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SYMMETRIES AND FIRST INTEGRALS OF A SECOND ORDER EVOLUTIONARY OPERATOR EQUATION

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Abstract. A constructive method for finding some first integrals of a given evolutionary operator equation is suggested.

1 Introduction

First integrals play an important role in mathematics, mechanics, physics because they have various applications. Usually they are used to prove the uniqueness of classical solutions of partial differential equations (see [2, 9]). P. Lax [3] applied some conservation laws to prove existence of wave solutions for the Korteweg - de Vries equation. First integrals of evolutionary equations can be also used for investigation of stability of motion in the case of some systems with infinite number of degrees of freedom (see [10]). So more attention has been paid to development of methods of constructing first integrals and many important results have been obtained [1, 4].

In the paper we use a method based on application of transformation of variables to establish invariance of the evolutionary operator equation of the following type

$$N(u) \equiv P_{2u,t}u_{tt} + P_{3u,t}u_t^2 + P_{1u,t}u_t + Q(t,u) = 0, \tag{1}$$

$$u \in D(N) \subseteq U \subseteq V, \quad t \in [t_0, t_1] \subset \mathbb{R}; \quad u_t \equiv D_t u \equiv \frac{d}{dt}u, \qquad u_{tt} \equiv \frac{d^2}{dt^2}u.$$

Here $\forall t \in [t_0, t_1]$, $\forall u \in U_1$ $P_{iu,t} : U_1 \to V_1\, (i = \overline{1,3})$ are linear operators; $Q : [t_0, t_1] \times U_1 \to V_1$ is an arbitrary operator; $D(N)$ is the domain of definition of the operator N,

$$D(N) = \{u \in U : u|_{t=t_0} = \varphi_1,\ u|_{t=t_1} = \varphi_2,$$
$$u_t|_{t=t_0} = \varphi_3,\ u_t|_{t=t_1} = \varphi_4,\ \varphi_i \in U_1\, (i = \overline{1,4})\}; \tag{2}$$

$U = C^2([t_0, t_1]; U_1)$, $V = C([t_0, t_1]; V_1)$, U_1, V_1 are linear normed spaces, $U_1 \subseteq V_1$.

Assume that for every $t \in (t_0, t_1)$ and $g(t), u(t) \in U_1$ the functions $P_{1u,t}g(t)$, $P_{3u,t}g(t)$ are continuously differentiable and $P_{2u,t}g(t)$ is twice continuously differentiable on (t_0, t_1).

Any function $u \in D(N)$ is called a solution of problem (1) if it satisfies equation (1).

In the sequel, we shall write

$$N(u) \equiv P_{2u}u_{tt} + P_{3u}u_t^2 + P_{1u}u_t + Q(u) = 0,$$

bearing in mind that the operators P_{1u}, P_{2u}, P_{3u} and Q also depend on t.

In the paper we shall use notations and notions of [5-8].

Consider a nonlocal bilinear form

$$\Phi(\cdot,\cdot) \equiv \int_{t_0}^{t_1} < \cdot,\cdot > dt : V \times V \to \mathbb{R} \tag{3}$$

such that the bilinear mapping $\Phi_1(\cdot,\cdot) \equiv < \cdot,\cdot >$ satisfies the following conditions:

$$< v_1(t), v_2(t) > = < v_2(t), v_1(t) > \qquad \forall v_1(t), v_2(t) \in V_1,$$

$$D_t < v(t), g(t) > = < D_t v(t), g(t) > + < v(t), D_t g(t) > \;\; \forall v, g \in C^1([t_0, t_1]; V_1).$$

Definition 1. The operator $N : D(N) \subset U \to V$ is said to be B_u-potential on the set $D(N)$ relative to bilinear form (3), if there exist a functional $F_N : D(F_N) = D(N) \to \mathbb{R}$ and an operator $B_u : D(B_u) \subset V \to V$ such that

$$\delta F_N[u, h] = \Phi(N(u), B_u h) \qquad \forall u \in D(N), \quad \forall h \in D(N'_u, B_u).$$

If $B_u \equiv I$ is the identical operator then the operator N is called potential on $D(N)$ relative to bilinear form (3).

The following theorem is needed for the sequel.

