Short communications

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SOLVABILITY CONDITIONS FOR THE FIRST ORDER ELLIPTIC SYSTEMS ON THE PLANE

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Abstract. Unique solvability conditions are obtained for an elliptic system of two first order real partial differential equations with unbounded coefficients.

The general elliptic system of two first order real partial differential equations on the plane $\mathbb{R}^2$ can be written in the form (see [1], p. 106)

$$\begin{cases} -v_y + a_{11}(x,y)u_x + a_{12}(x,y)u_y + a_1(x,y)u + a_2(x,y)v = f(x,y) \\ u_y + a_{21}(x,y)v_x + a_{22}(x,y)v_y + a_3(x,y)u + a_4(x,y)v = g(x,y). \end{cases} \tag{1}$$

Everywhere in this work we assume that $a_{kj}\,(k,j=1,2)$ are continuously differentiable and bounded functions, and $a_l\,(l=\overline{1,4})$ are continuous. System (1) arises in a number of problems in fluid dynamics, gas dynamics, in the theory of quasiconformal mappings and in the theory of surfaces and membranes. Well-known particular cases of systems of form (1), such as the generalized Cauchy-Riemann system and Beltramy-type system, as well as more general elliptic systems have been studied in many papers. A systematic presentation of the results, as well as the literature's review are presented in the monographs [1-3]. The main results are concerned with the use of a number of properties of analytic functions that are extended to solutions of elliptic systems with variable coefficients based on some relationship between them. In the theory developed in [1-3] it is assumed in the majority of cases, that the coefficients $a_l\,(l=\overline{1,4})$ in a certain sense tend to 0 at infinity.

Considerably less papers are devoted to the case in which the coefficients $a_l\,(l=\overline{1,4})$ are separated [1] from 0. In [4-6] qualitative properties of solutions of a generalized Cauchy-Riemann system and its unique solvability were studied. In [4, 5] the Liouville theorem for generalized analytic functions, proved in [1], was essentially used. Note

[1] that is there exists $\delta > 0$ such that $a_l(x,y) \geq \delta$ for all $(x,\,y) \in \mathbb{R}^2$.

that if a_l $(l=\overline{1,4})$ are separated from 0 or are unbounded, then a solution is not always unique. For example, the system

$$w_z + 2z(1+|z|^2)w = 0$$

has the solutions

$$w_n = z^n exp[-(1+|z|^2]^2 \in L_2, \quad n = 1,\ 2,\ \dots\ .$$

Furthemore, the solution of the system $w_z + aw + b\overline{w} = 0$, where a, b are constants, is trivial if and only if $|b| - |a| > 0$ [4]. Therefore in [4-6] additional conditions on the behavior of the coefficients a_l $(l=\overline{1,4})$ were imposed to ensure the uniqueness of a solution.

The present communication is devoted to the existence and uniqueness problem for general system (1) when a_l $(l=\overline{1,4})$ are separated from 0 and are unbounded. Here we use the approach that is similar to the methods of [6, 7], where the case of constant senior coefficients a_{kj} $(k,j=1,2)$ was considered.

The ellipticity conditions for the system (1) have the form (see [1], chapter 2) for some $\Delta_0 > 0$:

$$a_{11}a_{21} > 0,\ \ 4a_{11}a_{21} - (a_{11}a_{22} + a_{12}a_{21})^2 \geq \Delta_0 \tag{2}$$

for all $(x,\ y) \in \mathbb{R}^2$. Denote

$$A_1 = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{21} \end{pmatrix},\ \ A_2 = \begin{pmatrix} a_{12} & -1 \\ 1 & a_{22} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},\ \ w = \begin{pmatrix} u \\ v \end{pmatrix},\ \ F = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Then the left-hand side of (1) can be written as

