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MINIMAX CONDITIONS FOR SCHATTEN IDEALS OF COMPACT OPERATORS

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Abstract. In this paper we study the validity of various types of minimax condition for operators in Schatten ideals of compact operators on separable Hilbert spaces.

1 Introduction

Let $f : X \times \Lambda \to \mathbb{R}$ be a real function on the product of non-empty sets X and Λ. The classical minimax problem is the problem to find suitable conditions on f that guarantee the validity of the equality

$$\inf_{x\in X}\left(\sup_{\lambda\in\Lambda} f(x,\lambda)\right) = \sup_{\lambda\in\Lambda}\left(\inf_{x\in X} f(x,\lambda)\right). \tag{1.1}$$

For an overview of the subject see [7]. Borenshtein and Shulman proved in [4] that if X is a compact metric space, Λ is a real interval and f is continuous, then (1.1) holds provided that, for each $x \in X$, the function $f(x,\cdot)$ is concave; and for each $\lambda \in \Lambda$, every local minimum of the function $f(\cdot,\lambda)$ is a global minimum. Some weaker conditions on f that ensure the validity of (1.1) were established by Saint Raymond in [9] and Ricceri in [8].

The minimax conditions has many interesting applications to the operator theory (see [1], [2], [4] and [6]). In [4], for example, the authors used (1.1) to prove Asplund-Ptak equality $\inf_{\lambda\in\mathbb{C}}\|A-\lambda B\| = \sup_{x\in\mathbf{B}(H)}\inf_{\lambda\in\mathbb{C}}\|Ax-\lambda Bx\|$, for operators A, B on a Hilbert space H and established that, for Banach spaces, it should be replaced by an inequality.

In this paper we study the validity of various types of minimax condition (1.1) for operators in Schatten ideals of compact operators. These minimax conditions are linked to the approximation of operators by finite-rank operators in the Schatten norms.

Let H be a separable Hilbert space. Let $B(H)$ be the C*-algebra of all bounded operators on H with operator norm $\|\cdot\|$ and let $C(H)$ be the closed ideal of all compact operators in $B(H)$. A two-sided ideal J of $B(H)$ is symmetrically normed (s. n.) if (see [5]) it is a Banach space in its own norm $\|\cdot\|_J$ and $\|AXB\|_J \le \|A\|\|X\|_J\|B\|$, for $A, B \in B(H)$ and $X \in J$. By Calkin theorem, all s. n. ideals lie in $C(H)$.

The most important class of s. n. ideals - the class of Schatten ideals - is defined in the following way (see § III.4 [5]). Let c_0 be the space of all sequences of real numbers converging to 0. Consider the following functions ϕ_p on c_0:

$$\phi_p(\xi) = \left(\sum_{i=1}^{\infty} |\xi_i|^p \right)^{1/p}, \text{ for } 1 \le p < \infty, \text{ where } \xi = (\xi_1, ..., \xi_n, ...) \in c_0.$$

For each $A \in C(H)$, the non-increasing sequence $s(A) = \{s_i(A)\}$ of all singular values of A, which are eigenvalues of the operator $(A^*A)^{1/2}$, belongs to c_0. For each $p \in [1, \infty)$, the set of compact operators

$$S^p = \{A \in C(H)\colon \phi_p(s(A)) < \infty\} \text{ with norm } \|A\|_p = \phi_p(s(A)) = \left(\sum_j s_j^p(A) \right)^{1/p} \tag{1.2}$$

is a Banach *-algebra and a symmetrically normed ideal of $B(H)$:

$$\|A^*\|_p = \|A\|_p \text{ and } \|BAC\|_p \le \|B\| \|A\|_p \|C\| \text{ for all } A \in S^p, \ B, C \in B(H). \tag{1.3}$$

The algebras S^p are called Schatten ideals. We will also write $S^\infty = C(H)$. Then

$$\|A\|_\infty = \|A\| = \sup s_j = s_1.$$

All S^p are separable algebras and the ideal of all finite rank operators in $B(H)$ is dense in each of them. Moreover,

$$S^q \subset S^p, \text{ if } q < p \le \infty, \text{ and } \|A\|_p \le \|A\|_q \text{ if } A \in S^q. \tag{1.4}$$

We extend the norms $\|\cdot\|_p$ to $B(H)$, by setting $\|A\|_p = \infty$, if $A \in B(H)$ and $A \notin S^p$. Thus

$$\|A\|_p < \infty \text{ if } A \in S^p, \text{ and } \|A\|_p = \infty \text{ if } A \notin S^p, \text{ for } p \in [1, \infty). \tag{1.5}$$

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of operators in $B(H)$. It converges to a bounded operator A in the weak operator topology (w.o.t), if

$$(A_n x, y) \to (Ax, y) \text{ for all } x, y \in H;$$

and in the strong operator topology, if

$$\|Ax - A_n x\| \to 0 \text{ for all } x \in H.$$

All Schatten ideals S^p, $p \in [1, \infty)$, share the following important property.

