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NIKOL'SKII-BESOV AND LIZORKIN-TRIEBEL SPACES CONSTRUCTED ON THE BASE OF THE MULTIDIMENSIONAL FOURIER-BESSEL TRANSFORM

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Abstract. In this paper we define the Nikol'skii-Besov and Lizorkin-Triebel spaces (B-Nikol'skii-Besov and B-Lizorkin-Triebel spaces) in the context of the Fourier-Bessel harmonic analysis. We establish some basic properties of the B-Nikol'skii-Besov and B-Lizorkin-Triebel spaces such as embedding theorems, the lifting property, and characterizing of the Bessel potentials in terms of the B-Lizorkin-Triebel spaces. We prove the inclusion and the density of the Schwartz space in the B-Nikol'skii-Besov and B-Lizorkin-Triebel spaces and prove an interpolation formula for these spaces by the real method. We also prove the Young inequality for the B-convolution operators in the B-Bessel potential spaces. Finally, we give some applications involving the Laplace-Bessel differential operator.

1 Introduction

Let $\mathbb{R}^n_{k,+} \equiv (0,\infty)^k \times \mathbb{R}^{n-k}$ be the part of the euclidean space $\mathbb{R}^n$ of points $x = (x_1,...,x_n) = (x', x'')$ defined by the inequalities $x_1 > 0, \dots, x_k > 0$, $1 \le k \le n$, where $x' = (x_1,...,x_k)$, and $x'' = (x_{k+1},...,x_n)$. Let $E(x,t) = \{y \in \mathbb{R}^n_{k,+}\ ;\ |x-y| < t\}$ and $^{\complement}E(x,t) = \mathbb{R}^n_{k,+} \backslash E(x,t)$. For any measurable set $A \subset \mathbb{R}^n_{k,+}$, define $|A|_\gamma = \int_A (x')^\gamma dx$, where $(x')^\gamma = x_1^{\gamma_1} \cdot \ldots \cdot x_k^{\gamma_k}$, $\gamma = (\gamma_1,...,\gamma_k)$ is a vector consisting of fixed positive numbers, also $|\gamma| = \gamma_1 + ... + \gamma_k$.

Let $B = (B_1, \dots, B_k)$, $B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$, $\gamma_i > 0$, $i = 1,\dots,k$, $1 \le k \le n$, be the (multidimensional) Bessel differential operator, and $\mathbb{S}_{k,+} \equiv \mathbb{S}(\mathbb{R}^n_{k,+})$ be the Schwarz space of all functions which are the restrictions to $\mathbb{R}^n_{k,+}$ of test functions that are even with respect to $x_1,\dots,x_k$ and decreasing sufficiently rapidly at infinity, together with all derivatives of the form

$$D^\alpha_\gamma \equiv B^{\alpha'} D^{\alpha''}_{x''} := B_1^{\alpha_1} \dots B_k^{\alpha_k} D_{k+1}^{\alpha_{k+1}} ... D_n^{\alpha_n} \quad \text{if} \ \ 1 \le k \le n-1,$$

and

$$D^\alpha_\gamma \equiv B^\alpha := B_1^{\alpha_1} \dots B_n^{\alpha_n} \quad \text{if} \ \ k = n,$$

i.e., for all $\varphi \in \mathbb{S}_{k,+}$, $\sup_{x\in\mathbb{R}^n_{k,+}} \left|x^\beta D^\alpha_\gamma \varphi(x)\right| < \infty$ if $1 \leq k \leq n-1$, and $\sup_{x\in\mathbb{R}^n_{n,+}} \left|x^\beta B^\alpha \varphi(x)\right| < \infty$ if $k = n$, where $D_i = \partial/\partial x_i$, $k+1 \leq i \leq n$, α and β are multi-indices and $x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$.

The closure of the space $\mathbb{S}_{k,+}$ in the norm

$$||f||_{L_{p,\gamma}} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx \right)^{1/p} < \infty$$

is denoted by $L_{p,\gamma} \equiv L_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1 \leq p < \infty$. The space of the essentially bounded measurable function on $\mathbb{R}^n_{k,+}$ is denoted by $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$.

The space $\mathbb{S}_{k,+}$ is equipped with the usual topology. We denote by $\mathbb{S}'_{k,+} \equiv \mathbb{S}'(\mathbb{R}^n_{k,+})$, the collection of all tempered distributions on $\mathbb{R}^n_{k,+}$, equipped with the strong topology. The Fourier-Bessel transform and its inverse on $\mathbb{S}_{k,+}$ (see [9, 13]) are defined by

$$F_\gamma f(x) = \int_{\mathbb{R}^n_{k,+}} f(y) \prod_{i=1}^{k} j_{\frac{\gamma_i - 1}{2}}(x_i y_i)\, e^{-i(x'' y'')}\, (y')^\gamma dy,$$

$$F_\gamma^{-1} f(x) = A_{k,\gamma} F_\gamma f(-x), \tag{1}$$

where $j_\nu(t)$ $(t > 0,\ \nu > -1/2)$ is connected with the Bessel function of the first kind $J_\nu(t)$ as follows

$$j_\nu(t) = 2^\nu \Gamma(\nu+1) \frac{J_\nu(t)}{t^\nu} \tag{2}$$

and

$$A_{k,\gamma} = 2^{-|\gamma|} \prod_{i=1}^{k} \Gamma^{-2}((\gamma_i + 1)/2).$$

It is well known that the Fourier-Bessel transform is closely related to the Laplace-Bessel differential operator

$$\Delta_\gamma = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \quad \gamma_1 > 0, \dots, \gamma_k > 0.$$

The expression (1) is a hybrid of the Hankel transform with respect to the variables $x_1, \dots, x_k$ and the ordinary Fourier transform with respect to the variables $x_{k+1}, \dots, x_n$. These transforms and related problems for singular PDE and integral operators were studied by B.M. Levitan, I.A. Kipriyanov and his collaborators, K. Trimeche and his collaborators, K. Stempak, A.D. Gadjiev, I.A. Aliev, V.S. Guliyev, and others.

