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ON ANALOGUES OF PERIODIC BOUNDARY VALUE PROBLEMS FOR THE LAPLACE OPERATOR IN A BALL

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Abstract. We consider a new class of boundary value problems for the Laplace operator in a multi-dimensional ball. These problems are analogues of classical periodic boundary value problems. We prove the correctness of these problems. We present a method of constructing of all eigenfunctions of the problem.

1. It is well-known that the Dirichlet and Neumann boundary value problems are the main problems of the theory of harmonic functions. In the one-dimensional case or in problems on a multi-dimensional parallelepiped, periodic boundary value problems also belong to main problems. But for case of a ball analogues of periodic problems were not constructed. We first constructed analogues of periodic boundary value problems for the Laplace operator in the multi-dimensional ball. We constructed functions of Green for the problems. We found all the eigenvalues and the eigenfunctions.

2. Let $\Omega = \{x \in R^n : r = |x| < 1\}$ be the unit ball, $\partial\Omega = \{x \in R^n : |x| = 1\}$ be the unit sphere. We denote $\partial\Omega_+ = \partial\Omega \cap \{x \in R^n : x_1 \geq 0\}$, $\partial\Omega_- = \partial\Omega \cap \{x \in R^n : x_1 \leq 0\}$, $I = \partial\Omega \cap \{x \in R^n : x_1 = 0\}$. For each point $x = (x_1, x_2, ..., x_n) \in \Omega$ we associate the *"opposite"* one $x^* = (-x_1, \alpha_2 x_2, ..., \alpha_n x_n) \in \Omega$, where $\alpha_j, j = 2, ..., n$ take one of the values ± 1. Obviously, if $x \in \partial\Omega_+$, then $x^* \in \partial\Omega_-$. Consider the following two types of problems $(k = 1, 2)$.

Find a function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfying the Poisson equation in Ω

$$-\Delta u(x) = f(x),\ x \in \Omega \tag{1}$$

and satisfying the following boundary conditions on the boundary of Ω

$$u(x) - (-1)^k u(x^*) = \tau(x),\ x \in \partial\Omega_+, \tag{2_k}$$

$$\frac{\partial u}{\partial r}(x) + (-1)^k \frac{\partial}{\partial r} u(x^*) = \mu(x),\ x \in \partial\Omega_+, \tag{3_k}$$

where $\tau \in C^{1+\varepsilon}(\partial\Omega_+)$, $\mu \in C^{\varepsilon}(\partial\Omega_+)$ and $f \in C^{\varepsilon}(\overline{\Omega})$, $0 < \varepsilon < 1$.

When $k = 1$ we call the problem *antiperiodic boundary value problem*, and when $k = 2$ the problem is called *periodic*.

3. The following compatibility conditions are necessary for the existence of a solution :

$$\tau(0,x_2,...,x_n)+(-1)^k\tau(0,\alpha_2x_2,...,\alpha_nx_n)=0,\ (0,x_2,...,x_n)\in I, \tag{4}$$

$$\frac{\partial\tau}{\partial x_j}(0,x_2,...,x_n)+(-1)^k\frac{\partial\tau}{\partial x_j}(0,\alpha_2x_2,...,\alpha_nx_n)=0,\ j=\overline{1,n},\ (0,x_2,...,x_n)\in I, \tag{5}$$

and

$$\mu(0,x_2,...,x_n)-(-1)^k\mu(0,\alpha_2x_2,...,\alpha_nx_n)=0,\ (0,x_2,...,x_n)\in I \tag{6}$$

Hereafter we assume that these conditions are satisfied.

4. The uniqueness of a solution is proved by using of the extremum principle and the Zaremba-Giraud principle.

Theorem 1.

a) The solution of the antiperiodic boundary value problem (1), (2_1), (3_1) *is unique.*

b) The solution of the periodic boundary value problem (1), (2_2), (3_2) *is unique up to an additive constant.*

5. The existence of a solution is proved by the method of the Green function.

Theorem 2. *Let* $f\in C^{\varepsilon}(\bar{\Omega})$, $\tau\in C^{1+\varepsilon}(\partial\Omega_+)$, $\mu\in C^{\varepsilon}(\partial\Omega_+)$, $0<\varepsilon<1$, *and the necessary compatibility conditions* $(4)-(6)$ *be satisfied.*

Then the solution of the antiperiodic problem (1), (2_1), (3_1) *exists, is unique and can be represented in the form:*

$$u(x)=\int_{\Omega}G_1(x,y)f(y)dy-\int_{\partial\Omega_+}\frac{\partial G_1}{\partial n_y}(x,y)\tau(y)ds_y+\int_{\partial\Omega_+}G_1(x,y)\mu(y)ds_y,$$

where $G_1(x,y)$ *is the Green function of the antiperiodic problem* (1), (2_1), (3_1)*:*

$$G_1(x,y)=\frac{1}{2}\left[G_D(x,y)+G_D(x,y^*)+G_N(x,y)-G_N(x,y^*)\right].$$

Here $G_D(x,y)$ *and* $G_N(x,y)$ *are the Green functions of the Dirichlet and Neumann problems respectively.*

Note that despite the fact that the Green function of the Neumann problem is defined up to an additive constant, the Green function of the antiperiodic problem is defined uniquely.

Theorem 3. *Let* $f\in C^{\varepsilon}(\bar{\Omega})$, $\tau\in C^{1+\varepsilon}(\partial\Omega_+)$, $\mu\in C^{\varepsilon}(\partial\Omega_+)$, $0<\varepsilon<1$, *and the compatibility conditions* $(4)-(6)$ *be satisfied.*

Then the periodic problem (1), (2_2), (3_2) *is solvable if and only if the following condition*

$$\int_{\Omega}f(y)ds_y=\int_{\partial\Omega_+}\mu(y)ds_y$$

holds.