Theorem 1 ([5])**.** *Consider the operator $N : D(N) \subset U \to V$ and the bilinear form $\Phi(\cdot,\cdot) : V \times V \to \mathbb{R}$ such that for any fixed elements $u \in D(N), \quad g, h \in D(N'_u, B_u)$ the function $\psi(\varepsilon) = \Phi(N(u + \varepsilon h), B_{u+\varepsilon h} g)$ belongs to the class $C^1[0, 1]$. For N to be B_u-potential on the convex set $D(N)$ relative to Φ it is necessary and sufficient to have*

$$\Phi(N'_u h, B_u g) + \Phi(N(u), B'_u(g; h)) = \Phi(N'_u g, B_u h) + \Phi(N(u), B'_u(h; g)) \tag{4}$$

$$\forall u \in D(N), \quad \forall g, h \in D(N'_u, B_u).$$

2 Conditions of B_u-potentiality and symmetries

Theorem 2. *Let D_t be skew-symmetric on $D(N'_u, B_u)$. The operator N of equation (1) is B_u-potential on $D(N)$ relative to bilinear form (3) if and only if $\forall u \in D(N), \forall h \in D(N'_u, B_u), \forall t \in [t_0, t_1]$ the following conditions are satisfied on $D(N'_u, B_u)$:*

$$B_u^* P_{2u} - P_{2u}^* B_u = 0, \tag{5}$$

$$u_t P_{3u}^* B_u - P_{2u}^{*\prime}(B_u(\cdot); u_t) - P_{2u}^* B'_u(\cdot; u_t) + B_u^* P_{3u}(u_t(\cdot)) = 0, \tag{6}$$

$$-2\frac{\partial}{\partial t}(P_{2u}^* B_u) + P_{1u}^* B_u + B_u^* P_{1u} = 0, \tag{7}$$

$$-\frac{\partial^2}{\partial t^2}(P_{2u}^*B_u)h+[B_u'(\cdot;h)]^*Q(u)-[B_u'(h;\cdot)]^*Q(u)+$$
$$+\frac{\partial}{\partial t}(P_{1u}^*B_u)h+B_u^*Q_u'h-Q_u'^*B_uh=0, \tag{8}$$

$$P_{1u}^{*\prime}(B_uh;u_t)+B_u^*P_{1u}'(u_t;h)-[P_{1u}'(u_t;\cdot)]^*B_uh+2u_t\frac{\partial}{\partial t}(P_{3u}^*B_u)h+$$
$$+P_{1u}^*B_u'(h;u_t)-2\frac{\partial}{\partial t}P_{2u}^{*\prime}(B_uh;u_t)+[B_u'(\cdot;h)]^*P_{1u}u_t-$$
$$-2\frac{\partial}{\partial t}(P_{2u}^*B_u'(h;u_t))-[B_u'(h;\cdot)]^*P_{1u}u_t=0, \tag{9}$$

$$B_u^*P_{2u}'(u_{tt};h)-P_{2u}^{*\prime}(B_uh;u_{tt})-[P_{2u}'(u_{tt};\cdot)]^*B_uh+2u_{tt}P_{3u}^*B_uh+$$
$$+[B_u'(\cdot;h)]^*P_{2u}u_{tt}-P_{2u}^*B_u'(h;u_{tt})-[B_u'(h;\cdot)]^*P_{2u}u_{tt}=0, \tag{10}$$

$$-P_{2u}^{*\prime\prime}(B_uh;u_t;u_t)+B_u^*P_{3u}'(u_t^2;h)-[P_{3u}'(u_t^2;\cdot)]^*B_uh+2u_tP_{3u}^{*\prime}(B_uh;u_t)+$$
$$+[B_u'(\cdot;h)]^*P_{3u}u_t^2-2P_{2u}^{*\prime}(B_u'(h;u_t);u_t)-P_{2u}^*B_u''(h;u_t;u_t)+$$
$$+2u_tP_{3u}^*B_u'(h;u_t)-[B_u'(h;\cdot)]^*P_{3u}u_t^2=0. \tag{11}$$

Proof. Using (1), we get

$$N_u'h=2P_{3u}(u_th_t)+P_{3u}'(u_t^2;h)+P_{2u}h_{tt}+P_{2u}'(u_{tt};h)+P_{1u}h_t+P_{1u}'(u_t;h)+Q_u'h.$$

In this case, condition (4) can be written in the form

$$\int_{t_0}^{t_1}(<2P_{3u}(u_th_t)+P_{3u}'(u_t^2;h)+P_{2u}h_{tt}+P_{2u}'(u_{tt};h)+P_{1u}h_t+P_{1u}'(u_t;h)+$$
$$+Q_u'h,B_ug>+<P_{2u}u_{tt}+P_{1u}u_t+P_{3u}u_t^2+Q(u),B_u'(g;h)>)dt=$$
$$=\int_{t_0}^{t_1}(<2P_{3u}(u_tg_t)+P_{3u}'(u_t^2;g)+P_{2u}g_{tt}+P_{2u}'(u_{tt};g)+P_{1u}g_t+P_{1u}'(u_t;g)+$$
$$+Q_u'g,B_uh>+<P_{2u}u_{tt}+P_{1u}u_t+P_{3u}u_t^2+Q(u),B_u'(h;g)>)dt,$$