The generalized convolution (B-convolution) $f \otimes g$ associated with the Δ_γ is defined by

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)\, T^y g(x)\, (y')^\gamma dy,$$

where T^y is the generalized shift operator (B-shift) defined by

$$T^y f(x) = C_{k,\gamma} \int_0^\pi \dots \int_0^\pi f\left((x', y')_\theta, x'' - y''\right)\, d\nu(\theta),$$

with

$$C_{k,\gamma} = \pi^{-\frac{k}{2}} \prod_{i=1}^{k} \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}, \quad d\nu(\theta) = \prod_{i=1}^{k} \sin^{\gamma_i-1}\theta_i d\theta_i, \quad 1 \le k \le n,$$

$$(x', y')_\theta = ((x_1, y_1)_{\theta_1}, \dots, (x_k, y_k)_{\theta_k}), \quad (x_i, y_i)_{\theta_i} = \sqrt{x_i^2 - 2x_i y_i \cos\theta_i + y_i^2}$$

(see for example [7, 8, 11]).

For the B-convolution $f \otimes g$ the Young inequality

$$\|f \otimes g\|_{r,\gamma} \le \|f\|_{p,\gamma} \|g\|_{q,\gamma}, \quad 1 \le p, q, r \le \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

holds. The B-convolution $f \otimes g$ plays an important role in the study of the B-Nikol'skii-Besov and B-Lizorkin-Triebel spaces.

It is well known that (see [1, 7, 11, 12])

$$F_\gamma (B_i^{\alpha_i} f)(x) = (-x_i^2)^{\alpha_i} F_\gamma f(x), \quad i = 1, \dots, k,$$

$$F_\gamma \left(D_i^{2\alpha_i} f\right)(x) = (-x_i^2)^{\alpha_i} F_\gamma f(x), \quad i = k+1, \dots, n,$$

$$F_\gamma \left(D_\gamma^{(\alpha', 2\alpha'')} f\right)(x) = (-1)^{|\alpha|} x^{2\alpha} F_\gamma f(x),$$

$$F_\gamma (\Delta_\gamma f)(x) = -|x|^2 F_\gamma f(x) \quad \text{and} \quad F_\gamma (f \otimes g) = F_\gamma f \; F_\gamma g,$$

$$F_\gamma (\lambda I - \Delta_\gamma f)(x) = \left(\lambda + |x|^2\right) F_\gamma f(x) \tag{3}$$

and

$$\|T^y f(\cdot)\|_{L_{p,\gamma}} \le \|f\|_{L_{p,\gamma}}, \quad \text{for all } y \in \mathbb{R}^n_{k,+}, \quad f \in L_{p,\gamma}(\mathbb{R}^n_{k,+}), \quad 1 \le p \le \infty.$$

In this paper, we study the Nikol'skii-Besov and Lizorkin-Triebel spaces $B^s_{p,q,\gamma}$ and $F^s_{p,q,\gamma}$ (B-Nikol'skii-Besov and B-Lizorkin-Triebel spaces) defined on the basis of the Fourier-Bessel transform F_γ given by the equality (1.1). Such spaces were studied by Altenburg [2], Assal and Ben Abdallah [3], Baez and Rodriguez [6], Betancor and Rodrнguez-Mesa [5], and Pathak and Pandey [14] associated with the Fourier-Bessel transform (Hankel transform) on the interval $I = (0, \infty)$. We prove embedding theorems between $B^s_{p,q,\gamma}$ and $F^s_{p,q,\gamma}$. We show the inclusion and the density of the Schwartz space in B-Nikol'skii-Besov and B-Lizorkin-Triebel spaces and we prove an interpolation formula for these spaces by the real method. We prove the Young inequality for the B-convolution operators in B-Bessel potential spaces. Then, we prove a one-to-one mapping property of the B-Bessel potential spaces $H^s_{p,\gamma}$ and of the B-Nikol′skiĭ-Besov $B^s_{p,q,\gamma}$. Some properties of the Rademacher functions are used to characterize $H^s_{p,\gamma}$ in terms of the B-Lizorkin-Triebel spaces $F^s_{p,q,\gamma}$; in particular, we characterize the B-Sobolev space $W^s_{p,\gamma}$. Finally, we give some applications to regularity results and to solving differential equations involving the Laplace-Bessel operator Δ_γ.

We use the letter C for a positive constant, independent of appropriate parameters and not necessarily the same at each occurrence.

2 Preliminaries

Definition 1. *Let $m \in \mathbb{S}'_{k,+}$. A generalized function m is called a Fourier-Bessel multiplier (B-multiplier) in $L_{p,\gamma}$, if for all $f \in \mathbb{S}_{k,+}$ the B-convolution $\left(F_\gamma^{-1} m(\xi)\right) \otimes f$ belongs to $L_{p,\gamma}$, and if*

$$\sup_{\|f\|_{L_{p,\gamma}}=1} \left\| \left(F_\gamma^{-1} m\right) \otimes f \right\|_{L_{p,\gamma}}$$

is finite. The linear space of all such m is denoted by $M_{p,\gamma} \equiv M_{p,\gamma}(\mathbb{R}^n_{k,+})$; the norm on $M_{p,\gamma}$ is the above supremum, we denote it by $\| \, . \, \|_{M_{p,\gamma}}$.

