Theorem 11.2.3) when $P^{(\alpha)}(\xi)/P(\xi) \to 0$ if $0 \neq \alpha \in N_0^n$ and $|\xi''| \to \infty$ while $\xi' = \xi_1$ remain bounded.

Finally, a polynomial $P(\xi)$ is called **almost hypoelliptic** (see [15]) if for a constant $C > 0$

$$|P^{(\alpha)}(\xi)|/[1 + |P(\xi)|] \leq C \ \ \forall \xi \in R^n, \ \forall \alpha \in N_0^n.$$

It is known that the Newton polyhedron of hypoelliptic polynomial is completely regular (see [24] or [16]) and the Newton polyhedron of an almost hypoelliptic polynomial is regular (see [15]).

In [9] the following statement was proved. Let f and its derivatives be square integrable on E^n with a certain exponential weight. Then all solutions of the equation $P(D)u = f$, which are square integrable with the same weight, are also such that all their derivatives are square integrable with this weight, if and only if the operator $P(D)$ is almost hypoelliptic.

During the whole work even numbers m and m_2 $(m > m_2)$ are fixed and we denote by $\Re = \Re(m, m_2) \subset R^n_+$ the polyhedron with the vertices $(0, ..., 0)$, $(m, 0, ..., 0)$ $, ..., (0, ..., 0, m)$ and $(m, m_2, 0, ..., 0)$. It is easy to verify that $\Re$ is a regular (but not completely regular) polyhedron in R^n_+ which is bounded by the $(n-1)$-dimensional coordinate hyperplanes $\xi_j = 0$ $(j = 1, ..., n)$ and the $(n-1)$-dimensional hyperplanes

$$P_1 = \{\xi : \xi \in R^n, \Delta_1(\xi) \equiv \frac{m - m_2}{m^2}\xi_1 + \frac{1}{m}(\xi_2 + ... + \xi_n) = 1\},$$

$$P_2 = \{\xi : \xi \in R^n, \Delta_2(\xi) \equiv \xi_1 + \xi_3 + ... + \xi_n = m\}.$$

Throughout this paper the notation $\alpha \in \Re$ means that $\alpha \in \Re \cap N^n_0$.

We shall study a linear differential operator $P(D)$ with constant coefficients and with the Newton polyhedron $\Re = \Re(m, m_2)$, where the characteristic polynomial $P(\xi)$ is nondegenerate (regular) with respect to the polyhedron $\Re$ (see [22] or [21]). This means that there exist positive constants μ_1 and μ_2 such that

$$1 + |P(\xi)| \geq \mu_1 \sum_{\alpha \in \Re} |\, \xi^\alpha \,| \quad \forall \xi \in R^n \tag{1.1}$$

and for all $k = 0, ...$

$$|D^k_1 P(\xi)| \leq \mu_2[\, 1 + \sum_{(\alpha_1 + k,\, \alpha'') \in \Re} |\, \xi^\alpha \,|\,] \quad \forall \xi \in R^n. \tag{1.2}$$

One can easily see that the polynomial $P(\xi)$, satisfying conditions (1.1), (1.2) is almost hypoelliptic (see [15], Theorem 3) and partially hypoelliptic with respect to $\xi'' = (\xi_2, ..., \xi_n)$ (see [14], Theorem 11.2.3).

It also satisfies Condition 1) and Condition 2) with $m = 1$ of Burenkov's Theorem. This follows since each multi-index (α_1, α'') for which $(\alpha_1 + k, \alpha'') \in \Re(m, m_2)$ does not belong to $P_1 \cup P_2$, hence

$$\lim_{\xi \to \infty} \frac{1 + \sum\limits_{(\alpha_1 + k, \alpha'') \in \Re} |\xi^\alpha|}{\sum\limits_{\alpha \in \Re} |\xi^\alpha|} = 0.$$

A positive function k defined in R^n is said to be a tempered weight function (see [14], Definition 10.1.1) if there exist positive constants C and M such that

$$k(\xi + \eta) \leq C(1 + |\xi|)^M\, k(\eta) \quad \forall \xi, \eta \in R^n.$$

The set of all such functions k will be denoted by K.

Let $S = S(R^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions in R^n and let $S'(R^n)$ be the set of all complex-valued tempered distributions on R^n. For $k \in K$ by B_k denote the set of all distributions $u \in S'$ such that (see [14], Definition 10.1.6) the Fourier transform $F(u)$ is a function and

$$||u||^2_k \equiv ||u||^2_{B_k} = \int |k(\xi)F(u)(\xi)|^2 d\xi < \infty.$$

It is easily shown that if $k_0 \in K$ and $k_j(\xi) = k_0(\xi)(1+|\xi_1|^j)$ then $k_j \in K$ $(j = 1, 2, \cdots)$.

In the sequel we shall use the following statement which we present in a suitable for us formulation (where, for $\kappa > 0$, $\Omega_\kappa = \{(x_1, x^{''}) = (x_1, x_2, ..., x_n) \in E^n;\ |x_1| < \kappa\}$).

Gårding-Malgrange Theorem (see [14], Theorem 11.2.5). Let $P(D)$ be a partially hypoelliptic operator with respect to the hyperplane $x^{''} = (x_2, ..., x_n) = 0$, $B_k^{loc}(G) = \{u \in S'; u \in B_k(G^{'})\ \forall G^{'} \subset G\}$ and $k_0 \in K$. If $u \in B_{k_j}^{loc}(\Omega_\kappa)$ $(j = 0, 1, ...)$ is a solution of equation $P(D)u = 0$, then $u \in C^\infty(\Omega_\kappa)$.

2 Some numerical inequalities and weighted estimates for the derivatives of functions

In the sequel we will introduce some weight functions and weighted multi-anisotropic Sobolev spaces connected with the polyhedron $\Re = \Re(m, m_2)$ and the domain $\Omega_\kappa = \{(x_1, x^{''}) = (x_1, x_2, ..., x_n) \in E^n;\ |x_1| < \kappa\}$ for a given $\kappa > 0$. Namely:

a) as a weight function we consider a function $g(t)$ of one variable $t \in R^1$ such that

1) $g \in C^\infty(-1, 1)$,

2) $0 \le g(t) \le 1$, $g(-t) = g(t)$ for $t \in R^1$, and $g(t) = 0$ for $|t| \ge 1$. Let $\kappa > 0$ and $g_\kappa(t) = g(t/\kappa)$ then it is obvious that

3) $g_\kappa^{(l)}(t) \equiv D^l[g_\kappa(t)] = \kappa^{-l}\,(D^l g)_\kappa(t)$ for $t \in (-\kappa, \kappa)$ and for all $l = 0, 1, ...$.

Here is an example of such function: $g(t) = 1/(2p)!(1 - t^{2p})$ for $t \in (-1, 1)$ and $g(t) = 0$ for $|t| \ge 1$ for any $p \in N$.

b) Let $\Re'$ be the set of multi-indices $\alpha \in N_0^n$ such that $(\alpha_1, \alpha'') \in \Re$, $(\alpha_1 + 1, \alpha'') \notin \Re$. We introduce an integer-valued function $d(\alpha)$ with the domain $\Re \cap N_0^n$, which satisfies the following conditions:

1) $d(\alpha_1 \pm l, \alpha'') = d(\alpha) \pm l$ for any $l \in N, \alpha_1 - l \in N_0$,

2) $d(\alpha) < m$ for $\alpha \in \Re \setminus \Re'$

and

3) $d(\alpha) = m$ for $\alpha \in \Re'$.

To construct such a function let us first construct the $(n-1)-$ dimensional hyperplane P_3 which passes through points $(m, m_2, 0, ..., 0)$, $(0, m_2, 0, ..., 0)$, $(0, 0, m, 0, ..., 0), ..., (0, 0, ..., 0, m)$ of the polygon $\Re = \Re(m, m_2)$. The equation of this hyperplane is

$$P_3:\ \Delta_3(\xi) \equiv \frac{\xi_2}{m_2} + \frac{\xi_3}{m} + ... + \frac{\xi_n}{m} = 1.$$

Thus the set $\Re$ is representable as the union of the following two sets: $\Re = \aleph_1 \cup \aleph_2$, where $\aleph_1 = \{\alpha \in \Re; \Delta_3(\alpha) \le 1\}$ and $\aleph_2 = \Re \setminus \aleph_1$. Let $\aleph_1^{'} = \aleph_1 \cap \Re^{'}$ and $\aleph_2^{'} = \aleph_2 \cap \Re^{'}$. Note that for $\alpha \in \aleph_1^{'}$ either $\alpha \in P_2$ or the point $(\alpha_1 + 1, \alpha^{''})$ is outside of $\Re$.

Let $a \in R^1$ and $[a]$ be the integer part of a. Denote by $[a]^{'} = [a] = a$, if a is integer and $[a]^{'} = [a] + 1$ otherwise. Put for any $\alpha \in \Re$

$$d(\alpha) = \Delta_2(\alpha) \equiv \alpha_1 + \alpha_3 + ... + \alpha_n \quad \alpha \in \aleph_1, \tag{1.3}$$

$$d(\alpha) = \alpha_1 + [\Delta_4(\alpha)]' \quad \alpha \in \aleph_2, \tag{1.4}$$

where

$$\Delta_4(\alpha) = m\frac{|\alpha''| - m_2}{m - m_2}.$$

A simple calcultation gives that such a definition of function d is correct since the values of functon $d(\alpha)$ for the points $\alpha \in P_3 \cap N_0^n$ defined by different formulas (1.3) or (1.4) coincide.