Theorem 1.1. [5, Theorem III.5.1]*Let $p \in [1, \infty)$ and let a sequence $\{A_n\}$ of operators from S^p converge to $A \in B(H)$ in* w.o.t. *If $\sup_n \|A_n\|_p = M < \infty$ for some $M > 0$, then $A \in S^p$ and $\|A\|_p \le M$.*

Theorem 1.1 implies the following result.

Corollary 1.1. *Let* $\{T_n\}$ *be a sequence of operators in* $B(H)$ *that converges to* $\mathbf{1}_H$ *in* s.o.t. *Let* $p \in [1, \infty)$ *and let* $A \in B(H)$. *The following conditions are equivalent.*

(i) *A belongs to* S^p.

(ii) *for some* $M_1 > 0$, *A satisfies*

$$\sup_n \|T_n A T_n\|_p = M_1 < \infty. \tag{1.6}$$

(iii) *for some* $M_2 > 0$, *A satisfies* $\sup_n \|T_n A\|_p = M_2 < \infty$.

Proof. By the uniform boundedness principle, there is $L > 0$ such that $\sup_n \|T_n\| \leq L$.

(i) $\to$ (iii). If $A \in S^p$ then, by (1.3), we have $\|T_n A\|_p \leq \|T_n\| \|A\|_p \leq L \|A\|_p$. Hence (iii) holds for $M_2 = L \|A\|_p$.

(iii) $\to$ (ii) follows from the fact that, by (1.3), $\|T_n A T_n\|_p \leq \|T_n A\|_p \|T_n\| \leq M_2 L$.

(ii) $\to$ (i). Let (1.6) hold. The sequence $\{T_n A T_n\}$ converges to A in s.o.t. Indeed, we have $\|z - T_n z\| \to 0$, as $n \to \infty$, for all $z \in H$. Hence, for each $x \in H$,

$$\begin{aligned} \|Ax - T_n A T_n x\| &\leq \|Ax - T_n A x\| + \|T_n A x - T_n A T_n x\| \\ &\leq \|Ax - T_n A x\| + \|T_n\| \|A\| \|x - T_n x\| \to 0, \text{ as } n \to \infty. \end{aligned}$$

Thus $\{T_n A T_n\}$ converges to A in w.o.t. As $\|T_n A T_n\|_p \leq M_1 < \infty$, all operators $T_n A T_n$ belong to S^p. It follows from Theorem 1.1 that $A \in S^p$ and $\|A\|_p \leq M_1$. □

Corollary 1.1 is partially stated in Theorem III.5.2 of [5] but only for *monotonically increasing* sequence of finite rank projections. We gave its proof here for the reader's convenience.

It should be noted that Corollary 1.1 does not hold for $p = \infty$, that is, for $S^\infty = C(H)$, since $\|\cdot\|_\infty$ coincides with the usual operator norm $\|\cdot\|$, so that (1.6) holds for all operators $A \in B(H)$ and not only for compact operators.

The following result shows that each ideal S^p, $p \in [1, \infty]$, (including $S^\infty = C(H)$) has an approximate identity.

Theorem 1.2. [5, Theorem III.6.3] *Let a sequence of bounded operators* $\{T_n\}$ *on H converge to* $\mathbf{1}_H$ *in* s.o.t. *Then* $\{T_n\}$ *is an approximate identity in all ideals* S^p, $p \in [1, \infty]$, *that is, for each* $A \in S^p$,

$$\|A - T_n A\|_p \to 0 \text{ and } \|A - T_n A T_n\|_p \to 0, \text{ as } n \to \infty. \tag{1.7}$$

Corollary 1.2. *Let a sequence of bounded operators* $\{T_n\}$ *on H converge to* $\mathbf{1}_H$ *in* s.o.t. *Suppose that* $\sup_n \|T_n\| \leq 1$. *Then, for each* $A \in B(H)$,

$$\sup_n \|T_n A T_n\|_p = \|A\|_p. \tag{1.8}$$

Proof. Let $A \in S^p$. Then, by (1.3), $\|T_n A T_n\|_p \leq \|T_n\| \|A\|_p \|T_n\| \leq \|A\|_p$. On the other hand, by (1.7), $\lim_{n\to\infty} \|T_n A T_n\|_p = \|A\|_p$. Hence $\sup_n \|T_n A T_n\|_p = \|A\|_p$.

Let $A \notin S^p$. Then, by (1.5), $\|A\|_p = \infty$. If $\sup_n \|T_n A T_n\|_p < \infty$ then it follows from Corollary 1.1 that $A \in S^p$, a contradiction. Hence $\sup_n \|T_n A T_n\|_p = \infty = \|A\|_p$. □

2 Some minimax condition for norms in S^p

Let $\{T_n\}$ be a sequence of bounded operators on H that converges to $\mathbf{1}_H$ in the s.o.t. For each $A \in B(H)$, consider the function $f_A(p,n) = \|T_nAT_n\|_p$, for $n \in \mathbb{N}$ and $p \in [1,\infty)$. In this section we will show that f_A satisfies the following minimax condition:

$$\inf_{p\in[1,\infty)} \sup_n f_A(p,n) = \sup_n \inf_{p\in[1,\infty)} f_A(p,n)$$

in the following two cases:

1) when $A \in \cup_{p\in[1,\infty)} S^p$,

2) when $A \notin \cup_{p\in[1,\infty)} S^p$ and $T_kAT_k \notin \cup_{p\in[1,\infty)} S^p$ for some k.