Theorem A (Mikhlin's theorem on B-multipliers [12]). *Let $m \in \mathbf{C}^s_{even}(\mathbb{R}^n_{k,+})$ (i.e., $m \in \mathbf{C}^s$ and m is even with respect to the variables $x_1, \dots, x_k$), where s is the least even number, larger than $\frac{1}{2}(n + |\gamma|)$. Assume that there exists a constant C such that for all $\xi \in \mathbb{R}^n_{k,+}$ and for all multi-indices α satisfying $2|\alpha| \le s$*

$$|\xi|^{2|\alpha|} \left| D_\gamma^{(\alpha',2\alpha'')} m(\xi) \right| \le C.$$

Then $m \in M_{p,\gamma}$ for $1 < p < \infty$.

Lemma 1. *If $1 \le p \le q \le 2$, then $M_{p,\gamma} \subset M_{q,\gamma}$. Also, if $1/p + 1/p' = 1$, $1 \le p \le \infty$, then $M_{p,\gamma} = M_{p',\gamma}$ (with equality of norms).*

Lemma 2. *Let l be an even number such that $l > (n + |\gamma|)/2$, and let $m \in L_{2,\gamma}$ and $D_\gamma^{(\alpha',2\alpha'')} m \in L_{2,\gamma}$, $2|\alpha| = l$. Then $m \in M_{p,\gamma}$, $1 \le p \le \infty$, and*

$$\|m\|_{M_{p,\gamma}} \le C \|m\|^{1-\theta}_{L_{2,\gamma}} \left(\sup_{2|\alpha|=l} \|D_\gamma^{(\alpha',2\alpha'')} m\|_{L_{2,\gamma}} \right)^\theta, \quad \theta = (n + |\gamma|)/2l.$$

Proof. Let $t > 0$. Applying Hölder's and Parseval's inequalities, we obtain

$$\int_{{}^{\complement}E(0,t)} \left|F_\gamma^{-1} m(x)\right| (x')^\gamma dx = \int_{{}^{\complement}E(0,t)} x^{-\alpha} x^\alpha \left|F_\gamma^{-1} m(x)\right| (x')^\gamma dx$$

$$\le \left(\int_{{}^{\complement}E(0,t)} x^{-2\alpha} (x')^\gamma dx \right)^{1/2} \left(\int_{{}^{\complement}E(0,t)} x^{2\alpha} \left|F_\gamma^{-1} m(x)\right|^2 (x')^\gamma dx \right)^{1/2}$$

$$= C t^{(n+|\gamma|-l)/2} \cdot \left(\int_{{}^{\complement}E(0,t)} x^{2\alpha} \left|F_\gamma^{-1} m(x)\right|^2 (x')^\gamma dx \right)^{1/2},$$

where ${}^{\complement}E(0,t) = \mathbb{R}^n_{k,+} \backslash E(0,t)$.

Taking into account that $x^{2\alpha} F_\gamma^{-1} m(x) = (-1)^{|\alpha|} F_\gamma^{-1} \left(D_\gamma^{(\alpha',2\alpha'')} m \right)(x)$ and

$$\left(\int_{{}^{\complement}E(0,t)} x^{2\alpha} |F_\gamma^{-1} m(x)|^2 (x')^\gamma dx \right)^{1/2} = \left(\int_{{}^{\complement}E(0,t)} \left| F_\gamma^{-1} \left(D_\gamma^{(\alpha',2\alpha'')} m(x) \right) \right|^2 (x')^\gamma dx \right)^{1/2}$$

$$\le C \left(\int_{{}^{\complement}E(0,t)} \left| D_\gamma^{(\alpha',2\alpha'')} m(x) \right|^2 (x')^\gamma dx \right)^{1/2} \le C \sup_{2|\alpha|=l} \|D_\gamma^{(\alpha',2\alpha'')} m\|_{L_{2,\gamma}},$$

we get

$$\int_{\complement_{E(0,t)}} \left|F_\gamma^{-1} m(x)\right| (x')^\gamma dx \le C\, t^{(n+|\gamma|-l)/2} \sup_{2|\alpha|=l} \|D_\gamma^{(\alpha',2\alpha'')} m\|_{L_{2,\gamma}}.$$

Similarly, we prove that

$$\begin{aligned}\int_{E(0,t)} \left|F_\gamma^{-1} m(x)\right| (x')^\gamma dx &\le \left(\int_{E(0,t)} (x')^\gamma dx\right)^{1/2} \left(\int_{E(0,t)} \left|F_\gamma^{-1} m(x)\right|^2 (x')^\gamma dx\right)^{1/2} \\ &\le C t^{(n+|\gamma|)/2} \left(\int_{E(0,t)} \left|F_\gamma^{-1} m(x)\right|^2 (x')^\gamma dx\right)^{1/2} \\ &\le C t^{(n+|\gamma|)/2} \|m\|_{L_{2,\gamma}}.\end{aligned}$$