Let us prove that the function $d(\alpha)$ defined by formulas (1.3), (1.4) satisfies the Conditions 1) - 3). Condition 1) follows immediately by the definition of the function $d(\alpha)$. We prove Conditions 2) and 3) first for $\alpha \in \aleph_1$. Condition 2) in this case follows immediately by the definition of the function $d(\alpha)$ and the definition of set $\aleph_1$ as well. To prove Condition 3) in this case it suffices to show that $\aleph_1' \subset P_2$. Let $\alpha \in \aleph_1'$, i.e. $\alpha \in \aleph_1 \subset \Re$ and $(\alpha_1 + 1, \alpha'') \notin \Re$ then $\Delta_2(\alpha) \leq m$ and $(\alpha_1 + 1) + \alpha_3 + ... + \alpha_n > m$, i.e. $m - 1 < \Delta_2(\alpha) \leq m$. Since the number $\Delta_2(\alpha)$ is integer we have that $\Delta_2(\alpha) = m$, that is $\alpha \in P_2$. Let now $\alpha \in \aleph_2$. Let us remark that in this case a point $\alpha \in \aleph_2'$ can be an interior point of $\Re$. So α can be an element of P_1 or not.

First note that for $\alpha \in P_1 \cap N_0^n$ the number $\Delta_4(\alpha)$ is integer. Indeed in this case

$$\alpha_1\frac{m - m_2}{m} + \alpha_2 + ... + \alpha_n = m,$$

therefore

$$\alpha_1\frac{m - m_2}{m} + \alpha_2 + ... + \alpha_n - m_2 = m - m_2,$$

and

$$\alpha_1(m - m_2) + m(|\alpha''| - m_2) = m(m - m_2),$$

i.e.

$$\alpha_1 + m\frac{|\alpha''| - m_2}{m - m_2} = m,$$

whence it follows that $\Delta_4(\alpha) + \alpha_1 = m$, hence, the number $\Delta_4(\alpha)$ is integer and Condition 3) is proved for the points $\alpha \in P_1 \cap N_0^n$.

Let $\alpha \in \aleph_2 \setminus P_1$ and the number $\sigma > 0$ be chosen in such a way that $\alpha(\sigma) \equiv (\alpha_1 + \sigma, \alpha'') \in \Re \cap P_1$. If $\sigma \in N$ then $\alpha(\sigma) \in P \cap N_0^n$, $\alpha \notin \aleph_2'$ and we conclude by the part already proved that $d(\alpha(\sigma)) = m$. On the other hand $d(\alpha) = \alpha_1 + [\Delta_4(\alpha)]' \leq \alpha_1 + [\Delta_4(\alpha)] + 1$, i.e. $d(\alpha) < m - [\sigma] + 1$.

If $[\sigma] \neq 0$ then firstly $(\alpha(\sigma), \alpha'') \notin \aleph_2'$ and secondly from this it follows that $d(\alpha) < m$ which proves Condition 2) in this case as well.

Let now $[\sigma] = 0$, i.e. $0 < \sigma < 1$. Then $(\alpha_1 + 1, \alpha'') \notin \Re$, and $\alpha \in \aleph_2'$, $\alpha_1 + \Delta_4(\alpha) + 1 > m$ by the definition of the set $\aleph_2'$, i.e. $\alpha_1 + \Delta_4(\alpha) > m - 1$.

On the other hand since $\alpha \in \aleph_2 \setminus P_2$ and $[a]' < a + 1$ we obtain $\alpha_1 + \Delta_4(\alpha) < m$ and $d(\alpha) < \alpha_1 + \Delta_4(\alpha) + 1 < m - 1$. Thus we get $m - 1 < d(\alpha) < m + 1$. Because the number $d(\alpha)$ is integer we have $d(\alpha) = m$ which proves that the function $d(\alpha)$ satisfies Conditions 1) – 3).

The following lemma is a generalization of Lemma 1.3 in [10] and Lemma 1.1 in [17].

Lemma 2.1. *Let* $M \geq 2$ *and* $\kappa_0 > 0$, $p(\kappa)$ *and* $a_j(\kappa)$ $(j = 1, ..., M)$ *be non - negative functions such that* $p(\kappa) < 1$ *for* $\kappa \geq \kappa_0$ *and*

$$(1 - p(\kappa))\, a_j(\kappa) \leq \frac{1}{2}\,(1 + p(\kappa))\, a_{j-1} + \frac{1}{2}\, a_{j+1}(\kappa) \quad (j = 1, ..., M)\cdot \tag{2.1}$$

Then there exist a number $\kappa_1 \geq \kappa_0$ *and functions* $\{\delta_j(\kappa)\}$ *bounded for* $\kappa \geq \kappa_1$ *and* $\{\sigma_j(\kappa)\}$ *such that* $\delta_j(\kappa) \to 0$ *as* $p(\kappa) \to 0$ *and for all* $\kappa \geq \kappa_1$

$$a_j(\kappa) \leq (\frac{j}{M} + \delta_j(\kappa))\, a_M(\kappa) + \sigma_j(\kappa)\, a_0(\kappa) \qquad j = 1, ..., M - 1. \tag{2.2}$$

In particular for some $\kappa_2 \geq \kappa_1$ *and* $\sigma_0 > 0$

$$a_j(\kappa) \leq a_M(\kappa) + \sigma_0\, a_0(\kappa), \qquad j = 1, ..., M - 1. \tag{2.3}$$

Proof. The proof is by induction on M. For $M = 2$ and $j = 1$ we have from (2.1)

$$a_1(\kappa) \leq \frac{1}{2(1 - p(\kappa))} a_2(\kappa) + \frac{1 + p(\kappa)}{2(1 - p(\kappa))} a_0(\kappa)\cdot$$

Given any $\kappa > 0$ we write

$$\delta_1(\kappa) = \frac{p(\kappa)}{2(1 - p(\kappa))}, \quad \sigma_1(\kappa) = \frac{1 + p(\kappa)}{2(1 - p(\kappa))}\cdot$$

These are bounded functions for $\kappa \geq \kappa_0$ such that $1/2 + \delta_1(\kappa) = 1/2(1 - p(\kappa))$ and $\delta_1(\kappa) \to 0$ as $p(\kappa) \to 0$. This proves inequality (2.2) for $M = 2$.

Let $l \geq 2\cdot$ Assuming that inequalities (2.2) hold for $M \leq l$, let us prove that they hold for $M = l + 1$.

From (2.1) for $M = l + 1, j = l$ and from (2.2) for $M = l$, $j = l - 1$ we have for any $\kappa \geq \kappa_0$

$$(1 - p(\kappa))\, a_l(\kappa) \leq \frac{1}{2}\,(1 + p(\kappa))\, a_{l-1} + \frac{1}{2}\, a_{l+1}(\kappa) \leq$$

$$\leq \frac{1}{2}\, a_M(\kappa) + \frac{1}{2}\,(1 + p(\kappa))\, \Big[\frac{l - 1}{l} + \delta_{l-1}(\kappa)\Big]\, a_l(\kappa) +$$

$$+ \Big[\frac{1}{2}\,(1 + p(\kappa))\, \sigma_{l-1}(\kappa)\Big]\, a_0(\kappa).$$

Transferring corresponding terms from the right-hand to the left-hand side and denoting

$$\delta_l^{'} = 1 - p(\kappa) - \frac{1}{2}\,(1 + p(\kappa))\, \Big[\frac{l - 1}{l} + \delta_{l-1}(\kappa)\Big]; \ \sigma_l^{'}(\kappa) = \frac{1}{2}\,(1 + p(\kappa))\, \sigma_{l-1}(\kappa)$$

we obtain

$$\delta_l^{'}\, a_l(\kappa) \le \frac{1}{2}\, a_M(\kappa) + \sigma_l^{'}(\kappa)\, a_0(\kappa).$$

Choose a number $\kappa_1 \ge \kappa_0$ such that $\delta_l^{'} \ge 1/4$ for all $\kappa \ge \kappa_1$. Then

$$a_l(\kappa) \le \frac{a_M(\kappa)}{2\delta_l^{'}(\kappa)} + 4\sigma_l^{'}(\kappa), \quad \kappa \ge \kappa_1,$$

which implies that for $\kappa \ge \kappa_1$

$$a_l(\kappa) \le \left[\frac{l}{M} + \delta_l(\kappa)\right] a_M(\kappa) + \sigma_l(\kappa)\, a_0(\kappa),$$

where

$$\delta_l(\kappa) = \frac{l}{2(l+1)}\, \delta_l^{'}(\kappa); \quad \sigma_l(\kappa) = 4\sigma_l^{'}(\kappa)$$

and a simple computation gives $\delta_l(\kappa) \to 0$ as $p(\kappa) \to 0$, i.e. we get one of inequalities (2.2) for $M = l+1$ and $j = l = M-1$. From this and by the inductive assumption it follows that inequalities (2.2) are proved for any $M \in N$.