We will also show that the inverse minimax condition

$$\inf_n \sup_{p\in[1,\infty)} f_A(p,n) = \sup_{p\in[1,\infty)} \inf_n f_A(p,n)$$

holds for all operators A in $B(H)$. The following lemma contains simple norm equalities.

Lemma 2.1. *Let $A \in S^q$, for some $q \in [1,\infty)$. Then*

$$\lim_{q\le p\to\infty} \|A\|_p = \|A\|. \tag{2.1}$$

Let a sequence of bounded operators $\{T_n\}$ on H converge to $\mathbf{1}_H$ in the s.o.t. *If a sequence $\{p_n\}$ in $[q,\infty)$ satisfies $\lim_{n\to\infty} p_n = \infty$, then*

$$\lim_{n\to\infty} \|T_nAT_n\|_{p_n} = \|A\|. \tag{2.2}$$

Proof. Let $\{s_j\}$ be the non-increasing sequence of all singular values of A (the eigenvalues of the operator $(A^*A)^{1/2}$). Set $\alpha_j = \frac{s_j}{s_1}$. Then all $\alpha_j \le 1$. As $A \in S^q$, the series $\sum_{j=1}^\infty s_j^q = s_1^q \sum_{j=1}^\infty \alpha_j^q$ converges. Hence we can find N such that $\sum_{j=N}^\infty \alpha_j^q < 1$, so that $\sum_{j=N}^\infty \alpha_j^p \le \sum_{j=N}^\infty \alpha_j^q < 1$, for all $p > q$. Therefore,

$$\sum_{j=1}^\infty s_j^p = s_1^p \sum_{j=1}^\infty \alpha_j^p = s_1^p \sum_{j=1}^{N-1} \alpha_j^p + s_1^p \sum_{j=N}^\infty \alpha_j^p \le s_1^p(N-1) + s_1^p = Ns_1^p.$$

Thus

$$s_1 \le \left(\sum_{j=1}^\infty s_j^p\right)^{1/p} = \|A\|_p \le s_1 N^{1/p} \to s_1, \text{ as } p\to\infty.$$

Hence $\lim_{p\to\infty} \|A\|_p = s_1 = \left\|(A^*A)^{1/2}\right\| = \|A\|$ which completes the proof of (2.1).

As all $p_n \ge q$, it follows from (1.4) that A belongs to all S^{p_n} and

$$\left|\|A\| - \|T_nAT_n\|_{p_n}\right| \le \left|\|A\| - \|A\|_{p_n}\right| + \left|\|A\|_{p_n} - \|T_nAT_n\|_{p_n}\right|.$$

By (2.1), $\lim_{n\to\infty} \left|\|A\| - \|A\|_{p_n}\right| = 0$, as $\lim_{n\to\infty} p_n = \infty$. We also have that

$$\left|\|A\|_{p_n} - \|T_nAT_n\|_{p_n}\right| \le \|A - T_nAT_n\|_{p_n} \overset{(1.4)}{\le} \|A - T_nAT_n\|_q.$$

By Theorem 1.2, $\|A - T_nAT_n\|_q \to 0$, as $n\to\infty$. Thus (2.2) holds. $\square$

In the following proposition we evaluate

$$\inf_{p\in[1,\infty)}\left(\sup_n \|T_nAT_n\|_p\right) \text{ and } \sup_n \inf_{p\in[1,\infty)} \|T_nAT_n\|_p .$$

Proposition 2.1. *Let $A \in B(H)$. Let a sequence of bounded operators $\{T_n\}$ on H converge to $\mathbf{1}_H$ in* s.o.t. *Suppose that* $\sup_n \|T_n\| \leq 1$.

(i) *If $A \in \cup_{p\in[1,\infty)}S^p$ then* $\inf_{p\in[1,\infty)}\left(\sup_n \|T_nAT_n\|_p\right) = \|A\|$.

(ii) *If $A \notin \cup_{p\in[1,\infty)}S^p$ then* $\inf_{p\in[1,\infty)}\left(\sup_n \|T_nAT_n\|_p\right) = \infty$.

(iii) *If $T_nAT_n \in \cup_{p\in[1,\infty)}S^p$, for each n (for example, all T_n are finite rank operators, or A belongs to some S^q), then*

$$\sup_n \inf_{p\in[1,\infty)} \|T_nAT_n\|_p = \|A\| . \tag{2.3}$$

(iv) *If, for some k, $T_kAT_k \notin \cup_{p\in[1,\infty)}S^p$ then* $\sup_n \inf_{p\in[1,\infty)} \|T_nAT_n\|_p = \infty$.

Proof. As $\sup_n \|T_n\| \leq 1$, it follows from (1.8) that, for each $A \in B(H)$, we have

$$\inf_{p\in[1,\infty)}\left(\sup_n \|T_nAT_n\|_p\right) = \inf_{p\in[1,\infty)} \|A\|_p .$$

(i) If $A \in S^q$, for some $q \in [1,\infty)$, then, taking into account (1.4), we have

$$\inf_{p\in[1,\infty)} \|A\|_p = \lim_{q\leq p\to\infty} \|A\|_p \overset{(2.1)}{=} \|A\|$$

which completes the proof of (i).