or

$$\int_{t_0}^{t_1}\{<2B_u^*P_{3u}(u_th_t)+B_u^*P_{3u}'(u_t^2;h)+B_u^*P_{2u}h_{tt}+B_u^*P_{2u}'(u_{tt};h)+B_u^*P_{1u}h_t+$$
$$+B_u^*P_{1u}'(u_t;h)+B_u^*Q_u'h,g>+<[B_u'(\cdot;h)]^*(P_{2u}u_{tt}+P_{1u}u_t+P_{3u}u_t^2+Q(u)),g>-$$
$$-<-2D_t(u_tP_{3u}^*B_uh)+[P_{3u}'(u_t^2;\cdot)]^*B_uh+D_t^2(P_{2u}^*B_uh)+[P_{2u}'(u_{tt};\cdot)]^*B_uh-$$
$$-D_t(P_{1u}^*B_uh)+[P_{1u}'(u_t;\cdot)]^*B_uh+Q_u'^*B_uh,g>-$$
$$-<[B_u'(h;\cdot)]^*(P_{2u}u_{tt}+P_{1u}u_t+P_{3u}u_t^2+Q(u)),g>\}dt=0 \tag{12}$$
$$\forall u\in D(N),\quad \forall g,h\in D(N_u',B_u).$$

Taking into account the second Gâteaux derivative, we obtain

$$D_t^2(P_{2u}^*B_uh)=D_t[D_t(P_{2u}^*B_uh)]=D_t\Big[P_{2u}^*B_uh_t+\frac{\partial}{\partial t}(P_{2u}^*B_u)h+P_{2u}^{*\prime}(B_uh;u_t)+$$

$$+P_{2u}^*B_u'(h;u_t)\Big]=\frac{\partial^2}{\partial t^2}(P_{2u}^*B_u)h+2\frac{\partial}{\partial t}P_{2u}^{*\prime}(B_uh;u_t)+2\frac{\partial}{\partial t}(P_{2u}^*B_u'(h;u_t))+$$

$$+2\frac{\partial}{\partial t}(P_{2u}^*B_u)h_t+P_{2u}^{*\prime\prime}(B_uh;u_t;u_t)+2P_{2u}^{*\prime}(B_u'(h;u_t);u_t)+2P_{2u}^{*\prime}(B_uh_t;u_t)+$$

$$+P_{2u}^{*\prime}(B_uh;u_{tt})+P_{2u}^*B_u''(h;u_t;u_t)+2P_{2u}^*B_u'(h_t;u_t)+P_{2u}^*B_u'(h;u_{tt})+P_{2u}^*B_uh_{tt}. \quad (13)$$

Further,

$$D_t[P_{1u}^*B_uh]=\frac{\partial}{\partial t}(P_{1u}^*B_u)h+P_{1u}^{*\prime}(B_uh;u_t)+P_{1u}^*B_u'(h;u_t)+P_{1u}^*B_uh_t; \quad (14)$$

$$D_t[u_tP_{3u}^*B_uh]=u_{tt}P_{3u}^*B_uh+u_t\frac{\partial}{\partial t}(P_{3u}^*B_u)h+u_tP_{3u}^{*\prime}(B_uh;u_t)+$$

$$+u_tP_{3u}^*B_u'(h;u_t)+u_tP_{3u}^*B_uh_t. \quad (15)$$

From (12) – (15) it follows that

$$\int_{t_0}^{t_1}<2B_u^*P_{3u}(u_th_t)+B_u^*P_{3u}'(u_t^2;h)+B_u^*P_{2u}h_{tt}+B_u^*P_{2u}'(u_{tt};h)+B_u^*P_{1u}h_t+$$

$$+B_u^*P_{1u}'(u_t;h)+B_u^*Q_u'h+[B_u'(\cdot;h)]^*(P_{2u}u_{tt}+P_{1u}u_t+P_{3u}u_t^2+Q(u))-$$

$$-\frac{\partial^2}{\partial t^2}(P_{2u}^*B_u)h-2\frac{\partial}{\partial t}P_{2u}^{*\prime}(B_uh;u_t)-2\frac{\partial}{\partial t}(P_{2u}^*B_u'(h;u_t))-$$

$$-2\frac{\partial}{\partial t}(P_{2u}^*B_u)h_t-P_{2u}^{*\prime\prime}(B_uh;u_t;u_t)-2P_{2u}^{*\prime}(B_u'(h;u_t);u_t)-2P_{2u}^{*\prime}(B_uh_t;u_t)-$$