If this condition holds, then the solution exists, is unique up to an additive constant and can be represented in the form:

$$u(x)=\int_{\Omega}G_2(x,y)f(y)ds_y-\int_{\partial\Omega_+}\frac{\partial G_2}{\partial n_y}(x,y)\tau(y)ds_y+\int_{\partial\Omega_+}G_2(x,y)\mu(y)ds_y+Const,$$

where $G_2(x,y)$ *is the Green function of the periodic problem* (1), (2_2), (3_2) *defined by the equality*

$$G_2(x,y) = \frac{1}{2}\left[G_D(x,y) - G_D(x,y^*) + G_N(x,y) + G_N(x,y^*)\right] + Const.$$

Theorem 4. *The problems* $(1), (2_k), (3_k)$ *are self-adjoint,* $k = 1, 2$.

6. Now we present a method of constructing of all eigenfunctions of the problems $(1), (2_k), (3_k)$, $k = 1, 2$. We start with considering two auxiliary problems for the equation

$$-\Delta u(x) = \lambda u(x), \tag{7}$$

We consider the Dirichlet problem

$$-\Delta v(x) = \lambda v(x),\ |x| < 1;\ v(x) = 0,\ |x| = 1, \tag{8}$$

and the Neumann problem

$$-\Delta v(x) = \lambda v(x),\ |x| < 1;\ \frac{\partial v}{\partial r}(x) = 0,\ |x| = 1. \tag{9}$$

Both of the problems are self-adjoint, hence root subspaces consist only of eigenfunctions. The Dirichlet and Neumann problems possess the following *symmetry properties of eigenfunctions.*

Lemma 1. *All eigenfunctions of the Dirichlet problem* (8) *and of the Neumann problem* (9) *can be chosen so that they have one of the symmetry properties :*

$$v(x) + v(x^*) = 0 \tag{10}$$

or

$$v(x) - v(x^*) = 0. \tag{11}$$

Theorem 5. *The system of all eigenfunctions of the antiperiodic problem* (1), (2_1), (3_1) *consists only of the eigenfunctions of the Dirichlet problem* (8) *having the symmetry property* (11), *and of the eigenfunctions of the Neumann problem* (9) *having the symmetry property* (10).

Theorem 6. *The system of all eigenfunctions of the periodic problem* (1), (2_2), (3_2) *consists only of the eigenfunctions of the Dirichlet problem (8) having the symmetry property* (10), *and of the eigenfunctions of the Neumann problem* (9) *having the symmetry property* (11).

7. We give two examples for the two-dimensional case.

Example 1. In the unit disc $0 < r < 1$ we consider the antiperiodic problem for equation (7) with the boundary conditions

$$\begin{cases} u(1,\varphi) + u(1,2\pi-\varphi) = 0,\ 0 \le \varphi \le \pi, \\ \frac{\partial u}{\partial r}(1,\varphi) - \frac{\partial u}{\partial r}(1,2\pi-\varphi) = 0,\ 0 \le \varphi \le \pi. \end{cases}$$

Here $x = (r, \varphi),\ x^* = (r, 2\pi - \varphi)$. The eigenfunctions of the problem are expressed via the Bessel function $J_n(\mu r)$ and have the form:

$$\begin{array}{l} u_{nk0}(r,\varphi) = J_n\left(\mu_{k0}^{(n)} r\right)\cos n\varphi, \\ u_{nk1}(r,\varphi) = J_n\left(\mu_{k1}^{(n)} r\right)\sin n\varphi, \end{array},\ n = 0, 1, 2, ...$$

Here $\mu_{k0}^{(n)}$ are the roots of the equation $J_n(\mu) = 0$ and $\mu_{k1}^{(n)}$ are the roots of $J_n'(\mu) = 0$.

So, we take here **all the eigenvalues** of the Dirichlet and Neumann problems but to each of them only one eigenfunction corresponds.

Example 2. In the unit disc $0 < r < 1$ we consider the antiperiodic problem for equation (7) with the boundary conditions

$$\begin{cases} u(1,\varphi) + u(1,\varphi+\pi) = 0,\ 0 \le \varphi \le \pi, \\ \frac{\partial u}{\partial r}(1,\varphi) - \frac{\partial u}{\partial r}(1,\varphi+\pi) = 0,\ 0 \le \varphi \le \pi. \end{cases}$$

Here $x = (r, \varphi),\ x^* = (r, \varphi + \pi)$. The eigenfunctions of the problem have the form:

$$\begin{array}{l} u_{nk0}(r,\varphi) = J_n\left(\mu_{k}^{(n)} r\right)\cos n\varphi \\ u_{nk1}(r,\varphi) = J_n\left(\mu_{k}^{(n)} r\right)\sin n\varphi \end{array},\ n = 0, 1, 2, ...,$$

where by $\mu_k^{(n)}$ for $n = 2j$ we denote the roots of the equation $J_n(\mu) = 0$, and for $n = 2j + 1$ we denote the roots of the equation $J_n'(\mu) = 0$.

So, we take only **a part ("half") of the eigenvalues** $\mu_k^{(n)}$ of the Dirichlet problem for $n = 2j$ and **a part ("half") of the eigenvalues** $\mu_k^{(n)}$ of the Neumann problem for $n = 2j + 1$. To each eigenvalue two eigenfunctions correspond.

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