(ii) If $A \notin S^p$, for all $p \in [1,\infty)$, then $\|A\|_p = \infty$ and we have $\inf_{p\in[1,\infty)} \|A\|_p = \infty$ which proves (ii).

(iii) Fix n. Then T_nAT_n belongs to some $S^{q(n)}$. Hence $T_nAT_n \in S^p$, for all $p \geq q(n)$, and

$$\inf_{p\in[1,\infty)} \|T_nAT_n\|_p = \inf_{q(n)\leq p\in[1,\infty)} \|T_nAT_n\|_p \overset{(1.4)}{=} \lim_{p\to\infty} \|T_nAT_n\|_p \overset{(2.1)}{=} \|T_nAT_n\| .$$

Therefore

$$\sup_n \inf_{p\in[1,\infty)} \|T_nAT_n\|_p = \sup_n \|T_nAT_n\| \leq \sup_n \|T_n\|\,\|A\|\,\|T_n\| = \|A\| .$$

Thus in order to prove (2.3) it suffices to show that $\|A\| = \lim_{n\to\infty} \|T_nAT_n\|$.

Given $\varepsilon > 0$, we can find $x \in H$ such that $\|x\| = 1$ and $0 \leq \|A\| - \|Ax\| < \varepsilon$. Then, as $T_n \to \mathbf{1}_H$ in the s.o.t., we have

$$\begin{aligned}\|T_nAT_nx - Ax\| &\leq \|T_nA(T_nx - x)\| + \|T_nAx - Ax\| \\ &\leq \|T_n\|\,\|A\|\,\|T_nx - x\| + \|T_nAx - Ax\| \to 0, \text{ as } n \to \infty.\end{aligned}$$

Choose $N \in \mathbb{N}$ such that $\|T_n A T_n x - Ax\| < \varepsilon$, for all $n > N$. Then, as $\|T_n A T_n x\| \leq \|T_n A T_n\| \leq \|A\|$, we have

$$0 \leq \|A\| - \|T_n A T_n\| \leq \|A\| - \|T_n A T_n x\| \leq (\|A\| - \|Ax\|) + (\|Ax\| - \|T_n A T_n x\|) \\ < \varepsilon + \|Ax - T_n A T_n x\| < 2\varepsilon.$$

Since we can choose ε arbitrary small, we have that $\lim_{n\to\infty} \|T_n A T_n\| = \|A\|$. Thus (2.3) is proved.

(iv) If, for some k, the operator $T_k A T_k$ does not belong to all S^p, then $\|T_k A T_k\|_p = \infty$ for all $p \in [1, \infty)$. Hence $\inf_{p\in[1,\infty)} \|T_k A T_k\|_p = \infty$. Therefore $\sup_n \inf_{p\in[1,\infty)} \|T_n A T_n\|_p = \infty$ and the proof is complete. □

Making use of Propositions 2.1, we obtain

Theorem 2.1. *Let A be a bounded operator on Hilbert spaces H. Let $\{T_n\}$ be a sequence of bounded operators on H that converges to $\mathbf{1}_H$ in the* s.o.t. *Suppose that $\sup_n \|T_n\| \leq 1$. Then*

(i) *If $A \in \cup_{p\in[1,\infty)} S^p$ then the minimax condition holds*:

$$\inf_{p\in[1,\infty)} \sup_n \|T_n A T_n\|_p = \sup_n \inf_{p\in[1,\infty)} \|T_n A T_n\|_p = \|A\|.$$

(ii) *If $A \notin \cup_{p\in[1,\infty)} S^p$ and $T_k A T_k \notin \cup_{p\in[1,\infty)} S^p$, for some k, then the minimax condition trivially holds*:

$$\inf_{p\in[1,\infty)} \sup_n \|T_n A T_n\|_p = \sup_n \inf_{p\in[1,\infty)} \|T_n A T_n\|_p = \infty$$

(iii) *If $A \notin \cup_{p\in[1,\infty)} S^p$ but each $T_n A T_n \in \cup_{p\in[1,\infty)} S^p$, then the minimax condition does not hold*:

$$\inf_{p\in[1,\infty)} \sup_n \|T_n A T_n\|_p = \infty, \text{ while } \sup_n \inf_{p\in[1,\infty)} \|T_n A T_n\|_p = \|A\|.$$

Remark 2.1. Using the same arguments as above, we obtain that the results of Theorem 2.1 hold if $T_n A T_n$ is replaced by $T_n A$.

Unlike the minimax condition in Theorem 2.1 that only holds for some operators in $B(H)$, its inverse minimax condition holds for all operators in $B(H)$. To prove this, we need the following lemma.