$$-P_{2u}^{*\prime}(B_uh;u_{tt})-P_{2u}^*B_u''(h;u_t;u_t)-2P_{2u}^*B_u'(h_t;u_t)-P_{2u}^*B_u'(h;u_{tt})-P_{2u}^*B_uh_{tt}-$$

$$-[P_{2u}'(u_{tt};\cdot)]^*B_uh+2u_{tt}P_{3u}^*B_uh+2u_t\frac{\partial}{\partial t}(P_{3u}^*B_u)h+2u_tP_{3u}^{*\prime}(B_uh;u_t)+$$

$$+2u_tP_{3u}^*B_u'(h;u_t)+2u_tP_{3u}^*B_uh_t-[P_{3u}'(u_t^2;\cdot)]^*B_uh+P_{1u}^*B_uh_t+\frac{\partial}{\partial t}(P_{1u}^*B_u)h+$$

$$+P_{1u}^{*\prime}(B_uh;u_t)+P_{1u}^*B_u'(h;u_t)-[P_{1u}'(u_t;\cdot)]^*B_uh-Q_u'^*B_uh-$$

$$-[B_u'(h;\cdot)]^*(P_{2u}u_{tt}+P_{1u}u_t+P_{3u}u_t^2+Q(u)),g>\ dt=0.$$

Thus condition (12) is represented in the form

$$\int_{t_0}^{t_1}<(B_u^*P_{2u}-P_{2u}^*B_u)\,h_{tt}+(2B_u^*P_{3u}(u_t(\cdot))+B_u^*P_{1u}+2u_tP_{3u}^*B_u-$$

$$-2\frac{\partial}{\partial t}(P_{2u}^*B_u)-2P_{2u}^{*\prime}(B_u(\cdot);u_t)-2P_{2u}^*B_u'(\cdot;u_t)+P_{1u}^*B_u)\,h_t+B_u^*P_{3u}'(u_t^2;h)+$$

$$+B_u^*P_{2u}'(u_{tt};h)+B_u^*P_{1u}'(u_t;h)+B_u^*Q_u'h+[B_u'(\cdot;h)]^*P_{2u}u_{tt}+[B_u'(\cdot;h)]^*P_{1u}u_t+$$
$$+[B_u'(\cdot;h)]^*P_{3u}u_t^2+[B_u'(\cdot;h)]^*Q(u)+2u_{tt}P_{3u}^*B_uh+2u_t\frac{\partial}{\partial t}(P_{3u}^*B_u)h+$$
$$+2u_tP_{3u}^{*\prime}(B_uh;u_t)+2u_tP_{3u}^*B_u'(h;u_t)-[P_{3u}'(u_t^2;\cdot)]^*B_uh-\frac{\partial^2}{\partial t^2}(P_{2u}^*B_u)h-$$
$$-2\frac{\partial}{\partial t}P_{2u}^{*\prime}(B_uh;u_t)-2\frac{\partial}{\partial t}(P_{2u}^*B_u'(h;u_t))-P_{2u}^{*\prime\prime}(B_uh;u_t;u_t)-2P_{2u}^{*\prime}(B_u'(h;u_t);u_t)-$$
$$-P_{2u}^{*\prime}(B_uh;u_{tt})-P_{2u}^*B_u''(h;u_t;u_t)-P_{2u}^*B_u'(h;u_{tt})-[P_{2u}'(u_{tt};\cdot)]^*B_uh+$$
$$+\frac{\partial}{\partial t}(P_{1u}^*B_u)h+P_{1u}^{*\prime}(B_uh;u_t)+P_{1u}^*B_u'(h;u_t)-[P_{1u}'(u_t;\cdot)]^*B_uh-Q_u'^*B_uh-$$
$$-[B_u'(h;\cdot)]^*P_{2u}u_{tt}-[B_u'(h;\cdot)]^*P_{1u}u_t-[B_u'(h;\cdot)]^*P_{3u}u_t^2-[B_u'(h;\cdot)]^*Q(u),g>dt=0$$
$$\forall u\in D(N),\quad \forall g,h\in D(N_u',B_u).$$