Inequalities (2.3) follow by inequalities (2.1) and by already proved properties of the functions $\{\delta_j(\kappa)\}$ and $\{\sigma_j(\kappa)\}$. □

Let the polyhedron $\Re = \Re(m, m_2)$, the set $\Re^{'}$, the domain Ω_κ (for a given $\kappa > 0$), have the same meaning as in the introduction, $\alpha^{''} = (\alpha_2, \cdots, \alpha_n) \in N_0^{n-1}$ and $|\alpha^{''}| = \alpha_2 + \cdots + \alpha_n \le m$. Then $(0, \alpha^{''}) \in \Re$ and either $(0, \alpha^{''}) \in \Re^{'}$ or to each of such multi-index $\alpha^{''}$ corresponds a unique number $\alpha_1^{'} = \alpha_1^{'}(\alpha^{''}) \in N_0$ (which we shall call the limiting value of $\alpha^{''}$) such that $\alpha^{'} = (\alpha_1^{'}, \alpha^{''}) \in \Re^{'}$, or which is the same $\alpha^{'} \in \Re$, $(\alpha_1^{'}+1, \alpha^{''}) \notin \Re$. Also by the definition of the polyhedron $\Re$

a) $(j, \alpha^{''}) \in \Re$ for all $j = 0, 1, \cdots, \alpha_1^{'}$,

b) the polyhedron $\Re$ contains such and only such multi-indices $(j, \alpha^{''})$ for which $|\alpha^{''}| \le m$ and $j = 0, 1, \cdots, \alpha_1^{'}$.

Lemma 2.2 *Let $\alpha^{''} \in N_0^{n-1}$, $|\alpha^{''}| \le m$, $\alpha_1^{'} = \alpha_1^{'}(\alpha^{''})$ be corresponding limiting value of $\alpha^{''}$ and $\alpha^{'} = (\alpha_1^{'}, \alpha^{''})$. Then there exist positive numbers C and κ_1 such that for any $\kappa \ge \kappa_1$ and for all $\varphi \in C_0^\infty(\Omega_\kappa)$*

$$\sum_{j=0}^{\alpha_1^{'}} ||D^{(j,\,\alpha^{''})}\varphi \cdot g_\kappa^{d(j,\,\alpha^{''})}(x_1)||_{L_2(\Omega\kappa)} \le C \cdot [||D^{\alpha^{'}}\varphi \cdot g_\kappa^{m}(x_1)||_{L_2(\Omega\kappa)}||+$$

$$+\theta(\alpha^{''}) \cdot ||D^{\alpha^{''}}\varphi \cdot g_\kappa^{d(0,\,\alpha^{''})}(x_1)||_{L_2(\Omega\kappa)}], \tag{2.4}$$

where $||\cdot||_{L_2(\Omega\kappa)}$ has the usual meaning, $\theta(\alpha^{''}) = 0$ if $(0, \alpha^{''}) \in \Re$ and $\theta(\alpha^{''}) = 1$ otherwise.

Proof. Fist note that $\alpha_1^{'} = 0$ and $d(0, \alpha^{''}) = m$ for $(0, \alpha^{''}) \in \Re^{'}$ (see the proof of the properties of the function $d(\alpha)$). Therefore in this case inequality (2.4) turns into equality for $C = 1$ and $\theta(\alpha^{''}) = 0$. Thus we can assume that $(0, \alpha^{''}) \notin \Re^{'}$, i.e. $\alpha_1^{'} > 0$.

If $\alpha_1^{'} = 1$ then the sum in the left-hand side of (2.4) consists of two items. The item for $j = 0$ coincides with the second item of the right-hand side of (2.4) for $\theta(\alpha^{''}) = 1$ and the item for $j = 1$ coincides with the first item of the right-hand side of (2.4) for $C = 1$. This means that in case $\alpha_1^{'} = 1$ the inequality (2.4) is valid for any $C \geq 1$. Hereinafter we suppose that $\alpha_1^{'} \geq 2$.

Arguing as above we get estimates for the items corresponding to the values $j = 0$ and $j = \alpha_1^{'}$.

Thus we can assume that $\alpha_1^{'} \geq 2, \ 1 \leq j \leq \alpha_1^{'} - 1$.

Let us introduce the following notation $\alpha(j) = (j, \alpha^{''}), \ d_j = d(\alpha(j)) \ (j = 1, \cdots, \alpha_1^{'} - 1)$.

Integrating by parts in the variable x_1, applying Fubini's theorem and the property $g_\kappa(-x_1) = g_\kappa(x_1)$ of the weight function g we obtain for each $j : 1 \leq j \leq \alpha_1^{'} - 1$, for any $\kappa > 0$, and for all $\varphi \in C_0^\infty(\Omega_\kappa)$

$$||D^{\alpha(j)} \varphi g_\kappa^{d_j}||^2_{L_2(\Omega_\kappa)} = \int\limits_{\Omega_\kappa} |D^{\alpha(j)} \varphi(x)|^2 \ g_\kappa^{2d_j}(x_1) dx =$$

$$= \int\limits_{E^{n-1}} \int\limits_{-\kappa}^{\kappa} |i^{-|\alpha(j)|} \frac{\partial^{|\alpha(j)|} \varphi(x)}{\partial x_1^j \ \partial(x'')^{\alpha''}}|^2 \ g_\kappa^{2d_j}(x_1) dx =$$

$$= \int\limits_{E^{n-1}} \int\limits_{-\kappa}^{\kappa} \frac{\partial^{|\alpha(j)|} \varphi(x)}{\partial x_1^j \ \partial(x'')^{\alpha''}} \frac{\partial^{|\alpha(j)|} \overline{\varphi}(x)}{\partial x_1^j \partial(x'')^{\alpha''}} \ g_\kappa^{2d_j}(x_1) \ dx =$$

$$= - \int\limits_{E^{n-1}} \int\limits_{-\kappa}^{\kappa} \frac{\partial^{|\alpha(j)|-1} \varphi(x)}{\partial x_1^{j-1} \partial(x'')^{\alpha''}} \ \frac{\partial^{|\alpha(j)+1} \overline{\varphi}(x)}{\partial x_1^{j+1} \partial(x'')^{\alpha''}} \ g_\kappa^{2d_j}(x_1) \ dx +$$

$$+ \frac{2d_j}{\kappa} \int\limits_{E^{n-1}} \int\limits_{-\kappa}^{\kappa} \frac{\partial^{|\alpha(j)|-1} \varphi(x)}{\partial x_1^{(j-1)} \partial(x'')^{\alpha''}} \ \frac{\partial^{|\alpha(j)|} \overline{\varphi}(x)}{\partial x_1^j \partial(x'')^{\alpha''}} \ \ g_\kappa^{2d_j - 1}(x_1)(g^{'})_\kappa(x_1) dx \equiv$$

$$\equiv I_1 + I_2 \cdot \tag{2.5}$$

To evaluate I_1, we apply the property $2d_j = d_{j-1} + d_{j+1}$ of the function $d(\alpha)$ and the numerical inequality $|ab| \leq \frac{1}{2}(a^2 + b^2) \cdot$ We get

$$I_1 \leq \int\limits_{E^{n-1}} \int\limits_{-\kappa}^{\kappa} |D^{\alpha(j-1)} \varphi(x) \ g_\kappa^{d_{j-1}}(x_1)| \ |D^{\alpha(j+1)} \overline{\varphi}(x) \ g_\kappa^{d_{j+1}}(x_1)| dx \leq$$

$$\leq \frac{1}{2} \ [\ ||D^{\alpha(j-1)} \varphi \ g_\kappa^{d_{j-1}}||^2_{L_2(\Omega_\kappa)} + ||D^{\alpha(j+1)} \overline{\varphi} \ g_\kappa^{d_{j+1})}||^2_{L_2(\Omega_\kappa)} \ ]. \tag{2.6}$$

To evaluate I_2, note that $|\frac{x_1}{\kappa}| \leq 1$ for $x \in \Omega_\kappa$ and $d(\alpha) \leq m$ for $\alpha \in \Re$, therefore

$$|I_2| \leq \frac{2m}{\kappa} [\ ||D^{\alpha(j-1)} \varphi \ g_\kappa^{d_{j-1}}||^2_{L_2(\Omega_\kappa)} + ||D^{\alpha(j)} \varphi \ g_\kappa^{d_j}||^2_{L_2(\Omega_\kappa)} \ ] \cdot \tag{2.7}$$

From (2.6), (2.7) it follows that

$$||D^{\alpha(j)}\varphi\ g_\kappa^{d_j}||^2_{L_2(\Omega_\kappa)} \le \frac{1}{2}\,(1+\frac{4m}{\kappa})||D^{\alpha(j-1)}\varphi\ g_\kappa^{d_{j-1}}||^2_{L_2(\Omega_\kappa)}+$$

$$+\frac{1}{2}\,||D^{\alpha(j+1)}\varphi\ g_\kappa^{d_{j+1}}||^2_{L_2(\Omega_\kappa)}+\frac{2m}{\kappa}||D^{\alpha(j)}\varphi\ g_\kappa^{d_j}||^2_{L_2(\Omega_\kappa)}.$$