Lemma 2.2. *Let $f : X \times \Lambda \to \mathbb{R} \cup \infty$ be a function on the product of non-empty sets X and Λ. Suppose that there exists $\mu \in \Lambda$ such that*

$$\sup_{\lambda\in\Lambda} f(x, \lambda) = f(x, \mu) \text{ for each } x \in X. \tag{2.4}$$

Then

$$\inf_{x\in X} \left(\sup_{\lambda\in\Lambda} f(x, \lambda) \right) = \sup_{\lambda\in\Lambda} \left(\inf_{x\in X} f(x, \lambda) \right) = \inf_{x\in X} f(x, \mu). \tag{2.5}$$

Proof. For any function $f : X \times \Lambda \to \mathbb{R} \cup \infty$, we always have

$$\inf_{x\in X}\left(\sup_{\lambda\in\Lambda} f(x,\lambda)\right) \geq \sup_{\lambda\in\Lambda}\left(\inf_{x\in X} f(x,\lambda)\right). \tag{2.6}$$

Indeed, $f(y,\lambda) \geq \inf_{x\in X} f(x,\lambda)$, for each $\lambda \in \Lambda$ and $y \in X$. Hence $\sup_{\lambda\in\Lambda} f(y,\lambda) \geq \sup_{\lambda\in\Lambda}\left(\inf_{x\in X} f(x,\lambda)\right)$, for all $y \in Y$, which implies (2.6).

Suppose now that (2.4) holds. Then $\inf_{x\in X}\left(\sup_{\lambda\in\Lambda} f(x,\lambda)\right) = \inf_{x\in X} f(x,\mu)$. Hence

$$\inf_{x\in X}\left(\sup_{\lambda\in\Lambda} f(x,\lambda)\right) = \inf_{x\in X} f(x,\mu) \leq \sup_{\lambda\in\Lambda}\left(\inf_{x\in X} f(x,\lambda)\right).$$

Combining this with (2.6), we obtain (2.5). □

Theorem 2.2. *For a set X, let $\{A_x\}_{x\in X}$ be a family of operators in $B(H)$. Then the following minimax condition holds*:

$$\inf_{x\in X}\left(\sup_{p\in[1,\infty)} \|A_x\|_p\right) = \sup_{p\in[1,\infty)}\left(\inf_{x\in X} \|A_x\|_p\right) = \inf_{x\in X} \|A_x\|_1 .$$

In particular, if $\{T_n\}$ is a sequence of operators in $B(H)$ then, for each operator $A \in B(H)$, the following minimax condition holds:

$$\inf_n\left(\sup_{p\in[1,\infty)} \|T_nAT_n\|_p\right) = \sup_{p\in[1,\infty)}\left(\inf_n \|T_nAT_n\|_p\right) = \inf_n \|T_nAT_n\|_1 .$$

Proof. Set $f(x,p) = \|A_x\|_p$ for all $x \in X$ and $p \in [1,\infty)$. Then, for each $x \in X$, it follows from (1.4) that

$$\sup_{p\in[1,\infty)} f(x,p) = \sup_{p\in[1,\infty)} \|A_x\|_p = \|A_x\|_1 = f(x,1).$$

Setting $\Lambda = [1,\infty)$ and $\mu = 1$ in Lemma 2.2, we obtain

$$\inf_{x\in X}\left(\sup_{p\in[1,\infty)} \|A_x\|_p\right) = \sup_{p\in[1,\infty)}\left(\inf_{x\in X} \|A_x\|_p\right) = \inf_{x\in X} \|A_x\|_1$$

which concludes the proof. □

Remark 2.2. Note that we can not apply Lemma 2.2 to prove Theorem 2.1. Indeed, to do this, we have to set $X = [1,\infty)$, $\Lambda = \mathbb{N}$ and $f(p,n) = \|T_nAT_n\|_p$. Then (see (2.4)) we have to find $\mu \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} \|T_nAT_n\|_p = \sup_{n\in\mathbb{N}} f(p,n) = f(p,\mu) = \|T_\mu AT_\mu\|_p, \text{ for each } p \in [1,\infty).$$

As, by (1.8), $\sup_{n\in\mathbb{N}} \|T_nAT_n\|_p = \|A\|_p$, this means that there is $\mu \in \mathbb{N}$ such that, for each $p \in [1,\infty)$, $\|A\|_p = \|T_\mu AT_\mu\|_p$ which is, generally speaking, not true.

For $x, y \in H$, consider the rank one operator $x \otimes y$ on H that acts by the formula $(x \otimes y)z = (z, x)y$ for all $z \in H$. For any selfadjoint operator $T \in B(H)$, we have

$$T(x \otimes y)T = Tx \otimes Ty \text{ and } \|x \otimes y\|_p = \|x\| \, \|y\| \,, \tag{2.7}$$

for all $p \in [1, \infty)$. Thus if $\|x\| = 1$ then $x \otimes x$ is the projection on the one-dimensional subspace $\mathbb{C}x$.

Definition 2.1. We say that a family of nonzero subspaces $\{L_n\}$ of H is approximately intersecting if, for every $\varepsilon > 0$, there is $x_\varepsilon \in H$ such that

$$\|x_\varepsilon\| = 1 \text{ and } dist(x_\varepsilon, L_n) := \min_{y \in L_n} \|x_\varepsilon - y\| \leq \varepsilon \text{ for all } n. \tag{2.8}$$

Let P_n be the projections on L_n. If there exists $0 \neq x \in H$ such that $P_n x = x$, for all n, then, clearly, the family of subspaces $\{L_n\}$ is approximately intersecting.