This is satisfied identically if and only if

$$(B_u^*P_{2u}-P_{2u}^*B_u)\,h_{tt}+(2B_u^*P_{3u}(u_t(\cdot))+B_u^*P_{1u}+2u_tP_{3u}^*B_u-$$
$$-2\frac{\partial}{\partial t}(P_{2u}^*B_u)-2P_{2u}^{*\prime}(B_u(\cdot);u_t)-2P_{2u}^*B_u'(\cdot;u_t)+P_{1u}^*B_u)\,h_t+B_u^*P_{3u}'(u_t^2;h)+$$
$$+B_u^*P_{2u}'(u_{tt};h)+B_u^*P_{1u}'(u_t;h)+B_u^*Q_u'h+[B_u'(\cdot;h)]^*P_{2u}u_{tt}+[B_u'(\cdot;h)]^*P_{1u}u_t+$$
$$+[B_u'(\cdot;h)]^*P_{3u}u_t^2+[B_u'(\cdot;h)]^*Q(u)+2u_{tt}P_{3u}^*B_uh+2u_t\frac{\partial}{\partial t}(P_{3u}^*B_u)h+$$
$$+2u_tP_{3u}^{*\prime}(B_uh;u_t)+2u_tP_{3u}^*B_u'(h;u_t)-[P_{3u}'(u_t^2;\cdot)]^*B_uh-\frac{\partial^2}{\partial t^2}(P_{2u}^*B_u)h-$$
$$-2\frac{\partial}{\partial t}P_{2u}^{*\prime}(B_uh;u_t)-2\frac{\partial}{\partial t}(P_{2u}^*B_u'(h;u_t))-P_{2u}^{*\prime\prime}(B_uh;u_t;u_t)-2P_{2u}^{*\prime}(B_u'(h;u_t);u_t)-$$
$$-P_{2u}^{*\prime}(B_uh;u_{tt})-P_{2u}^*B_u''(h;u_t;u_t)-P_{2u}^*B_u'(h;u_{tt})-[P_{2u}'(u_{tt};\cdot)]^*B_uh+$$
$$+\frac{\partial}{\partial t}(P_{1u}^*B_u)h+P_{1u}^{*\prime}(B_uh;u_t)+P_{1u}^*B_u'(h;u_t)-[P_{1u}'(u_t;\cdot)]^*B_uh-Q_u'^*B_uh-$$
$$-[B_u'(h;\cdot)]^*P_{2u}u_{tt}-[B_u'(h;\cdot)]^*P_{1u}u_t-[B_u'(h;\cdot)]^*P_{3u}u_t^2-[B_u'(h;\cdot)]^*Q(u)=0$$
$$\forall u\in D(N),\quad \forall h\in D(N_u',B_u).$$

The necessary and sufficient conditions for this equality to be valid are that equations (5) – (11) be satisfied. □

Consider a one-parametric group of transformations

$$G:\begin{cases}\bar t=t+\varepsilon\varphi(t,u),\\ \overline{u}(\bar t)=u(t)+\varepsilon\psi(t,u),\end{cases}\tag{16}$$

where φ,ψ are some operators.

Using transformation (16), one can define a function $\overline{u}(t,\varepsilon)$ such that

$$\overline{u}=u+\varepsilon S(u),\tag{17}$$

where $S(u)=\psi(t,u)-u_t\varphi(t,u)$. In this case, the operator S is called a generator of transformation (17).

Definition 2. Transformation (17) is said to be a symmetry of the equation

$$N(u) = 0, \tag{18}$$

if function $\overline{u}$ (17) is a solution of (18) for any sufficiently small parameter ε and any solution u of this equation.

The following theorem is needed for the sequel.

Theorem 3 (Savchin V.M.)**.** *Transformation* (17) *is a symmetry of equation* (18) *if and only if*

$$[N, S](u) \equiv N'_u S(u) - S'_u N(u) \overset{(18)}{=} 0. \tag{19}$$

Definition 3. A functional $J[t, u]$ is called a first integral of equation (1) under conditions (2) if it does not depend on t for any solution $u(t)$ of problem (1) – (2).

Theorem 4. *Suppose that S_1, S_2 are generators of symmetries of equation* (1) *and the operator N is B_u-potential on $D(N)$ relative to bilinear form* (3)*. Then*

$$J[t, u] = D_t < P_{2u} S_2(u), B_u S_1(u) > -$$

$$- < 2P_{3u}(u_t S_2(u)) + 2P_{2u} D_t S_2(u) + P_{1u} S_2(u), B_u S_1(u) > \tag{20}$$

is a first integral of the given equation.