Hence it follows that

$$(1-\frac{2m}{\kappa})\,||D^{\alpha(j)}\varphi\ g_\kappa^{d_j}||^2_{L_2(\Omega_\kappa)} \le$$

$$\le \frac{1}{2}\,(1+\frac{4m}{\kappa})||D^{\alpha(j-1)}\varphi\ g_\kappa^{d_{j-1}}||^2_{L_2(\Omega_\kappa)}+\frac{1}{2}\,||D^{\alpha(j+1)}\varphi\ g_\kappa^{d_{j+1}}||^2_{L_2(\Omega_\kappa)}.$$

Application Lemma 2.1 and summing up the obtained inequalities in $j=1,\cdots,\alpha_1^{'}$ we get the required inequaliy (2.4). □

Corollary 2.1. *Applying here Lemma* 2.1 *for all* $\alpha^{''}\in N_0^{n-1}$, $|\alpha|\le m$ *and summing up corresponding inequalities* (2.4) *we get for any* $\kappa\ge\kappa_1\ge 2m$ *and for all* $\varphi\in C_0^\infty(\Omega_\kappa)$, *with a constant* $C>0$

$$\sum_{\alpha\in\Re}||D^\alpha\varphi\cdot g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa)} \le C[\sum_{\alpha\in\Re'}||D^\alpha\varphi\cdot g_\kappa^m||_{L_2(\Omega_\kappa)}+$$

$$+\sum_{|\alpha^{''}|\le m}\theta(\alpha^{''})||D^{\alpha^{''}}\varphi\cdot g_\kappa^{d(0,\,\alpha^{''})}||_{L_2(\Omega_\kappa)}]\cdot \tag{2.8}$$

Corollary 2.2. *By applying inequalities* (2.8), *Corollary* 1.3 *in* [16] *(see also Theorem* 2.3 *in* [19]*) and the Leibnitz formula we get for all* $\varphi\in C_0^\infty(\Omega_\kappa)$ *and for every* $\kappa\ge\kappa_2$ *with constants* $C_1>0$ *and* $\kappa_2\ge\kappa_1$

$$\sum_{\alpha\in\Re}||D^\alpha\varphi\, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa)} \le C[\sum_{\alpha\in\Re\,'}||D^\alpha(\varphi\, g_\kappa^m)||_{L_2(\Omega_\kappa)}+$$

$$+\sum_{|\alpha^{''}|\le m}||D^{\alpha^{''}}\varphi\, g_\kappa^{d(0,\,\alpha^{''})}||_{L_2(\Omega_\kappa)}]\quad \forall\varphi\in C_0^\infty(\Omega\kappa)\cdot \tag{2.9}$$

Lemma 2.3. *Let the symbol* $P(\xi)$ *of the operator* $P(D)$ *with Newton's polyhedron* $\Re=\Re(m,m_2)$ *satisfy conditions* (1.1), (1.2). *Then there exist positive numbers* κ_0 *and* C *such that for all* $\kappa\ge\kappa_0$ *and* $\varphi\in C_0^\infty(\Omega_\kappa)$,

$$\sum_{\alpha\in\Re}||D^\alpha\varphi\, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa)} \le C[||P(D)\varphi\, g_\kappa^m||_{L_2(\Omega_\kappa)}+$$

$$+\sum_{|\alpha^{''}|\le m}||D^{\alpha^{''}}\varphi\, g_\kappa^{d(0,\,\alpha^{''})}||_{L_2(\Omega_\kappa)}]\quad \forall\varphi\in C_0^\infty(\Omega_\kappa)\cdot \tag{2.10}$$

Proof. Let us choose number $\kappa_2 > 0$ such that inequalities (2.8) and (2.9) hold for any $\kappa \geq \kappa_2$. By applying the Parseval equality, estimate (2.9) and property (1.1) of the operaror P we obtain with a constant $C_1 = C_1(\kappa_2, \mu_1) > 0$ for any $\kappa \geq \kappa_2$

$$\sum_{\alpha \in \Re} ||D^{\alpha} \varphi\, g_{\kappa}^{d(\alpha)}||_{L_2(\Omega_\kappa)} \leq C_1[\, ||P(D)(\varphi\, g_{\kappa}^{m})||_{L_2(\Omega_\kappa)} + ||\varphi\, g_{\kappa}^{m}||_{L_2(\Omega_\kappa)} +$$

$$+ \sum_{|\alpha''| \leq m} ||D^{\alpha''} \varphi\, g_{\kappa}^{d(0,\, \alpha'')}||_{L_2(\Omega_\kappa)}\,] \quad \forall \varphi \in C_0^{\infty}(\Omega_\kappa). \tag{2.11}$$

It is obvious that it sufficies to estimate only the first term of the right-hand side of (2.11). For this purpose by applying the Leibnitz formula, properties 1) - 3) of the function $d(\alpha)$ (see Introduction), the estimate (1.2) and the Parseval equality, we obtain with a constant $C_2 > 0$ for any $\kappa \geq \kappa_2$ and for all $\varphi \in C_0^{\infty}(\Omega_\kappa)$,

$$||P(D)(\varphi\, g_{\kappa}^{m})||_{L_2(\Omega_\kappa)} \leq ||(P(D)\varphi)\, g_{\kappa}^{m})||_{L_2(\Omega_\kappa)} +$$

$$+ \sum_{j \geq 1} \frac{1}{j!} ||\, [P^{(j,\, 0'')}(D)\varphi]\, (D_1^j g_{\kappa}^{m})||_{L_2(\Omega_\kappa)} \leq ||\, [P(D)\varphi]\, g_{\kappa}^{m}||_{L_2(\Omega_\kappa)} +$$

$$+ C_2 \mu_2 \sum_{\beta \in (P); \beta_1 \geq 1} \sum_{j=1}^{\beta_1} (\frac{2}{\kappa})^j\, ||D^{(\beta_1 - j,\, \beta'')} \varphi\, g_{\kappa}^{m-j}||_{L_2(\Omega_\kappa)} \leq$$

$$\leq ||(P(D)\varphi)\, g_{\kappa}^{m})||_{L_2(\Omega_\kappa)} + \frac{2}{\kappa}\, C_2\, m\, \mu_2 \sum_{\beta \in (P)} ||D^{\beta} \varphi\, g_{\kappa}^{d(\beta)}||_{L_2(\Omega_\kappa)}.$$

Choose a number κ_0 such that $\kappa_0 > 2C_2 m \mu_2$.

Since $(P) \subset \Re$, we get the inequality (2.10) for any $\kappa \geq \kappa_0$ by transferring last term of this inequality from the right-hand to the left-hand side, dividing both parts by arising positive coefficient and applying inequality (2.11). □

For $k \in N_0$ by A_k denote the set of multi-indices $\alpha \in N_0^n$, for which $(\alpha_1 - k, \alpha'') \in \Re'$, and by $\Re_k$ Newton's polyhedron of set $\Re \bigcup A_k$. It is obvious that $\Re_0 = \Re$.

At last we prove the main result of this section.

Lemma 2.4. *Let the assumptions of Lemma* 2.3 *hold. Then for each* $k \in N_0$ *there exist numbers* $a_j > 0 \quad (j = 0, 1, \cdots, k)$ *such that for any* $\kappa \geq \kappa_0$

$$\sum_{\beta \in \Re_k} ||D^{\beta} \varphi\, g_{\kappa}^{d(\alpha)}||_{L_2(\Omega_\kappa)} \leq \sum_{j=0}^{k} a_j\, ||D_1^j (P(D)\varphi)\, g_{\kappa}^{m+j}||_{L_2(\Omega_\kappa)} +$$

$$+ a_{k+1} \sum_{|\alpha''| \leq m} ||D^{\alpha''} \varphi\, g_{\kappa}^{d(0,\, \alpha'')}||_{L_2(\Omega_\kappa)}] \quad \forall \varphi \in C_0^{\infty}(\Omega_\kappa). \tag{2.12}$$

Proof. The proof is by induction on $k\cdot$ Since $\Re_0 = \Re$, the inequality (2.12) follows from (2.10) for $k = 0$. Assuming that inequalities (2.12) hold for $k \le r$, let us prove that they hold for $k = r + 1\cdot$ Applying the Leibnitz formula we get

$$\sum_{\beta \in \Re_{r+1}} ||D^\beta \varphi\, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa)} = (\sum_{\beta \in \Re_{r+1} \setminus \Re_r} + \sum_{\beta \in \Re_r}) ||D^\beta \varphi\, g_\kappa^{d(\beta)}||_{L_2(\Omega_\kappa)} =$$

$$= \sum_{\alpha \in \Re} ||D^{(\alpha_1+r+1,\, \alpha'')} \varphi\, g_\kappa^{d(\alpha_1+r+1,\, \alpha'')}||_{L_2(\Omega_\kappa)} + \sum_{\beta \in \Re_r} ||D^\beta \varphi\, g_\kappa^{d(\beta)}||_{L_2(\Omega_\kappa)}\cdot \qquad (2.13)$$

By the inductive assumption inequality (2.12) holds for $k = r$, and by the definition of the function $d(\alpha)$ (see Condition 1) of the function $d(\alpha)$ in Introduction) $d(\alpha_1 + r + 1, \alpha'') = d(\alpha) + r + 1$. Therefore the second summand in the right-hand side of (2.13) is estimated by the right-hand side of (2.12) for $k = r$ and thereby by the right-hand side of (2.12) for $k = r + 1$.