Lemma 2.3. *A family of nonzero subspaces $\{L_n\}$ of H is approximately intersecting if and only if, for each $\varepsilon > 0$, there is $x_\varepsilon \in H$ such that*

$$\|x_\varepsilon\| = 1 \textit{ and } \|P_n x_\varepsilon\| \geq 1 - \varepsilon \textit{ for all } n. \tag{2.9}$$

Proof. Let $\{L_n\}$ be approximately intersecting. Then, for each $\varepsilon > 0$, there is $x_\varepsilon \in H$ such that (2.8) holds for all n. As $\|x_\varepsilon - P_n x_\varepsilon\| = \min_{y \in L_n} \|x_\varepsilon - y\| \leq \varepsilon$, we have $\|P_n x_\varepsilon\| \geq \|x_\varepsilon\| - \|x_\varepsilon - P_n x_\varepsilon\| \geq 1 - \varepsilon$ for all n.

Conversely, let for each $\varepsilon > 0$, there is $x_\varepsilon \in H$ such that (2.9) holds for all n. As $x_\varepsilon = P_n x_\varepsilon + (\mathbf{1} - P_n)x_\varepsilon$ and $(\mathbf{1} - P_n)x_\varepsilon, P_n x_\varepsilon$ are orthogonal, we have $1 = \|x_\varepsilon\|^2 = \|P_n x_\varepsilon\|^2 + \|(\mathbf{1} - P_n)x_\varepsilon\|^2$. Hence

$$\|x_\varepsilon - P_n x_\varepsilon\|^2 = 1 - \|P_n x_\varepsilon\|^2 \leq 1 - (1 - \varepsilon)^2 = 2\varepsilon - \varepsilon^2 \text{ for all } n.$$

Thus $\left\|x_{\varepsilon^2/2} - P_n x_{\varepsilon^2/2}\right\| \leq \varepsilon$, for all n, and (2.8) holds. □

We will now verify the following minimax conditions in Schatten ideals for a family of projections.

Theorem 2.3. *Let $\{P_n\}_{n=1}^\infty$ be projections in $B(H)$, $P_n \neq \mathbf{1}$, and let $1 \leq q < \infty$. Suppose that a sequence $\{p_n\}_{n=1}^\infty$ in (q, ∞) satisfies $\lim_{n \to \infty} p_n = \infty$. Then*

(i) $\inf_{X \in S^q, \|X\|_q = 1} \left(\sup_n \|P_n X P_n\|_{p_n} \right) = \sup_n \left(\inf_{X \in S^q, \|X\|_q = 1} \|P_n X P_n\|_{p_n} \right) = 0.$

(ii) *The inverse minimax condition*

$$\inf_n \left(\sup_{X \in S^q, \|X\|_q = 1} \|P_n X P_n\|_{p_n} \right) = \sup_{X \in S^q, \|X\|_q = 1} \left(\inf_n \|P_n X P_n\|_{p_n} \right) = 1, \tag{2.10}$$

holds if and only if the family of subspaces $\{L_n = P_n H\}_{n=1}^\infty$ is approximately intersecting.

Proof. (i) As all $P_n \neq \mathbf{1}$, we can choose for each n the operator $X_n \in S^q$ such that $\|X_n\|_q = 1$ and $P_n X_n P_n = 0$. Then we have that RHS $= 0$.

Set $p = \inf\{p_n\}$. Since $\lim_{n\to\infty} p_n = \infty$, we have $q < p \le p_n \to \infty$, as $n \to \infty$, and

$$\|P_n X P_n\|_{p_n} \le \|P_n\| \, \|X\|_{p_n} \, \|P_n\| = \|X\|_{p_n} \overset{(1.4)}{\le} \|X\|_p . \tag{2.11}$$

Let $X_k = \{k^{-1/q}, ..., k^{-1/q}, 0, ...\}$ be the diagonal operators with first k elements equal to $k^{-1/q}$ and the rest equal 0. Then all $\|X_k\|_q = 1$ and $\|X_k\|_p = \left(\frac{k}{k^{p/q}}\right)^{1/p} = k^{\frac{1}{p}-\frac{1}{q}} \to 0$, as $k \to \infty$. Hence

$$\inf_{X\in S^q, \|X\|_q=1} \left(\sup_n \|P_n X P_n\|_{p_n}\right) \le \inf_k \left(\sup_n \|P_n X_k P_n\|_{p_n}\right) \overset{(2.11)}{\le} \inf_k \|X_k\|_p = 0$$

and (i) is proved.