Proof. We have

$$< N'_u h, B_u g > + < N(u), B'_u(g; h) >=< 2P_{3u}(u_t h_t) + P'_{3u}(u_t^2; h) + P_{2u} h_{tt} +$$

$$+ P'_{2u}(u_{tt}; h) + P_{1u} h_t + P'_{1u}(u_t; h) + Q'_u h, B_u g > +$$

$$+ < P_{2u} u_{tt} + P_{1u} u_t + P_{3u} u_t^2 + Q(u), B'_u(g; h) >=$$

$$= 2 < h_t, u_t P^*_{3u} B_u g > + < h, [P'_{3u}(u_t^2; \cdot)]^* B_u g > + < h_{tt}, P^*_{2u} B_u g > +$$

$$+ < h, [P'_{2u}(u_{tt}; \cdot)]^* B_u g > + < h_t, P^*_{1u} B_u g > + < h, [P'_{1u}(u_t; \cdot)]^* B_u g > +$$

$$+ < Q'^*_u B_u g, h > + < h, [B'_u(g; \cdot)]^* (P_{2u} u_{tt} + P_{1u} u_t + P_{3u} u_t^2 + Q(u)) >=$$

$$= 2D_t < h, u_t P^*_{3u} B_u g > -2 < h, u_{tt} P^*_{3u} B_u g > -2 < h, u_t \frac{\partial}{\partial t}(P^*_{3u} B_u) g > -$$

$$-2 < h, u_t P^{*\prime}_{3u}(B_u g; u_t) > -2 < h, u_t P^*_{3u} B'_u(g; u_t) > -2 < h, u_t P^*_{3u} B_u g_t > +$$

$$+ < h, [P'_{3u}(u_t^2; \cdot)]^* B_u g > + D_t^2 < h, P^*_{2u} B_u g > -2D_t < h, \frac{\partial}{\partial t}(P^*_{2u} B_u) g > -$$

$$-2D_t < h, P^{*\prime}_{2u}(B_u g; u_t) > -2D_t < h, P^*_{2u} B'_u(g; u_t) > -2D_t < h, P^*_{2u} B_u g_t > +$$

$$+ < h, \frac{\partial^2}{\partial t^2}(P^*_{2u} B_u) g > + < h, 2\frac{\partial}{\partial t} P^{*\prime}_{2u}(B_u g; u_t) > +2 < h, \frac{\partial}{\partial t} P^*_{2u} B'_u(g; u_t) > +$$

$$+2 < h, \frac{\partial}{\partial t} P^*_{2u} B_u g_t > + < h, P^{*\prime\prime}_{2u}(B_u g; u_t; u_t) > +2 < h, P^{*\prime}_{2u}(B'_u(g, u_t); u_t) > +$$

$$+2 < h, P^{*\prime}_{2u}(B_u g_t; u_t) > + < h, P^{*\prime}_{2u}(B_u g; u_{tt}) > + < h, P^*_{2u} B''_u(g; u_t; u_t) > +$$

$$+2 < h, P^*_{2u}B'_u(g_t; u_t) > + < h, P^*_{2u}B'_u(g; u_{tt}) > + < h, P^*_{2u}B_u g_{tt} > +$$
$$+ < h, [P'_{2u}(u_{tt}; \cdot)]^* B_u g > + D_t < h, P^*_{1u}B_u g > - < h, \frac{\partial}{\partial t}(P^*_{1u}B_u)g > -$$
$$- < h, P^{*\prime}_{1u}(B_u g; u_t) - < h, P^*_{1u}B'_u(g; u_t) > - < h, P^*_{1u}B_u g_t > +$$
$$+ < h, [P'_{1u}(u_t; \cdot)]^* B_u g > + < h, Q'^*_u B_u g > +$$
$$+ < h, [B'_u(g; \cdot)]^*(P_{2u}u_{tt} + P_{1u}u_t + P_{3u}u_t^2 + Q(u)) > .$$

Taking into account conditions (5) – (11), we obtain

$$< N'_u h, B_u g > + < N(u), B'_u(g; h) >=< N'_u g, B_u h > + < N(u), B'_u(h; g) > +$$
$$+D_t[D_t < P_{2u}g, B_u h > - < 2P_{3u}(u_t g) + 2P_{2u}g_t + P_{1u}g, B_u h >]. \qquad (21)$$

Substituting $S_1(u)$ for h and $S_2(u)$ for g in (21) and taking into consideration (19), we obtain that a first integral of the given equation is represented in form (20). □

Remark 2.1. Suppose that the operator N of equation (1) is not B_u-potential on $D(N)$ relative to bilinear form (3), S_1 is a generator of symmetry of this equation and there exists an operator S_2 such that $N'^*_u B_u S_2(u) \overset{(1)}{=} 0$. Then

$$J[t, u] = D_t < P_{2u}S_1(u), B_u S_2(u) > -$$
$$- < 2P_{3u}(u_t S_1(u)) + 2P_{2u}D_t S_1(u) + P_{1u}S_1(u), B_u S_2(u) > \qquad (22)$$

a the first integral of the given equation.