We choose t so that

$$\|m\|_{L_{2,\gamma}} = t^{-l} \sup_{2|\alpha|=l} \|D_\gamma^{(\alpha',2\alpha'')} m\|_{L_{2,\gamma}},$$

and by virtue of Lemma 1, we conclude that for $1 \le p \le 2$,

$$\begin{aligned}\|m\|_{M_{p,\gamma}} \le \|m\|_{M_{1,\gamma}} &= \int_{\mathbb{R}^n_{k,+}} \left|F_\gamma^{-1} m(x)\right| (x')^\gamma dx \\ &\le C \|m\|_{L_{2,\gamma}}^{1-\theta} \left(\sup_{2|\alpha|=l} \|D_\gamma^{(\alpha',2\alpha'')} m\|_{L_{2,\gamma}}\right)^\theta.\end{aligned}$$

For $2 < p \le \infty$, by virtue of Lemma 1, we have

$$\begin{aligned}\|m\|_{M_{p,\gamma}} = \|m\|_{M_{p',\gamma}} &\le \|m\|_{M_{1,\gamma}} \\ &\le C \|m\|_{L_{2,\gamma}}^{1-\theta} \left(\sup_{2|\alpha|=l} \|D_\gamma^{(\alpha',2\alpha'')} m\|_{L_{2,\gamma}}\right)^\theta.\end{aligned}$$

□

3 Characterization of the B-Nikol'skii-Besov spaces

In this section, we define the B-Nikol'skii-Besov and B-Lizorkin-Triebel spaces and we prove some relations between them. We give a characterization of $B^s_{p,q,\gamma}$ in terms of $b^s_{p,q,\gamma}$ and an embedding theorem between $B^s_{p,q,\gamma}$ and $F^s_{p,q,\gamma}$.

Definition 2. *Let $s \in \mathbb{R}$, for $1 \le p < \infty$ we define the sequence spaces l^s_p as*

$$l^s_p = \Big\{u \ :\ u = \{u_j\}_{j=0}^\infty,\ \|u\|_{l^s_p} = \Big(\sum_{j=0}^\infty \left(2^{jsp}|u_j|^p\right)\Big)^{1/p} < \infty\Big\}, \tag{4}$$

and for $p = \infty$

$$l^s_\infty = \Big\{u \ : u = \{u_j\}_{j=0}^\infty,\ \|u\|_{l^s_\infty} = \sup_j 2^{js}|u_j| < \infty\Big\}.$$

In the case of $s = 0$ we denote l^0_p by l_p.

Definition 3. *Let Φ be the collection of all systems $\{\varphi_j(x)\}_{j=0}^{\infty} \subset \mathbb{S}_{k,+}$ with the following properties*

i) $\varphi_j(x) \in \mathbb{S}_{k,+}$, $F_\gamma \varphi_j(x) \geq 0$ for $j = 0, 1, 2, 3, ...$;

ii) supp $F_\gamma \varphi_j \subset A_j \equiv \{x \in \mathbb{R}^n_{k,+} : \sqrt{2^{j-1} - 1} \leq |x| \leq \sqrt{2^{j+1} - 1}\}$ for $j = 1, 2, 3, ...$ and supp $F_\gamma \varphi_0 \subset \{x \in \mathbb{R}^n_{k,+} : |x| \leq 1\}$;

iii) exists a positive number C such that

$$\left| D_\gamma^{(\alpha', 2\alpha'')} F_\gamma \varphi_j(x) \right| \leq C|x|^{-|\alpha|} \quad for \quad j = 1, 2, ..., \quad 0 \leq |\alpha| \leq [(|\gamma| - 1)/2] + 2 \ ;$$

iv) $\sum\limits_{j=0}^{\infty} F_\gamma \varphi_j(x) = 1$ for every $x \in \mathbb{R}^n_{k,+}$.

It is clear that Φ is not empty.

In what follows we define the B-Nikol'skii-Besov spaces $B^s_{p,q,\gamma}$ and $b^s_{p,q,\gamma}$ on the basis of the Fourier-Bessel transform F_γ.

Definition 4. *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $s \in \mathbb{R}$. Then for any system of functions $\{\varphi\}_{j=0}^{\infty} \in \Phi$, the B-Nikol'skii-Besov spaces are defined by*

$$B^s_{p,q,\gamma} \equiv B^s_{p,q,\gamma}(\mathbb{R}^n_{k,+}) = \left\{ f \in \mathbb{S}'_{k,+} : \|f\|_{B^s_{p,q,\gamma}} = \|\{\varphi_j \otimes f\}\|_{l^s_q(L_{p,\gamma})} < \infty \right\},$$

where

$$\| \cdot \|_{l^s_q(L_{p,\gamma})} = \| \| \cdot \|_{L_{p,\gamma}} \|_{l^s_q} = \left(\sum_{j=0}^{\infty} \left(2^{sj} \| \cdot \|_{L_{p,\gamma}} \right)^q \right)^{1/q} .$$

Definition 5. *For $s \in \mathbb{R}$, $1 \leq p \leq \infty$, $1 \leq q < \infty$, we define*

$$b^s_{p,q,\gamma} \equiv b^s_{p,q,\gamma}(\mathbb{R}^n_{k,+}) = \Bigg\{ f : f \in \mathbb{S}_{k,+}, \quad f \underset{\mathbb{S}_{k,+}}{=} \sum_{i=0}^{\infty} a_i(x),$$

$$\|\{a_i\}\|_{l^s_q(L_{p,\gamma})} = \left(\sum_{i=0}^{\infty} \left(2^{si} \|a_i(\cdot)\|_{L_{p,\gamma}} \right)^q \right)^{1/q} < \infty \Bigg\}$$

and for $q = \infty$, we set

$$b^s_{p,\infty,\gamma} = \left\{ f : f \in \mathbb{S}_{k,+}, \quad f \underset{\mathbb{S}_{k,+}}{=} \sum_{i=0}^{\infty} a_i(x), \ \|\{a_i\}\|_{l^s_\infty(L_{p,\gamma})} = \sup_i 2^{si} \|a_i(\cdot)\|_{L_{p,\gamma}} < \infty \right\},$$

where supp$F_\gamma a_i \subset A_i$ for $i = 1, 2, 3, \ldots$ and supp$F_\gamma a_0 \subset \{\xi \in \mathbb{R}^n_{k,+} : |\xi| \leq 1\}$.