(ii) First note that it follows from (2.6) that

$$\inf_n \left(\sup_{X\in S^q, \|X\|_q=1} \|P_n X P_n\|_{p_n}\right) \ge \sup_{X\in S^q, \|X\|_q=1} \left(\inf_n \|P_n X P_n\|_{p_n}\right)$$

always holds. As $\|P_n X P_n\|_{p_n} \le \|P_n\| \, \|X\|_{p_n} \, \|P_n\| = \|X\|_{p_n} \le \|X\|_q = 1$, we have

$$1 \ge \inf_n \left(\sup_{X\in S^q, \|X\|_q=1} \|P_n X P_n\|_{p_n}\right).$$

Thus in order to prove (2.10) we only need to show that

$$\sup_{X\in S^q, \|X\|_q=1} \left(\inf_n \|P_n X P_n\|_{p_n}\right) \ge 1. \tag{2.12}$$

Let the spaces $\{L_n = P_n H\}_{n=1}^{\infty}$ be approximately intersecting. Then, by (2.9), for each $\varepsilon > 0$, there is $x_\varepsilon \in H$ such that $\|x_\varepsilon\| = 1$ and $\|P_n x_\varepsilon\| \ge 1 - \varepsilon$ for all n. Set $X_\varepsilon = x_\varepsilon \otimes x_\varepsilon$. Then, by (2.7), $\|X_\varepsilon\|_q = \|x_\varepsilon \otimes x_\varepsilon\|_q = \|x_\varepsilon\|^2 = 1$, $P_n X_\varepsilon P_n = P_n x_\varepsilon \otimes P_n x_\varepsilon$ and

$$\|P_n X_\varepsilon P_n\|_{p_n} = \|P_n x_\varepsilon \otimes P_n x_\varepsilon\|_{p_n} = \|P_n x_\varepsilon\|^2 \ge (1-\varepsilon)^2.$$

Hence

$$\sup_{X\in S^q, \|X\|_q=1} \left(\inf_n \|P_n X P_n\|_{p_n}\right) \ge \sup_\varepsilon \left(\inf_n \|P_n X_\varepsilon P_n\|_{p_n}\right) \ge \sup_\varepsilon (1-\varepsilon)^2 = 1$$

which proves (2.12).

Conversely, suppose now that (2.10) holds. Let us prove that the spaces $\{L_n = P_n H\}_{n=1}^{\infty}$ are approximately intersecting. It follows from the last equality in (2.10) that, for each $\varepsilon > 0$, there is $X_\varepsilon \in S^q$ such that $\|X_\varepsilon\|_q = 1$ and $\|P_n X_\varepsilon P_n\|_{p_n} \ge 1 - \varepsilon$, for all n. Let $p = \inf\{p_n\}$. Then $q < p$ and

$$\|X_\varepsilon\|_p \ge \|P_n X_\varepsilon P_n\|_p \ge \|P_n X_\varepsilon P_n\|_{p_n} \ge 1 - \varepsilon \text{ for all } n. \tag{2.13}$$

Let $s_1 \geq s_2 \geq ...$ be the singular values of X_ε, that is, the eigenvalues of the operator $(X_\varepsilon^* X_\varepsilon)^{1/2}$. Then it follows from (1.2) and (2.13) that

$$\sum_{n=1}^{\infty} s_n^p = \|X_\varepsilon\|_p^p \geq (1-\varepsilon)^p.$$

Therefore, as $s_n^p \leq s_1^{p-q} s_n^q$ and $\|X_\varepsilon\|_q = (\sum_{n=1}^{\infty} s_n^q)^{1/q} = 1$, we have

$$(1-\varepsilon)^p \leq \|X_\varepsilon\|_p^p = \sum_{n=1}^{\infty} s_n^p \leq s_1^{p-q} \sum_n s_n^q = s_1^{p-q} \|X\|_q^q = s_1^{p-q}.$$

Thus $s_1 \geq (1-\varepsilon)^{\frac{p}{p-q}}$.

Consider the Schmidt decomposition of the operator X_ε:

$$X_\varepsilon = \sum_k s_k x_k(\varepsilon) \otimes y_k(\varepsilon),$$

where $\{x_k(\varepsilon)\}_{k=1}^{\infty}$ and $\{y_k(\varepsilon)\}_{k=1}^{\infty}$ are some orthonormal systems of vectors in H (see [5, Sec. II.2.2]). Then $B_\varepsilon = s_1 x_1(\varepsilon) \otimes y_1(\varepsilon)$ is a rank one operator and, for all n,

$$\|X_\varepsilon - B_\varepsilon\|_{p_n} \leq \|X_\varepsilon - B_\varepsilon\|_q = \left(\sum_{k=2}^{\infty} s_k^q\right)^{1/q} = \left(\sum_{k=1}^{\infty} s_k^q - s_1^q\right)^{1/q} = (1-s_1^q)^{1/q}.$$

Hence it follows from this and (2.13) that, for all n,

$$\begin{aligned}\|P_n B_\varepsilon P_n\|_{p_n} &> \|P_n X_\varepsilon P_n\|_{p_n} - \|P_n(X_\varepsilon - B_\varepsilon)P_n\|_{p_n} \geq (1-\varepsilon) - \|X_\varepsilon - B_\varepsilon\|_{p_n} \\ &\geq (1-\varepsilon) - (1-s_1^q)^{1/q}.\end{aligned}$$

Making use of (2.7), we obtain that, for all n,

$$\|P_n B_\varepsilon P_n\|_{p_n} = \|s_1 P_n x_1(\varepsilon) \otimes P_n y_1(\varepsilon)\|_{p_n} = |s_1| \, \|P_n x_1(\varepsilon)\| \, \|P_n y_1(\varepsilon)\|.$$