Indeed, in this case

$$N'^*_u B_u g = D_t^2[P^*_{2u}B_u g] + [P'_{2u}(u_{tt}; \cdot)]^* B_u g - D_t[P^*_{1u}B_u g] + [P'_{1u}(u_t; \cdot)]^* B_u g -$$
$$-2D_t[u_t P^*_{3u}B_u g] + [P'_{3u}(u_t^2; \cdot)]^* B_u g + Q'^*_u B_u g.$$

Then

$$< h, N'^*_u B_u g > - < N'_u h, B_u g >=< h, D_t^2[P^*_{2u}B_u g] + [P'_{2u}(u_{tt}; \cdot)]^* B_u g -$$
$$-D_t[P^*_{1u}B_u g] + [P'_{1u}(u_t; \cdot)]^* B_u g - 2D_t[u_t P^*_{3u}B_u g] + [P'_{3u}(u_t^2; \cdot)]^* B_u g + Q'^*_u B_u g > -$$
$$- < P_{2u}h_{tt} + P'_{2u}(u_{tt}; h) + P_{1u}h_t + P'_{1u}(u_t; h) + 2P_{3u}(u_t h_t) + P'_{3u}(u_t^2; h) +$$
$$+Q'_u h, B_u g >=< h, D_t^2[P^*_{2u}B_u g] + [P'_{2u}(u_{tt}; \cdot)]^* B_u g - D_t[P^*_{1u}B_u g] + [P'_{1u}(u_t; \cdot)]^* B_u g -$$
$$-2D_t[u_t P^*_{3u}B_u g] + [P'_{3u}(u_t^2; \cdot)]^* B_u g + Q'^*_u B_u g > -D_t < h_t, P^*_{2u}B_u g > +$$
$$+ < h_t, D_t[P^*_{2u}B_u g] > - < h, [P'_{2u}(u_{tt}; \cdot)]^* B_u g > -D_t < h, P^*_{1u}B_u g > +$$
$$+ < h, D_t[P^*_{1u}B_u g] > - < h, [P'_{1u}(u_t; \cdot)]^* B_u g > -2D_t < h, u_t P^*_{3u}B_u g > +$$
$$+2 < h, D_t[u_t P^*_{3u}B_u g] > - < h, [P'_{3u}(u_t^2; \cdot)]^* B_u g > - < h, Q'^*_u B_u g >=$$
$$=< h, D_t^2[P^*_{2u}B_u g] > -D_t[D_t < h, P^*_{2u}B_u g > - < h, D_t[P^*_{2u}B_u g >] +$$
$$+D_t < h, D_t[P^*_{2u}B_u g] > - < h, D_t^2[P^*_{2u}B_u g] > -D_t < h, P^*_{1u}B_u g > -$$

$$-2D_t < h, u_t P^*_{3u} B_u g >= -D_t^2 < P_{2u} h, B_u g > +2D_t < h, D_t[P^*_{2u} B_u g] > -$$
$$-D_t < P_{1u} h, B_u g > -2D_t < P_{3u}(u_t h), B_u g >= -D_t^2 < P_{2u} h, B_u g > +$$
$$+2D_t^2 < P_{2u} h, B_u g > -2D_t < P_{2u} h_t, B_u g > -D_t < P_{1u} h, B_u g > -$$
$$-2D_t < P_{3u}(u_t h), B_u g >= D_t^2 < P_{2u} h, B_u g > -2D_t < P_{2u} h_t, B_u g > -$$
$$-D_t < P_{1u} h, B_u g > -2D_t < P_{3u}(u_t h), B_u g > . \qquad (23)$$

Substituting $S_1(u)$ for h and $S_2(u)$ for g in (23), we obtain that a first integral of the given equation is represented in form (22).

3 Examples

1. Consider the following partial differentional equation

$$N(u) \equiv u_{tt} + 2\beta v(t) u_{tx} + u_{xxxx} + v^2(t) u_{xx} + \beta v'(t) u_x = 0, \qquad (24)$$
$$(x,t) \in Q_T = (a,b) \times (0,T).$$

Define $D(N)$ by

$$D(N) = \{u \in U = C^{2,4}_{t,x}(\overline{Q_T}) : u|_{t=0} = \phi_1(x),\ u|_{t=T} = \phi_2(x)\ (x \in (a,b)), \qquad (25)$$
$$u|_{x=a} = \psi_1(t),\ u|_{x=b} = \psi_2(t),\ u_x|_{x=a} = \psi_3(t),\ u_x|_{x=b} = \psi_4(t),$$
$$u_{xx}|_{x=a} = \psi_5(t),\ u_{xx}|_{x=b} = \psi_6(t),\ u_{xxx}|_{x=a} = \psi_7(t),\ u_{xxx}|_{x=b} = \psi_8(t),$$
$$u_{tx}|_{x=a} = \psi_9(t),\ u_{tx}|_{x=b} = \psi_{10}(t), (t \in (0,T)\},$$

where $\phi_i, \psi_j (i = 1,2,\ j = \overline{1,10})$ are given continuous functions.