Thus it suffices to evaluate the first summand in the right-hand side of (2.13). Applying once more the Leibnitz formula we obtain

$$\sum_{\alpha \in \Re} ||D^{(\alpha_1+r+1,\, \alpha'')} \varphi\, g_\kappa^{d(\alpha_1+r+1,\, \alpha'')}||_{L_2(\Omega_\kappa)} =$$

$$= \sum_{\alpha \in \Re;\, \alpha_1=0} ||D^{\alpha''} [D_1^{r+1} \varphi\, g_\kappa^{r+1}]\, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa)} +$$

$$+ \sum_{\alpha \in \Re;\, \alpha_1 \ge 1} ||D^{\alpha} [D_1^{r+1} \varphi\, g_\kappa^{r+1}]\, g_\kappa^{d(\alpha)} -$$

$$- \sum_{j=1}^{\alpha_1} C_{\alpha_1}^j [D^{(\alpha_1 - j + r+1,\, \alpha'')} \varphi]\, (D_1^j g_\kappa^{r+1})\, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa)} \le$$

$$\le \sum_{\alpha \in \Re} ||D^{\alpha} [D_1^{r+1} \varphi\, g_\kappa^{r+1}]\, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa)} +$$

$$+ \sum_{\alpha \in \Re;\, \alpha_1 \ge 1} \sum_{j=1}^{\alpha_1} C_{\alpha_1}^j ||[D^{(\alpha_1 - j + r+1,\, \alpha'')} \varphi]\, (D_1^j g_\kappa^{r+1})\, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa)}\cdot \qquad (2.14)$$

Let $l_j = max\{r + 1 - j, 0\} \quad (j = 1, \cdots, \alpha_1)$, then

a) $d(\alpha) + l_j \ge d(\alpha_1 - j + r + 1, \alpha'') \quad (j = 1, \cdots, \alpha_1)$

b) since $|g_\kappa(x_1)| \le 1$ and $|x_1|/\kappa \le 1$ for $x \in \Omega_\kappa$, hence with some constants $b_j > 0$

$$|D_1^j\; g_\kappa^{r+1}(x_1)| \le b_j\, g_\kappa^{l_j}(x_1); \;\; |x_1| \le \kappa, \;\; (j = 1, \cdots, \alpha_1)$$

$$|\, D_1^j\; g_\kappa^{r+1}(x_1)\, g_\kappa^{d(\alpha)}| \le b_j\, g_\kappa^{d(\alpha)+l_j}(x_1) \le b_j\, g_\kappa^{d(\alpha_1 - j + r+1, \alpha'')}(x_1),$$

where $\beta^j \equiv (\alpha_1 - j + r + 1, \alpha'') \in \Re_r \;\; (j = 1, \cdots, \alpha_1)\cdot$

Thus the second summand in the right-hand side of (2.14) is estimated by the left-hand side of (2.12) for $k = r$, which in turn, by the inductive assumption, is estimated by the right-hand side of (2.12) for $k = r + 1\cdot$

Since $D_1^{r+1}\varphi\, g_\kappa^{r+1} \in C_0^\infty(\Omega_\kappa)$, it follows from Lemma 2.3 that for the first summand in the right-hand side of (2.14) we get with a constant $C_1 > 0$

$$\sum_{\alpha \in \Re} ||D^\alpha [D_1^{r+1}\varphi\, g_\kappa^{r+1}]\, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa)} \leq C_1\, ||P(D)[D_1^{r+1}\varphi\, g_\kappa^{r+1}]\, g_\kappa^m||_{L_2(\Omega_\kappa)} +$$

$$+ \sum_{|\alpha''| \leq m} ||D^{\alpha''} [D_1^{r+1}\varphi\, g_\kappa^{r+1}]\, g_\kappa^{d(0,\, \alpha'')}||_{L_2(\Omega_\kappa)}. \tag{2.15}$$

Applying the generalized Leibnitz formula (see [14], Theorem 11.1.7) we obtain for the first component in the right-hand side of (2.15)

$$P(D)[D_1^{r+1}\varphi\, g_\kappa^{r+1}] = P(D)[D_1^{r+1}\varphi]\, g_\kappa^{r+1} + \sum_{l=1}^{m} \frac{1}{l!} P^{(l,\, 0'')}(D)[D_1^{r+1}\varphi]\, D_1^l g_\kappa^{r+1}.$$

From here we get with a constant $C_2 > 0$

$$|P(D)[D_1^{r+1}\varphi\, g_\kappa^{r+1}]| \leq |P(D)[D_1^{r+1}\varphi]\, g_\kappa^{r+1}| +$$

$$+ C_2 \sum_{l=1}^{m} (\frac{2}{\kappa})^l\ |P^{(l,0'')}(D)[D_1^{r+1}\varphi]\, D_1^l g_\kappa^{r+1}|.$$

From which it follows for every $\kappa > 2$

$$||P(D)[D_1^{r+1}\varphi\, g_\kappa^{r+1}]\, g_\kappa^m||_{L_2(\Omega_\kappa)} \leq ||P(D)[D_1^{r+1}\varphi] g_\kappa^{r+1+m}||_{L_2(\Omega_\kappa)} +$$

$$+ C_2\, \frac{2}{\kappa} \sum_{l=1}^{m} ||P^{(l,\, 0'')}(D)[D_1^{r+1}\varphi]\, g_\kappa^{r+1+m}||_{L_2(\Omega_\kappa)}. \tag{2.16}$$

Let

$$P^{(l,\, 0'')}(D) = \sum_{\nu \in (P)} \gamma_\nu^l\, D^{(\nu - l,\, \nu'')}\ \ (l = 1, \cdots, m).$$

Then

$$P^{(l,\, 0'')}(D)[D_1^{r+1}\varphi]\, g_\kappa^{r+1+m-l} = [\sum_{\nu \in (P)} \gamma_\nu^l\, D^{(\nu_1 + r + 1 - l,\, \nu'')}\varphi]\, g_\kappa^{r+1+m-l}.$$

Since $r+1+m-l \leq r+1+m-1$ for all $\nu \in \Re$ $(l = 1, \cdots, m)$ and $(r+m,\, \nu'') \in \Re_r$, it follows that $(r+1+m-l,\, \nu'') \in \Re_r$ for all $\nu \in \Re$ and $l = 1, \cdots, m$.

On the other hand since $0 \leq g_\kappa(x_1) \leq 1$ for $x \in \Omega_\kappa$, we see that $g_\kappa^{r+1+m-l}(x_1) \leq g_\kappa^{r+1+\nu_1-l}(x_1)$ for $x \in \Omega_\kappa$. Therefore we get from here with a constant $C_3 > 0$ being independent of r

$$\sum_{l=1}^{m} ||P^{(l,\, 0'')}(D)[D_1^{r+1}\varphi]\, g_\kappa^{r+1+m-l}||_{L_2(\Omega_\kappa)} \leq C_3 \sum_{\beta \in \Re_r} ||D^\beta \varphi\, g_\kappa^{d(\beta)}||_{L_2(\Omega_\kappa)}.$$

This means that the second summand in the right-hand side of (2.16) is estimated by the left-hand side of (2.12) for $k = r$, which in turn, by the inductive assumtion, is estimated by the right-hand side of (2.12).

The first component in the right-hand side of (2.16) coincides with last term of the first summand in the right-hand side of (2.12) for $k = r+1\cdot$ The result is that the first component in the right - hand side of (2.15) is estimated by the right - hand side of (2.12) for $k = r+1\cdot$

For the second summand in the right-hand side of (2.15) we have

$$\sum_{|\alpha''|\leq m} ||D^{\alpha''}[D_1^{r+1}\varphi\, g_\kappa^{r+1}]\, g_\kappa^{d(0,\,\alpha'')}||_{L_2(\Omega_\kappa)} \leq \sum_{|\alpha''|\leq m} ||D^{(r+1,\,\alpha'')}\varphi\, g_\kappa^{d(0,\,\alpha'')+r+1}||_{L_2(\Omega_\kappa)}\cdot$$

Since $m \geq 2$ and $d(0,\,\alpha'') + r + 1 = d(r+1,\,\alpha'')$, it follows $(r+1,\,\alpha'') \in \Re_r$ for all $\alpha'' \in N_0^n, |\alpha''| \leq m$ and therefore we get with a constant $C_4 > 0$

$$\sum_{|\alpha''|\leq m} ||D^{(r+1,\,\alpha'')}\varphi\, g_\kappa^{d(0,\,\alpha'')+r+1}||_{L_2(\Omega_\kappa)} \leq C_4 \sum_{\beta\,\in\,\Re_r} ||D^\beta\varphi\, g_\kappa^{d(\beta)}||_{L_2(\Omega_\kappa)}\cdot$$

By the inductive assumption the right-hand side of this inequality is estimated by the right-hand side of (2.12) for $k = r$. It follows from the last two inequalities that the second summand in the right-hand side of (2.15) is estimated by the right-hand side of (2.12) for $k = r+1$ as well. □

3 Function spaces and the main result

Let the functions $g(t), d(\alpha)$, the domain Ω_κ, and for each $k \in N_0$ the polyhedron $\Re_k$ have the same meaning as above. Denote by $H_k = H_k(\Re_k, g, d, \Omega_\kappa)$ the set of all function u locally integrable on Ω_κ, with finite norms

$$||u||_{H_k} \equiv \sum_{\alpha\in\Re_k} ||D^\alpha u\, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa)}\cdot \tag{3.1}$$

It is obvious that for any $k \in N_0$ and any functions $d(\alpha)$ and $g(t)$, satisfying stated above conditions, the set H_k with the norm (3.1) is complete normed space, coinciding with the weighted Sobolev space $W_{2,g}^m(\Omega_\kappa)$ for $m_2 = 0\cdot$ For $m_2 \neq 0$ the space H_k is often called multianisotropic weighted Sobolev space.