By $f(x) \underset{\mathbb{S}_{k,+}}{=} \sum\limits_{j=0}^{\infty} a_j(x)$ it is meant that $\sum\limits_{j=0}^{\infty} a_j(x)$ converges in $\mathbb{S}'_{k,+}$ to f. The norm of the function f in $b^s_{p,q,\gamma}$ is defined by

$$\|f\|_{b^s_{p,q,\gamma}} = \inf_{f = \sum a_i} \|\{a_i\}\|_{l^s_q(L_{p,\gamma})}.$$

Now, we can state the following theorem.

Theorem 1. *Let* $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$, $s \in \mathbb{R}$, $1 < p < \infty$ *and* $1 \le q \le \infty$. *Then*

$$B^s_{p,q,\gamma} = b^s_{p,q,\gamma}$$

and the corresponding norms are equivalent.

Proof. First, we prove that $B^s_{p,q,\gamma} \subset b^s_{p,q,\gamma}$. Let $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$, then we have

$$\Big(\sum_{j=0}^{\infty} F_\gamma \varphi_j\Big)(\xi) = 1.$$

Thus, for $f \in B^s_{p,q,\gamma}$

$$f = F_\gamma^{-1} F_\gamma f = F_\gamma^{-1}\left(\sum_{j=0}^{\infty} F_\gamma \varphi_j \cdot F_\gamma f\right)$$
$$\underset{\mathbb{S}'_{k,+}}{=} \sum_{j=0}^{\infty} F_\gamma^{-1}\left(F_\gamma \varphi_j \cdot F_\gamma f\right) = \sum_{j=0}^{\infty} \varphi_j \otimes f.$$

If we take $a_j = \varphi_j \otimes f$, then we get

$$\|f\|_{b^s_{p,q,\gamma}} \le \|\{a_j\}\|_{l^s_q(L_{p,\gamma})} = \|\{\varphi_j \otimes f\}\|_{l^s_q(L_{p,\gamma})} = \|f\|_{B^s_{p,q,\gamma}}.$$

Hence, we have $B^s_{p,q,\gamma} \subset b^s_{p,q,\gamma}$.

Conversely, we show that $b^s_{p,q,\gamma} \subset B^s_{p,q,\gamma}$. Let $f \in b^s_{p,q,\gamma}$ and $f = \sum\limits_{j=0}^{\infty} a_j$ in the sense of the convergence in $\mathbb{S}'_{k,+}$.

Let $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$. Then

$$(\varphi_j \otimes f)(x) \underset{\mathbb{S}'_{k,+}}{=} \sum_{i=0}^{\infty} (\varphi_j \otimes a_i)(x) = \sum_{i=j-1}^{j+1} (\varphi_j \otimes a_i)(x)$$

since $\varphi_j \otimes a_i = F_\gamma^{-1}(F_\gamma \varphi_j \cdot F_\gamma a_i) = 0$, for $i > j+1$ and $i < j-1$. Furthermore, if we define $\varphi_j = a_j = 0$ for $j < 0$, then we have

$$\|f\|_{B^s_{p,q,\gamma}} = \|\{\varphi_j \otimes f\}\|_{l^s_q(L_{p,\gamma})} \le \sum_{r=-1}^{1} \|\{\varphi_j \otimes a_{j+r}\}\|_{l^s_q(L_{p,\gamma})}. \tag{5}$$

On the other hand, by Theorem A with $1 < p < \infty$ we get

$$\|\{\varphi_j \otimes a_{j+r}\}\|_{L_{p,\gamma}} \le C\,\|a_{j+r}\|_{L_{p,\gamma}}, \tag{6}$$

where C is a suitable positive constant.

Now, by taking the norm of l^s_q in (6) it follows that

$$\|\{\varphi_j \otimes a_{j+r}\}\|_{l^s_q(L_{p,\gamma})} \le C\,\|a_{j+r}\|_{l^s_q(L_{p,\gamma})}.$$

Then from (5) we obtain

$$\|f\|_{B^s_{p,q,\gamma}} = \|\{\varphi_j \otimes f\}\|_{l^s_q(L_{p,\gamma})} \leq C \sum_{r=-1}^{1} \|\{a_{j+r}\}\|_{l^s_q(L_{p,\gamma})} \leq C \|\{a_j\}\|_{l^s_q(L_{p,\gamma})}. \tag{7}$$

Taking the infimum on the right-hand side of (7) we get

$$\|f\|_{B^s_{p,q,\gamma}} \leq C \|f\|_{b^s_{p,q,\gamma}}.$$

□

Remark 1. Note that by Theorem 1, the spaces $B^s_{p,q,\gamma}$ are independent of systems $\{\varphi_j\}_{j=0}^\infty \in \Phi$.