Let us note that operator N (24) is potential on domain of definitions (25) relative to the classical bilinear form

$$\Phi(v,g) = \int_0^T \int_a^b v(x,t) g(x,t)\, dx dt.$$

Indeed, in this case

$$P_2 = I, \qquad P_1 = 2\beta v(t) D_x, \qquad P_1^* = -2\beta v(t) D_x, \qquad \frac{\partial P_1^*}{\partial t} = -2\beta v'(t) D_x,$$

$$P_3 = 0, \qquad Q'_u = D_x^4 + v^2(t) D_x^2 + \beta v'(t) D_x, \qquad Q'^*_u = D_x^4 + v^2(t) D_x^2 - \beta v'(t) D_x$$

and

(5) $\Longrightarrow I - I = 0,$
(6) $\Longrightarrow 0 = 0,$
(7) $\Longrightarrow 2\beta v(t) D_x - 2\beta v(t) D_x = 0,$
(8) $\Longrightarrow -2\beta v'(t) D_x + D_x^4 + v^2(t) D_x^2 + \beta v'(t) D_x - D_x^4 - v^2(t) D_x^2 + \beta v'(t) D_x = 0,$
(9) $\Longrightarrow 0 = 0,$
(10) $\Longrightarrow 0 = 0,$
(11) $\Longrightarrow 0 = 0.$

Suppose that $\psi_i \equiv 0\,(i = \overline{1,10})$ in (25). Taking into account that $S_1(u) = u_x$ and $S_2(u) = u_{xx}$ are generators of symmetries of equation (24) and using (20), we obtain a first integral of the given equation in the form

$$J[t,u] = \int\limits_a^b u_{xx}(u_{tx} + \beta v(t) u_{xx})dx.$$

Remark 3.1. Note that equation (24) can be represented in the divergence form

$$N(u) \equiv D_t u_t + D_x[2\beta v(t) u_t + u_{xxx} + v^2(t) u_x + \beta v'(t) u] = 0.$$

It is well known that

$$J_1[t,u] = \int\limits_a^b u_t dx$$

is also a first integral of (24).

2. Let us consider the equation of in the form

$$N(u) \equiv auu_{tt} + buu_{xx} + au_t^2 + bu_x^2 = 0, \tag{26}$$

$$(x,t) \in Q_T = (0,l) \times (0,T),$$

where a, b are constants.

Define $D(N)$ by

$$D(N) = \{u \in U = C^2(\overline{Q_T}) : u|_{t=0} = \varphi_1(x),\ u|_{t=T} = \varphi_2(x)\ (x \in (0,l)), \tag{27}$$

$$u|_{x=0} = \psi_1(t),\ \ u|_{x=l} = \psi_2(t)\ \ (t \in (0,T)\},$$

where $\varphi_1, \varphi_2, \psi_1, \psi_2$ are given continuous functions.

Here

$$P_{2u} = auI, \quad P_3 = aI, \quad P_1 = 0, \quad Q(u) = buu_{xx} + bu_x^2.$$

It is easy to check that operator N (26) is not B_u-potential on $D(N)$ (27) relative to the bilinear form

$$\Phi(v,g) = \int\limits_0^T \int\limits_0^l v(x,t) g(x,t)\, dxdt,$$

if $B_u = u(D_x + I)$.

By straightforward computations we find that $S_1(u) = u_x$ is a generator of symmetry of equation (26) and $S_2(u) = u$ satisfies the condition $N_u'^{*} B_u S_2(u) \overset{(26)}{=} 0$.

Suppose that $\psi_i \equiv 0$ $(i = 1,2)$ in (27) and using (22), we obtain that

$$J[t,u] = \int\limits_0^l u^2 u_t u_x dx$$

is a first integral of the given equation.

Remark 3.2. Let us note that equation (26) can be written in the divergence form

$$N(u) \equiv D_t(auu_t) + D_x(buu_x) = 0.$$

It is well known that in this case

$$J_1[t, u] = \int_0^l uu_t dx$$

is also a first integral of (26).

4 Conclusion

In the paper we investigate B_u-potentiality of a given operator N and obtain formulas for finding some first integrals of the evolutionary operator equation. All theoretical results are illustrated by some examples. It should be noted that the considered method allows us constructing first integrals different from the known ones.

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