First we need some properties of spaces H_k.

Lemma 3.1. *For each $k \in N_0$ and any $\kappa > 0$*

a) the norm

$$||u||'_{H_k} = \sum_{\alpha\in\,\Re_k} ||D^\alpha[u g_\kappa^{d(\alpha)}]||_{L_2(\Omega_\kappa)} \tag{3.2}$$

is equivalent to the initial norm (3.1) *of the space H_k,*

b) the set $C_0^\infty(\Omega_\kappa)$ is dense in H_k,

c) H_k is semi - local (see [14], *Definition 10.1.18), i.e. if $\varphi \in C_0^\infty(\Omega_\kappa)$ and $u \in H_k$, then $\varphi u \in H_k\cdot$*

Proof. We start with part a). Applying the Leibnitz formula and property 1) of the function $d(\alpha)$, we get with a constant $C = C(k) > 0$

$$||u||^{'}_{H_k} \leq \sum_{\alpha \in \Re_k} \sum_{l=0}^{\alpha_1} C^l_{\alpha_1} ||(D^{(\alpha_1 - l,\, \alpha^{''})} u \, D^l g_\kappa^{d(\alpha)})||_{L_2(\Omega_\kappa)} \leq$$

$$\leq \sum_{\alpha \in \Re_k} \sum_{l=0}^{\alpha_1} C^l_{\alpha_1} \kappa^{-\,l} \frac{d(\alpha)!}{(d(\alpha) - l)!} ||(D^{(\alpha_1 - l,\, \alpha^{''})} u \, g_\kappa^{d(\alpha) - l})||_{L_2(\Omega_\kappa)} \leq$$

$$\leq C \sum_{\beta \in \Re_k} ||(D^\beta u) \, g_\kappa^{d(\beta)}||_{L_2(\Omega_\kappa)} = C \, ||u||_{H_k}, \tag{3.3}$$

where C does not depend on κ when $\kappa \geq 1$·

To prove the converse estimate first we show that for any multiindex $\alpha \in \Re_k$ there exists a number $C_1 = C_1(\alpha) > 0$ such that

$$||(D^\alpha u) \, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa)} \leq C_1 \sum_{l=0}^{\alpha_1} ||D^{(l,\, \alpha^{''})} (u \, g_\kappa^{d(l,\, \alpha^{''})})||_{L_2(\Omega_\kappa)}\cdot \tag{3.4}$$

Since g depends on only x_1, inequality (3.4) is hold for $\alpha_1 = 0$ and for any $C_1 \geq 1$. Let $\alpha_1 > 1$ and $\kappa \geq 1$. Applying once more the Leibnitz formula and property 1) of the function $d(\alpha)$, we get with a constant $C_2 = C_2(\alpha) > 0$

$$||(D^\alpha u) \, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa)} = ||(D^\alpha (u \, g_\kappa^{d(\alpha)}) - \sum_{l=0}^{\alpha_1} (D^{(\alpha_1 - l,\, \alpha^{''})} u) \, D^l g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa)} \leq$$

$$\leq ||D^\alpha (u \, g_\kappa^{d(\alpha)})||_{L_2(\Omega_\kappa)} + C_2 \sum_{l=0}^{\alpha_1 - 1} ||(D^{(l,\, \alpha^{''})} u) g_\kappa^{d(l,\, \alpha^{''})}||_{L_2(\Omega_\kappa)},$$

which means that estimate (3.4) for a multiindex $\alpha = (\alpha_1, \alpha^{''})$ will be proved once we prove it for the multiindex $(\alpha_1 - 1,\, \alpha^{''})$·

Continuing this process after $\alpha_1 - 1$ step we get estimate (3.4).

Summing up inequalities (3.4) on all $\alpha \in \Re_k$, we get with a constant $C_3 > 0$

$$||u||_{H_k} \leq C_3 \, ||u||^{'}_{H_k} \quad \forall u \in H_k\cdot$$

Taking into account (3.3) the last inequality proves part a).

For the proof of part *b)* we shall assume that the function $u \in H_k$ is fixed. Then by the definition of the improper Lebesgue integral for each $\varepsilon > 0$ there exist numbers $\delta \in (0, \kappa)$ and $M \geq 1$ such that

$$||u||_{\, H(\Re_k,\, g,\, \Omega_\kappa \setminus \Omega^M_{\kappa-\delta})} < \varepsilon, \tag{3.5}$$

where $\Omega^M_{\kappa-\delta} = \{x \in E^n, |x_1| < \kappa - \delta, |x_j| < M, \quad j = 2, \cdots, n\}$·

Let the numbers κ, δ and M be fixed . We construct nonnegative functions $\psi_{1,\,\delta} \in C_0^\infty(E^1)$ of variable $x_1 \in E^1$ and $\psi_2 \in C_0^\infty(E^{n-1})$ of variables $x^{''} = (x_2, \cdots, x_n) \in E^{n-1}$ such that

1) $\psi_{1,\delta}(x_1) = 1$ for $|x_1| < \kappa - \delta$, $\psi_{1,\delta} = 0$ for $|x_1| > \kappa - \delta/2$,

2) $\psi_2(x^{''}) = 1$ for $|x_j| < M$ $(j = 2, \cdots, n), \psi_2(x^{''}) = 0$ for $|x_j| \geq M + 1$ $(j = 2, ..., n)$, $\psi_2 \in C_0^\infty(E^{n-1})$,

3) for a number $b \geq 1$ and for all $x = (x_1, x^{''}) \in E^n$

$$\psi_{1,\delta}^{(j)}(x_1) \leq b\,\delta^{-j} \quad (j = 0, 1, \cdots, m); \ |D^{\alpha^{''}} \psi_2(x^{''})| \leq b \quad |\alpha^{''}| \leq m\cdot$$

It is obvious that such a function ψ_2 exist and satisfies Conditions 2), 3).

Let us construct the function $\psi_{1,\delta}$. Let χ_A be the characteristic function of set $A = A(\kappa, \delta) = \{|x_1| \leq \kappa - \frac{3}{4}\delta\}$ and $0 \leq \varphi \in C_0^\infty(-1, 1)$, $\int \varphi(x)dx = 1$, $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(\frac{x}{\varepsilon})$, put

$$\psi_{1,\delta}(x_1) = (\chi_A * \varphi_{\delta/4})(x_1) = \int\limits_{E^1} \chi_A(x_1 - t)\varphi_{\delta/4}(t)dt =$$

$$= \int\limits_{-\infty}^{\infty} \chi_A(z)\varphi_{\delta/4}(x_1 - z)dz\cdot \tag{3.6}$$

It is obvious that $\psi_{1,\delta} \in C_0^\infty(E^1)\cdot$ We show that $\psi_{1,\delta}$ satisfies condition 1).

Let $|x_1| \leq \kappa - \delta\cdot$ Because of $|t| \leq \delta/4$ and $|x_1 - t| \leq |x_1| + |t| \leq \kappa - \delta + \delta/4 = \kappa - \frac{3}{4}\delta$, then $\chi_A(x_1 - t) = 1$ and from (3.6) we have for $|x_1| \leq \kappa - \delta$

$$\psi_{1,\delta}(x_1) = \int\limits_{-\delta/4}^{\delta/4} \varphi_{\delta/4}(t)dt = \int\limits_{-\delta/4}^{\delta/4} (\delta/4)^{-1}\varphi(\frac{t}{\delta/4})dt = 1\cdot$$

Let $|x_1| \geq \kappa - \delta/2\cdot$ Then $|x_1 - t| \geq |x_1| - |t| > \kappa - \delta/2 - \delta/4 = \kappa - \frac{3}{4}\delta$, therefore $\chi_A(x_1 - t) = 0$ and it follows from (3.6) that $\psi_{1,\delta}(x_1) = 0\cdot$

Let us prove Property 3) of function $\psi_{1,\delta}\cdot$ From (3.6) and the definition of the function χ_A we have

$$\psi_{1,\delta}(x_1) = \int\limits_{-(\kappa - \frac{3}{4}\delta)}^{\kappa - \frac{3}{4}\delta} \varphi_{\delta/4}(x_1 - z)dz = (\frac{\delta}{4})^{-1} \int\limits_{-(\kappa - \frac{3}{4}\delta)}^{\kappa - \frac{3}{4}\delta} \varphi(\frac{x_1 - z}{\delta/4})dz\cdot$$