We define the B-Lizorkin-Triebel spaces as follows.

Definition 6. *Let $1 < p < \infty, 1 \leq q \leq \infty$ and $s \in \mathbb{R}_+$. For any system of functions $\{\varphi_j\}_{j=0}^\infty \in \Phi$, we define the B-Lizorkin-Triebel spaces by*

$$F^s_{p,q,\gamma} \equiv F^s_{p,q,\gamma}(\mathbb{R}^n_{k,+}) = \left\{ f \in \mathbb{S}'_{k,+} : \|f\|_{F^s_{p,q,\gamma}} = \|\{\varphi_j \otimes f\}\|_{L_{p,\gamma}(l^s_q)} < \infty \right\}, \tag{8}$$

where

$$\|\cdot\|_{L_{p,\gamma}(l^s_q)} = \left\| \|\cdot\|_{l^s_q} \right\|_{L_{p,\gamma}} = \left\| \left(\sum_{j=0}^{\infty} \left(2^{sj}(\cdot)\right)^q \right)^{1/q} \right\|_{L_{p,\gamma}}.$$

Theorem 2. *Let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$, then*

$$B^s_{p,\min(p,q),\gamma} \subset F^s_{p,q,\gamma} \subset B^s_{p,\max(p,q),\gamma} \tag{9}$$

where $\subset$ means continuous embedding.

Proof. We must show that

$$B^s_{p,p,\gamma} \subset F^s_{p,q,\gamma} \subset B^s_{p,q,\gamma} \tag{10}$$

for $p \leq q$, and

$$B^s_{p,q,\gamma} \subset F^s_{p,q,\gamma} \subset B^s_{p,p,\gamma} \tag{11}$$

for $q \leq p$. We will use the monotonicity of the l^s_q spaces and the trivial equality $B_{p,p,\gamma} \equiv F^s_{p,p,\gamma}$.

First we shall prove the left-hand-side embedding in (10). Let $f \in F^s_{p,q,\gamma}$ and $\{\varphi_j\}_{j=0}^\infty \in \Phi$, then we have

$$\|f\|_{B^s_{p,q,\gamma}} = \|\{\varphi_j \otimes f\}\|_{l^s_q(L_{p,\gamma})} = \left(\sum_{j=0}^{\infty} \left(2^{sj} \|\{\varphi_j \otimes f\}\|_{L_{p,\gamma}} \right)^q \right)^{1/q}$$

$$= \left(\sum_{j=0}^{\infty} 2^{sjq} \left(\int_{\mathbb{R}^n_{k,+}} \|\varphi_j \otimes f\|^p (x')^\gamma dx \right)^{q/p} \right)^{1/q}$$

$$= \left\| \left\{ \int_{\mathbb{R}^n_{k,+}} 2^{sjp} \|\varphi_j \otimes f\|^p (x')^\gamma dx \right\} \right\|^{1/p}_{l^s_{q/p}}.$$

By using Minkowski's inequality, we obtain

$$\|f\|_{B^s_{p,q,\gamma}} \le \left(\int_{\mathbb{R}^n_{k,+}} \left\| \left\{ 2^{sjp} \|\varphi_j \otimes f\|^p \right\} \right\|_{l^s_{q/p}} (x')^\gamma dx \right)^{1/p}$$
$$= \left\| \left(\sum_{j=0}^{\infty} \left(2^{sj} \|\varphi_j \otimes f\| \right)^q \right)^{1/q} \right\|_{L_{p,\gamma}}$$
$$= \|\{\varphi_j \otimes f\}\|_{L_{p,\gamma}(l^s_q)} = \|f\|_{F^s_{p,q,\gamma}} \le \|\{\varphi_j \otimes f\}\|_{L_{p,\gamma}(l^s_p)}$$
$$= \|\{\varphi_j \otimes f\}\|_{l^s_p(L_{p,\gamma})} = \|f\|_{B^s_{p,p,\gamma}}.$$

Now, we prove the right-hand-side embedding in (10). Let $f \in B^s_{p,q,\gamma}$. Applying Minkowski's inequality we have

$$\|f\|_{B^s_{p,p,\gamma}} = \|\{\varphi_j \otimes f\}\|_{l^s_p(L_{p,\gamma})} = \|\{\varphi_j \otimes f\}\|_{L_{p,\gamma}(l^s_p)} \le \|\{\varphi_j \otimes f\}\|_{L_{p,\gamma}(l^s_q)}$$
$$= \left\| \sum_{j=0}^{\infty} \left(2^{sj} \|\varphi_j \otimes f\| \right)^q \right\|^{1/q}_{L_{p/q,\gamma}} \le \left(\sum_{j=0}^{\infty} 2^{sjq} \left\| \|\varphi_j \otimes f\|^q \right\|_{L_{p/q,\gamma}} \right)^{1/p}$$
$$= \|\{\varphi_j \otimes f\}\|_{l^s_q(L_{p,\gamma})} = \|f\|_{B^s_{p,q,\gamma}}.$$

□

4 A new characterization of the B-Bessel potential spaces

In this section we prove a lifting property, a classical equality for the Fourier-Bessel transform, characterizing the Bessel potentials (B-Bessel potentials) associated with the Laplace-Bessel differential operator in terms of the B-Lizorkin-Triebel spaces. For this we need to recall the definition of the Bessel potentials given in [1, 7, 11].