$$lw = (A_1\partial_x + A_2\partial_y + A_3)w.$$

Assume that $D(l) = C_0^{(1)}(\mathbb{R}^2,\ \mathbb{R}^2)$, where $C_0^{(1)}(\mathbb{R}^2,\ \mathbb{R}^2)$ is the set of all continuously differentiable and compactly supported vector-functions, and denote by L the closure of l in L_2-norm. A function $w = \begin{pmatrix} u \\ v \end{pmatrix} \in D(L)$ is called a solution to (1), if there is a sequence $\{w_n\}_{n=1}^{\infty} \subseteq C_0^{(1)}(\mathbb{R}^2,\ \mathbb{R}^2)$ such that $||w_n - w||_2 \to 0$, $||lw_n - F||_2 \to 0$ as $n \to \infty$. Here $\|\cdot\|_2$ is the norm in $L_2 = L_2(\mathbb{R}^2,\ \mathbb{R}^2)$.

Theorem 1. *Assume that the functions* a_{kj} $(k,j=1,2)$ *are continuously differentiable and bounded, and* a_l $(l=\overline{1,4})$ *are continuous. Let* $\Delta_0,\ \delta_0,\ \delta_1 > 0$*, inequalities* (2) *and the inequalities*

$$2a_1 - [(a_{11})_x + (a_{12})_y] \geq \delta_0 > 0, \tag{3}$$

$$[a_2 - a_3]^2 - 4\{a_1 - \delta_1 - 0,5[(a_{11})_x + (a_{12})_y]\}\{-a_4 - \delta_1 - 0,5[(a_{21})_x - (a_{22})_y]\} \leq 0, \delta_1 > 0. \tag{4}$$

be satisfied for all $(x,\ y) \in \mathbb{R}^2$.

Then the solution w *of system* (1) *is unique and for* w *the following estimate holds:*

$$\|w\|_2 \leq C||Lw||_2, \tag{5}$$

where $C > 0$ is independent of w.

Let $w = \begin{pmatrix} u \\ v \end{pmatrix} \in C_0^{(1)}(\mathbb{R}^2, \mathbb{R}^2)$, and denote $\overline{w} = \begin{pmatrix} u \\ -v \end{pmatrix}$. The proof is based on the equality

$$(Lw, \overline{w})$$

$$= \int\limits_{\mathbb{R}^2} [a_1 - 0,5(a_{11})_x - 0,5(a_{12})_y]u^2 + [-a_4 - 0,5(a_{21})_x - 0,5(a_{22})_y]v^2 + (a_2 - a_3)uv\}dxdy,$$

which can be obtained by integrating by parts. By (3) and (4) it follows that

$$(Lw, \overline{w}) \geq ||w||_2^2, \tag{6}$$

which by the Schwartz inequality implies (5). Since the operator L is closed, inequality (5) also holds for the solution w of system (1), which implies the uniqueness.

Theorem 2. *Assume that a_{kj} $(k, j = 1, 2)$ are continuously differentiable and bounded functions, and a_l $(l = 1, 4)$ are continuous. Let Δ_0, δ, δ_1, ε_1, ε_2, $\varepsilon_3 > 0$, inequalities* (2) *and the following inequalities*

$$\begin{cases} a_1 - 0,5[|(a_{11})_x| + |(a_{12})_y|] - \varepsilon_1(|a_{11}| + |a_{12}|) \geq \delta, \\ \\ -a_4 - 0,5[|(a_{21})_x| + |(a_{22})_y|] - \varepsilon_2[|a_{21}| + |a_{22}|] \geq \delta, \\ \\ (a_2 - a_3)^2 - (4 - \varepsilon_3)\{a_1 - \delta_1 - 0,5[(a_{11})_x - (a_{22})_y]\}\times \\ \times\{a_4 - \delta_1 - 0,5[(a_{21})_x - (a_{22})_y]\} \leq 0, \end{cases} \tag{7}$$

hold for all $(x, y) \in \mathbb{R}^2$.

Then system (1) *has a solution w for any $F \in L_2$.*

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