Therefore

$$\psi_{1,\delta}^{(j)}(x_1) = (\frac{\delta}{4})^{-1} \int\limits_{-(\kappa - \frac{3}{4}\delta)}^{\kappa - \frac{3}{4}\delta} D_{x_1}^j \varphi(\frac{x_1 - z}{\delta/4})dz =$$

$$= (\frac{\delta}{4})^{-j-1} \int\limits_{-(\kappa - \frac{3}{4}\delta)}^{\kappa - \frac{3}{4}\delta} (D_{x_1}^j \varphi)(\frac{x_1 - z}{\delta/4})dz = (\frac{\delta}{4})^{-j} \int\limits_{x_1 - (\kappa - \frac{3}{4}\delta)}^{x_1 + (\kappa - \frac{3}{4}\delta)} \varphi^{(j)}(t)dt\cdot$$

Then

$$|\psi^{(j)}_{1,\,\delta}(x_1)| \le (\frac{\delta}{4})^{-j} \int\limits_{-\infty}^{\infty} |\varphi^{(j)}(t)|dt \equiv C_j\,\delta^{-j} \quad (j = 0, 1, \cdots, m)\cdot$$

Denoting by b the maximum of the numbers $\{C_j\}$, we get the property 3) of the function $\psi_{1,\,\delta}\cdot$

After the construction of functions $\psi_{1,\,\delta}$ and ψ_2, we put $v(x) = u(x)\,\psi_{1,\,\delta}(x_1)\,\psi_2(x'')\cdot$ Then $supp\, v = \Omega^M_{\kappa-\delta/2}\cdot$

Henceforth it is assumed that for all $\alpha \in \Re_k$ the functions $D^\alpha u$ are continued by zero outside of Ω_κ. We denote by $D^\alpha u$ the continued functions too.

Since $v(x) = u(x)$ for $x \in \Omega^M_{\kappa-\delta}$ and $D^\alpha u \in L_2$ for $\alpha \in \Re_k$, we obtain by (3.5)

$$\sum_{\alpha\in\Re_k} ||(D^\alpha v - D^\alpha u)\, g_\kappa^{d(\alpha)}||_{\,L_2(E^n)} = \sum_{\alpha\in\Re_k} ||(D^\alpha v - D^\alpha u)\, g_\kappa^{d(\alpha)}||_{\,L_2(E^n\setminus\Omega^M_{\kappa-\delta})} \le$$

$$\le \sum_{\alpha\in\Re_k} [\ ||(D^\alpha v\, g_\kappa^{d(\alpha)}||_{\,L_2(E^n\setminus\Omega^M_{\kappa-\delta})} + ||(D^\alpha u\, g_\kappa^{d(\alpha)}||_{\,L_2(\Omega_\kappa\setminus\Omega^M_{\kappa-\delta})}\ ] \le$$

$$\le \sum_{\alpha\in\Re_k} ||D^\alpha(u(x)\,\psi_{1,\,\delta}(x_1)\,\psi_2(x''))\, g_\kappa^{d(\alpha)}||_{\,L_2(\Omega_\kappa\setminus\Omega^M_{\kappa-\delta})} + \varepsilon\cdot \qquad (3.7)$$

Since $g_\kappa(x_1) \le (2\delta)/\kappa$ for $x \in supp\,(\psi_{1,\,\delta}\psi_2) \cap (\Omega_\kappa \setminus \Omega^M_{\kappa-\delta})$ and $g_\kappa^{d(\alpha)} \le g_\kappa^{d(\beta)}$ for $\beta \le \alpha$ applying the Leibnitz formula and Properties 1) – 3) of the functions $\psi_{1,\,\delta}, \psi_2$ we obtain for the first part in the right-hand side of (3.7) with a constant $C_1 = C_1(\kappa) > 0$

$$\sum_{\alpha\in\Re_k} ||D^\alpha(u(x)\,\psi_{1,\,\delta}(x_1)\,\psi_2(x''))\, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa\setminus\Omega^M_{\kappa-\delta})} \le$$

$$\le \sum_{\alpha\in\Re_k}\sum_{\beta\le\alpha} C_\alpha^\beta\, ||D^\beta u D_1^{\alpha_1-\beta_1}\psi_{1,\,\delta} D_2^{\alpha_2-\beta_2}\cdots D_n^{\alpha_n-\beta_n}\psi_2\, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa\setminus\Omega^M_{\kappa-\delta})} \le$$

$$\le \sum_{\alpha\in\Re_k}\sum_{\beta\le\alpha} C_\alpha^\beta\, b^{|\alpha-\beta|}\,\delta^{-(\alpha_1-\beta_1)}\,(\frac{\delta}{\kappa})^{\,\alpha_1-\beta_1}\, ||D^\beta u\, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa\setminus\Omega^M_{\kappa-\delta})} \le$$

$$\le C_1 \sum_{\beta\in\Re_k} ||D^\beta u\, g_\kappa^{d(\beta)}||_{L_2(\Omega_\kappa\setminus\Omega^M_{\kappa-\delta})} \le C_1\,\varepsilon\cdot$$

From here and (3.7) we get

$$\sum_{\alpha\in\Re_k} ||(D^\alpha v - D^\alpha u)\, g_\kappa^{d(\alpha)}||_{L_2(E^n)} \le (C_1 + 1)\,\varepsilon\cdot \qquad (3.8)$$

Let $h > 0$, $S_h = \{x \in E^n; |\,x| < h\}$, $\chi \in C_0^\infty(S_1)$, $\chi(x) \ge 0$, $\int \chi(x)dx = 1$, $\chi_h(x) = h^{-2}\cdot\chi(x/h)$ and $v_h = v * \chi_h\cdot$

One can easily to see that $v_h \in C^\infty(E^n)$ for $h > 0$, where $v_h(x) = 0$ for $x \notin supp\ v \cup \overline{S_h}$. On the other hand since $supp\ v \cup \overline{S_h} \subset \Omega_\kappa$ for $h \in (0, \delta/4)$ we have $v_h \in C_0^\infty(\Omega_\kappa)$ for $h \in (0, \delta/4)\cdot$

Since $g_\kappa(x_1) \le 1$ and $u \in H_k$, we obtain $D^\alpha v \in L_2(E^n)$ for all $\alpha \in \Re_k$, where (see, for instance, [1] , 6.3.(2)) $D^\alpha(v_h) = (D^\alpha v)_h\cdot$ Then by Young's inequality and by the continuity in the mean of functions from L_2 we get

$$\sum_{\alpha\in \Re_k} ||D^\alpha(v_h - v)\, g_\kappa^{d(\alpha)}||_{L_2(E^n)} \le \sum_{\alpha\in \Re_k} ||D^\alpha(v_h - v)||_{L_2(E^n)} =$$

$$= \sum_{\alpha\in \Re_k} ||(D^\alpha v)_h - D^\alpha v||_{L_2(E^n)} \le \sum_{\alpha\in \Re_k} \sup_{|y|<h} ||D^\alpha v(x-y) - D^\alpha v(x)||_{L_2(E^n)} \to 0$$

as $h \to 0\cdot$

Because $\varepsilon > 0$ is arbitrary we get the proof of part b) of the Lemma from (3.5) - (3.8).

We obtain the proof of part c) if for any $\alpha \in \Re_k$ we denote by $\phi_\alpha(x) = \varphi(x)/g_\kappa^{d(\alpha)}(x_1)$ for $x \in supp\,\varphi$ and $\phi_\alpha(x) = 0$ for $x \notin supp\,\varphi$, and note that $\phi_\alpha \in C_0^\infty(E^n)$. □

Let $P(D)$ be a regular partially hypoelliptic (with respect to hyperplane $x^{''} = (x_2, ..., x_n) = 0$ of the space E^n) operator with Newton polyhedron $\Re = \Re(m, m_2)$, the symbol $P(\xi)$ of which satisfies conditions (1.1) - (1.2). Denote

$$N(P,\kappa) = \{u; D^{(0,\alpha^{''})} u \in L_2(\Omega_\kappa), |\alpha^{''}| \le m, P(D)u = 0 \text{ on } \Omega_\kappa\}.$$

Let $\varphi \in C_0^\infty(-1,1)$, $\varphi \ge 0$, $\int \varphi(t)dt = 1$, $h > 0$ and $\varphi_h(x) = h^{-1}\varphi(t/h)\cdot$ Arguing as above it is assumed that the functions u are continued by zero outside of Ω_κ and the continued functions we denote by u.