The B-Bessel potentials J^s_γ, $s > 0$, generated by the multidimensional Fourier-Bessel transform are defined as negative fractional powers of the differential operator

$$I - \Delta_\gamma = I - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i},$$

where I is the identity operator. With the help of the Fourier-Bessel transform F_γ negative fractional powers of $I - \Delta_\gamma$ may be defined by

$$\left(I - \Delta_\gamma \right)^{-s/2} u = F_\gamma^{-1} \left((1 + |\xi|^2)^{-s/2} \right) F_\gamma u(x).$$

The B-Bessel potentials

$$J^s_\gamma = \left(I - \Delta_\gamma \right)^{-s/2},$$

initially defined in terms of the Fourier–Bessel transform by (1), can be represented as integral operators of B-convolution type

$$J_\gamma^s u(x) = G_{s,\gamma} \otimes u(x) = \int_{\mathbb{R}^n_{k,+}} T^y G_{s,\gamma}(x) u(y)(y')^\gamma dy$$

with the kernels

$$G_{s,\gamma}(x) = A_{k,\gamma} \frac{\pi^{(n-k)/2}}{\Gamma(s/2)} \prod_{i=1}^{k} \Gamma\Big(\frac{\gamma_j+1}{2}\Big) \int_0^\infty e^{-t-\frac{|x|^2}{4t}}\, t^{-(n+|\gamma|-s)/2}\,\frac{dt}{t}$$

($A_{k,\gamma}$ is the same as that in (2)).

For $s \in \mathbb{R}$ and $1 \le p < \infty$ we define the B-Bessel potential spaces as

$$H^s_{p,\gamma} \equiv H^s_{p,\gamma}(\mathbb{R}^n_{k,+}) = \left\{\phi \in \mathbb{S}'_{k,+} : J_\gamma^{-s}\phi \in L_{p,\gamma}\right\}.$$

The norm in $H^s_{p,\gamma}$ is defined by

$$\|\phi\|_{s,p,\gamma} = \|\phi\|_{H^s_{p,\gamma}} = \left\|J_\gamma^{-s}\phi\right\|_{L_{p,\gamma}}.$$

Moreover, $\mathbb{S}_{k,+}$ is dense in $H^s_{p,\gamma}$ (see [1]).

In view of the Parseval formula

$$\|v\|_{L_{2,\gamma}} = \|F_\gamma v\|_{L_{2,\gamma}} \tag{12}$$

it follows that

$$H^0_{2,\gamma} = L_{2,\gamma}.$$

The following lemma shows that the Young inequality in the B-Bessel potential spaces is valid.

Lemma 3. *Let $1 \le p,\, q \le \infty$, $s,\, s' \in \mathbb{R}$, $f \in H^s_{p,\gamma}$, $g \in H^s_{q,\gamma}$. If*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \ge 0,$$

then $f \otimes g \in H^{s+s'}_{r,\gamma}$ and the following inequality is valid

$$\|f \otimes g\|_{H^{s+s'}_{r,\gamma}} \le \|f\|_{H^s_{p,\gamma}} \|g\|_{H^{s'}_{q,\gamma}}.$$

Proof.

$$\begin{aligned} \|f \otimes g\|_{H^{s+s'}_{r,\gamma}} &= \|J_\gamma^{-s-s'}(f \otimes g)\|_{L_{r,\gamma}} \\ &= \|J_\gamma^{-s} f \otimes J_\gamma^{-s'} g\|_{L_{r,\gamma}} \\ &\le \|J_\gamma^{-s} f\|_{L_{p,\gamma}} \|J_\gamma^{-s'} g\|_{L_{q,\gamma}} \\ &= \|f\|_{H^s_{p,\gamma}} \|g\|_{H^{s'}_{q,\gamma}}. \end{aligned}$$

□

Lemma 4. *Let $f \in \mathbb{S}'_{k,+}$ and $\varphi_k \otimes f \in L_{p,\gamma}$, $k \geq 1$. Then for $1 \leq p \leq \infty$, $s \in \mathbb{R}$ we have*

$$\left\| J_\gamma^s \varphi_k \otimes f \right\|_{L_{p,\gamma}} \leq C 2^{ks} \left\| \varphi_k \otimes f \right\|_{L_{p,\gamma}} \quad \text{для} \;\; k \geq 1, \tag{13}$$

where the constant C does not depend on p and k.

If, furthermore, $\psi \otimes f \in L_{p,\gamma}$, then

$$\left\| J_\gamma^s \psi \otimes f \right\|_{L_{p,\gamma}} \leq C \left\| \psi \otimes f \right\|_{L_{p,\gamma}}, \tag{14}$$

where the constant C does not depend on p and k.

Proof. Note, that for all k the following equality is satisfied

$$\varphi_k \otimes f = \sum_{l=-1}^{1} \varphi_{k+l} \otimes \varphi_k \otimes f.$$