Hence, by the above inequality,

$$|s_1| \, \|P_n x_1(\varepsilon)\| \, \|P_n y_1(\varepsilon)\| \geq (1-\varepsilon) - (1-s_1^q)^{1/q}.$$

Therefore

$$\|P_n x_1(\varepsilon)\| \, \|P_n y_1(\varepsilon)\| \geq \frac{(1-\varepsilon) - (1-s_1^q)^{1/q}}{|s_1|}.$$

We have $\|P_n x_1(\varepsilon)\| \leq \|x_1(\varepsilon)\| = 1$ and $\|P_n y_1(\varepsilon)\| \leq \|y_1(\varepsilon)\| = 1$. We also have that $s_1 \geq (1-\varepsilon)^{\frac{p}{p-q}}$. Hence, if $\varepsilon \to 0$ then $s_1 \to 1$, so that $\|P_n x_1(\varepsilon)\| \, \|P_n y_1(\varepsilon)\| \to 1$ uniformly with respect to n. Therefore $\|P_n x_1(\varepsilon)\| \to 1$ uniformly with respect to n. Hence it follows by Lemma 2.3 that the family $\{L_n = P_n H\}_{n=1}^{\infty}$ is approximately intersecting. $\square$

3 Some minimax condition for norms on l_p spaces

The results of the previous section can be easily transferred to l_p spaces. Let M be the commutative C*-algebra of all bounded infinite sequences $m = (m_1, m_2, ..., m_i, ...)$ with norm $\|m\|_M = \sup_i |m_i|$ and pointwise multiplication. For each $p \in [1, \infty)$, let l_p be the subspace of M that consists of all sequences $x = (x_1, ..., x_n, ...)$ in M satisfying

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_n|^p \right)^{1/p} < \infty.$$

Then each l_p is a Banach *-algebra in $\|\cdot\|_p$, an ideal of M and

$$\|mx\|_p \le \|m\|_M \|x\|_p \text{ for all } m \in M \text{ and } x \in l_p. \tag{3.1}$$

The space l_2 is a Hilbert space with the scalar product $(x, y) = \sum_{i=1}^{\infty} x_n \overline{y_n}$.

Each $m = (m_1, ..., m_n, ...) \in M$ generates a bounded multiplication operator A_m on l_2 by the formula $T_m x = (m_1 x_1, ..., m_n x_n, ...)$, for $x \in l_2$, and $\|T_m\| = \|m\|_M$. Thus $T_m \in B(l_2)$, for each $m \in M$, and we can identify M with the maximal commutative subalgebra $\widetilde{M} = \{T_m\colon m \in M\}$ of $B(l_2)$. Moreover, if $m \in l_p$ then $T_m \in S^p(l_2)$ and $\|T_m\|_p = \|m\|_p$. Thus we can identify each l_p with the ideal $\widetilde{M} \cap S^p(l_2)$ of $\widetilde{M}$.

Let a sequence $\{\xi_n\}_{n=1}^{\infty}$ of elements in M poinwise converge to the identity $\mathbf{1} = \{1, ..., 1, ...\}$ in M and let $\|\xi_n\|_M \le 1$, for all n. Then the multiplication operators T_{ξ_n} converge to the identity operator $\mathbf{1}_{l_2}$ on l_2 in the strong operator tolology and $\|T_{\xi_n}\| \le 1$. Combining this and Theorems 2.1 and 2.2, we obtain

Corollary 3.1. (i) *Let $\{\xi_n\}$ be a sequence of elements in M that pointwise converge to the identity $\mathbf{1} = \{1, ..., 1, ...\}$ in M and let $\|\xi_n\|_M \le 1$, for all n. Then*

1) *if $m \in \cup_{p\in[1,\infty)} l_p$ then the following minimax condition holds:*

$$\inf_{p\in[1,\infty)} \sup_n \|\xi_n m\|_p = \sup_n \inf_{p\in[1,\infty)} \|\xi_n m\|_p = \|m\|;$$

2) *if $m \notin \cup_{p\in[1,\infty)} l_p$ and $\xi_k m \notin \cup_{p\in[1,\infty)} l_p$, for some k, then the following minimax condition trivially holds:*

$$\inf_{p\in[1,\infty)} \sup_n \|\xi_n m\|_p = \sup_n \inf_{p\in[1,\infty)} \|\xi_n m\|_p = \infty;$$

3) *if $m \notin \cup_{p\in[1,\infty)} l_p$ but each $\xi_n m \in \cup_{p\in[1,\infty)} l_p$, then the above minimax condition does not hold:*

$$\inf_{p\in[1,\infty)} \sup_n \|\xi_n m\|_p = \infty, \text{ while } \sup_n \inf_{p\in[1,\infty)} \|\xi_n m\|_p = \|m\|.$$

(ii) *For a set X, let $\{\xi_x\}_{x\in X}$ be a family of elements in M. Then the following minimax condition holds:*

$$\inf_{x\in X} \left(\sup_{p\in[1,\infty)} \|\xi_x\|_p \right) = \sup_{p\in[1,\infty)} \left(\inf_{x\in X} \|\xi_x\|_p \right) = \inf_{x\in X} \|\xi_x\|_1.$$

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