Next we put for any $h > 0$

$$u_h(x) = \int u(x_1 - y_1, x^{''})\varphi_h(y_1)dy_1\cdot$$

It is easy to verify that $D^\alpha u_h \in L_2$ for $\alpha \in \Re_k$ $(k = 0, 1, \cdots)$ and $D^{\alpha^{''}} u_h = (D^{\alpha^{''}} u)_h$ for $|\alpha^{''}| \le m\cdot$

Lemma 3.2. *Let $u \in N(P,\kappa)\cdot$ Then for any $l = 0, 1, \cdots$*

$$||(D_1^l P(D)u_h) g_\kappa^{m+l}||_{L_2(\Omega_\kappa)} \to 0 \tag{3.9}$$

as $h \to 0^+\cdot$

Proof. Since $P(D)u = 0$ for $u \in N(P,\kappa)$, we have that $D_1^l[P(D)u] = 0$ $(l = 0, 1, \cdots)$. Consequently $D_1^l P(D)u_h(x) = (D_1^l P(D)u)_h(x) = 0$ $(l = 0, 1, \cdots)$ for $x \in \Omega_{\kappa-h}$, (see [1] , 6.2.(2)). Therefore to prove the relation (3.9) it suffices to show that for any $l = 0, 1, \cdots$

$$||(D_1^l P(D)u_h) \cdot g_\kappa^{m+l}||_{L_2(\Omega_\kappa \setminus \Omega_{\kappa-h})} \to 0 \tag{3.10}$$

as $h \to +0\cdot$

Let

$$D_1^l P(D) = \sum_{\alpha \in (D_1^l P)} \gamma_\alpha^k D^\alpha$$

and

$$\gamma = max\{|\gamma_\alpha^l|, \ \alpha \in (D_1^l P)\}\cdot$$

Since $g_\kappa(x_1) \leq 2h/\kappa$ for $x \in \Omega_\kappa \backslash \Omega_{\kappa-h}$, by Young's inequality we obtain with a constant $C_l = C_l(\kappa) > 0$

$$||(D_1^l P(D) u_h)\, g_\kappa^{m+l}||_{L_2(\Omega_\kappa \backslash \Omega_{\kappa-h})} \leq (\frac{2h}{\kappa})^{m+l}\, ||D_1^l P(D) u_h||_{L_2(\Omega_\kappa \backslash \Omega_{\kappa-h})} \leq$$

$$\leq \gamma\, (\frac{2h}{\kappa})^{m+l} \sum_{\alpha \in (P)} ||D^{(\alpha_1+l, \alpha'')} u_h||_{L_2(\Omega_\kappa \backslash \Omega_{\kappa-h})} \leq$$

$$\leq \gamma\, (\frac{2h}{\kappa})^{m+l} \sum_{\alpha \in (\Re)} || \int_{E^{n-1}} (D^{\alpha''} u)(x_1 - y_1, x'')\, D_1^{\alpha_1+l} \varphi_h(y_1) dy_1 ||_{L_2(\Omega_\kappa \backslash \Omega_{\kappa-h})} \leq$$

$$\leq \gamma\, (\frac{2h}{\kappa})^{m+l} \sum_{\alpha \in \Re} ||D_1^{\alpha_1+l} \varphi_h||_{L_1} \sup_{y_1 < h} ||(D^{\alpha''} u)(x_1 - y_1, x'')||_{L_2(\Omega_\kappa \backslash \Omega_{\kappa-h})} \leq$$

$$\leq C_l (\frac{1}{h})^{m+l} (\frac{2h}{\kappa})^{m+l} \sum_{|\alpha''| \leq m} \sup_{y_1 < h} ||(D^{\alpha''} u)(x_1 - y_1, x'')||_{L_2(\Omega_\kappa \backslash \Omega_{\kappa-h})}\cdot \qquad (3.11)$$

By the definition of the set $N(P, \kappa)$ we have $D^{\alpha''} u \in L_2(E^n)$ for $|\alpha''| \leq m$. Therefore by Fubini's theorem we obtain

$$\omega_{\alpha''}(x_1) \equiv \int_{E^{n-1}} (D^{\alpha''} u)^2 (x) dx'' \in L_1(E^1); \quad \omega_{\alpha''}(x_1) = 0, \ |x_1| > \kappa$$

and by the continuity of the Lebesgue integral in measure we have for $h \to +0$, and $|\alpha''| \leq m$

$$\sup_{y_1 < h} ||(D^{\alpha''} u)(x_1 - y_1, x'')|| = \sup_{y_1 < h} ||\omega_{\alpha''}(x_1 - y_1)||_{L_1(\kappa - h < |x_1| < \kappa)}^{1/2} \to 0\cdot$$

Hence relation (3.10) is proved using (3.11). □

The main goal of this paper is to prove the following statement.

Theorem 3.1. *Let $\Re = \Re(m_1, m_2)$ be the Newton polyhedron of an operator $P(D)$ with the symbol $P(\xi)$ satisfying conditions* (1.1) - (1.2) *and the number $\kappa_0 > 0$ be chosen as in Lemma 2.3. Then*

a) $N(P, \kappa) \subset H(\Re_l, g, \Omega_\kappa)$ for any $\kappa \geq \kappa_0$ and $l = 0, 1, ...$

b) $N(P, \kappa) \subset C^\infty(\Omega_\kappa)$ for all $\kappa \geq \kappa_0$.

Proof. of the first part. Let $l \in N_0, \kappa \geq \kappa_0, u \in N(P,\kappa)\cdot$ We must prove that $u \in H(\Re_l, g, \Omega_\kappa)\cdot$ As above it is assumed that u being continued by zero outside of $\Omega_\kappa\cdot$

Let $h > 0$, $\varphi \in C_0^\infty(-1,1)$, $\int \varphi(t)dt = 1$, and $\varphi_h(t) = h^{-1}\varphi(t/h)\cdot$ We put

$$u_h(x) = u * \varphi_h = \frac{1}{h}\int_{E^1} u(x_1 - t, x'')\,\varphi(\frac{t}{h})dt,$$

Applying Lemma 2.4 and part b) of Lemma 3.1, we obtain

$$\sum_{\beta \in \Re_l} ||(D^\beta u_h) g_\kappa^{d(\beta)}||_{L_2(\Omega_\kappa)} \leq \sum_{j=0}^{l} a_j\, ||D_1^j(P(D)u_h) g_\kappa^{m+j}||_{L_2(\Omega_\kappa)} +$$

$$+ a_{l+1} \sum_{|\beta''| \leq m} ||D^{\beta''} u_h\, g_\kappa^{d(0,\beta'')}||_{L_2(\Omega_\kappa)}$$

for $\kappa \geq \kappa_0\cdot$ Let $\{h_k\}$ be an arbitrary infinitesimal sequence. From this inequality and by Lemma 3.2 we obtain that $||u_{h_p} - u_{h_s}||_{H_k} \to 0$, as $p, s \to \infty$ i.e. u_{h_k} is a Cauchy sequence in H_l for every $l \in N_0\cdot$ Since the space H_l is complete the sequence $\{h_k\}$ converges. It is obvious that in $L_2(\Omega_\kappa)$ the sequence u_{h_l} converges to initial function $u\cdot$ On the other hand, since the operator of generalized differentiation is closed (see [1], Lemma 6.2), $u_{h_k} \to u$ as $k \to \infty$ in H_k too, where $u \in H_k\cdot$ The part a) is proved.

To prove the second part of the Theorem we put $k_0(\xi) = 1 + |P(\xi)|$, $k_j(\xi) = k_0(\xi).(1 + |\xi_1|)^j$ $(j = 1, 2, ...)$. Since the operator $P(D)$ satisfies the conditions (1.1)-(1.2), it is easy to verify that $k_j(\xi)$ $(j = 0, 1, \cdots)$ are tempered weight functions. On the other hand since the operator $P(D)$ is partially hypoelliptic with respect to hyperplane $x'' = (x_2, ..., x_n) = 0$, and taking account the Gårding - Malgrange Theorem (see Introduction), in order to prove the second part it suffices to show that

$$N(P,\kappa) \subset B_{2,\,k_j}^{loc}(\Omega_\kappa),\ \ \kappa \geq \kappa_0,\ \ j = 0, 1, \cdots.$$

Let $\varphi \in C_0^\infty(\Omega_\kappa)$. In view of Parseval's equality and the point b) of Lemma 3.1 we have with positive constants $C_1 = C_1(\Re_j)$, $C_2 = C_2(\Re, \varphi)$

$$||u\,\varphi||_{B_{2,\,k_j}}(\Omega_\kappa) = ||(1 + |P(\xi)|)(1 + |\xi_1|)^j F(u\,\varphi)(\xi)||_{L_2(E^n)} \leq$$

$$\leq C_1\,||u \cdot \varphi||_{H_j} \leq C_2\,||u||'_{H_j}.$$

It follows from this and the part a) of Lemma 3.1 that $u\,\varphi \in B_{2,\,k_j}(\Omega_\kappa)$ for any function $\varphi \in C_0^\infty(\Omega_\kappa)$, i.e. $u \in B_{2,\,k_j}^{loc}(\Omega_\kappa)$ for any $\kappa \geq \kappa_0$ and $j = 0, 1, \cdots$ □

Remark. Burenkov's Theorem quoted in Introduction cannot be applied to proving that $N(P,\kappa) \subset C^\infty$ because $N(P,\kappa) \not\subset [U_2]_1(\Omega_\kappa)$.

Theorem 3.1 shows that a priori assumption $u \in [U_2]_m(\Omega)$ in Burenkov's Theorem can be weakened at least for the class of operators under consideration. An interesting question arises. Is it possible to further weaken the assumption $u \in [U_2]_m(\Omega)$? Can it be replaced just by $u \in [L_2]_m(\Omega)$?

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