If we prove, that

$$\| F_\gamma (J_\gamma^s \varphi_{k+l}) \|_{M_{p,\gamma}} \leq C 2^{ks}, \quad l = 0, -1, 1, \tag{15}$$

then we obtain (13). In order to prove (15), observe that the function

$$\begin{aligned} F_\gamma \left\{ J_\gamma^s \varphi_{k+l} \right\} (\xi) &= (1 + |\xi|^2)^{s/2} F_\gamma \varphi_{k+l}(\xi) \\ &= (1 + |\xi|^2)^{s/2} \varphi \left(2^{-(k+l)} \xi \right) \end{aligned}$$

has the same norm in $M_{p,\gamma}$, as the function $2^{(k+l)s} \left(2^{-2(k+l)} + |\xi|^2 \right)^{s/2} \varphi(\xi)$. Indeed,

$$\begin{aligned} \| F_\gamma \left\{ J_\gamma^s \varphi_{k+l} \right\} \|_{M_{p,\gamma}} &= \| (1 + |\cdot|^2)^{s/2} \varphi (2^{-(k+l)} \cdot) \|_{M_{p,\gamma}} \\ &= \| (1 + 2^{(k+l)} |\xi|^2)^{s/2} \varphi(\cdot) \|_{M_{p,\gamma}} \\ &= \| 2^{(k+l)s} \left(2^{-2(k+l)} + |\cdot|^2 \right)^{s/2} \varphi(\cdot) \|_{M_{p,\gamma}}, \end{aligned}$$

and by virtue of Lemma 1 it can be shown that the above function belongs to $M_{p,\gamma}$, and also its norm does not exceed 2^{ks} $(k \geq 1)$. Thus inequality (15) is proved.

In order to prove (14) observe that

$$\psi \otimes f = (\psi + \varphi_1) \otimes \psi \otimes f,$$

and the fact $F_\gamma (J_\gamma^s \psi) \in M_{p,\gamma}$ follows obviously, in view of Lemma 2. □

Definition 7. *Let $s \in \mathbb{N}$ and $1 \leq p \leq \infty$. We define the B-Sobolev space by*

$$\begin{aligned} W_{p,\gamma}^s &\equiv W_{p,\gamma}^s (\mathbb{R}_{k,+}^n) \\ &= \left\{ f \in \mathbb{S}'_{k,+} \;:\; f \in L_{p,\gamma},\; B_i^s f \in L_{p,\gamma},\, 1 \leq i \leq k,\; D_j^{2s} f \in L_{p,\gamma},\, k+1 \leq j \leq n \right\}, \end{aligned}$$

and the norm in $W_{p,\gamma}^s$ is given by

$$\| f \|_{W_{p,\gamma}^s} = \| f \|_{L_{p,\gamma}} + \sum_{i=1}^{k} \| B_i^s f \|_{L_{p,\gamma}} + \sum_{i=k+1}^{n} \left\| D_i^{2s} f \right\|_{L_{p,\gamma}}. \tag{16}$$

Theorem 3. *For $s_1 < s_2$ and $1 \le p \le \infty$ we have $H^{s_2}_{p,\gamma} \subset H^{s_1}_{p,\gamma}$. Further, if $s \in \mathbb{N}$ and $1 < p < \infty$, then $H^s_{p,\gamma} = W^s_{p,\gamma}$, and the norm $\|f\|_{H^s_{p,\gamma}}$ is equivalent to $\|f\|_{W^s_{p,\gamma}}$ or*

$$\|f\|^1_{W^s_{p,\gamma}} = \|f\|_{L_{p,\gamma}} + \sup_{|\alpha'|+2|\alpha''|=s} \left\| D^{(\alpha',2\alpha'')}_\gamma f \right\|_{L_{p,\gamma}}.$$

Finally, $\mathbb{S}_{k,+}$ is dense in $H^s_{p,\gamma}$ $\quad(1 \le p < \infty)$.

Proof. Let $f \in H^s_{p,\gamma}$. We show that $J^{s_1-s_2}_\gamma$ maps $L_{p,\gamma}$ to $L_{p,\gamma}$.

In order to verify, that $J^{-\varepsilon}_\gamma : L_{p,\gamma} \to L_{p,\gamma}$, applying Lemma 1 and taking into account that $f = \psi \otimes f + \sum\limits_{k=1}^{\infty} \varphi_k \otimes f$ we get

$$\left\|J^{-\varepsilon}_\gamma f\right\|_{L_{p,\gamma}} \le \left\|J^{-\varepsilon}_\gamma \psi \otimes f\right\|_{L_{p,\gamma}} + \sum_{k=1}^{\infty} \left\|J^{-\varepsilon}_\gamma \varphi_k \otimes f\right\|_{L_{p,\gamma}}$$

$$\le C\left(\|\psi \otimes f\|_{L_{p,\gamma}} + \sum_{k=1}^{\infty} 2^{-\varepsilon k} \|\varphi_k \otimes f\|_{L_{p,\gamma}}\right) \le C\left(1 + \sum_{k=1}^{\infty} 2^{-\varepsilon k}\right) \|f\|_{L_{p,\gamma}},$$

where $\varepsilon = s_2 - s_1 > 0$.

Thus the first conclusion of the theorem follows, since

$$\|f\|_{H^{s_1}_{p,\gamma}} = \|J^{s_1}_\gamma f\|_{L_{p,\gamma}} = \|J^{s_1-s_2}_\gamma J^{s_2}_\gamma f\|_{L_{p,\gamma}} \le C \left\|J^{s_2}_\gamma f\right\| = C_{12} \|f\|_{H^{s_2}_{p,\gamma}}.$$

Next, we prove the second conclusion of the theorem. By using Theorem A we obtain $\xi^{2s}_j \left(1 + |\xi|^2\right)^{-s} \in M_{p,\gamma}$ $(1 < p < \infty)$, hence for $|\alpha'| + 2|\alpha''| = s$

$$\begin{aligned} \sum_{i=1}^{k} \|B^s_i f\|_{L_{p,\gamma}} + \sum_{i=k+1}^{n} \|D^{2s}_i f\|_{L_{p,\gamma}} &\le \sum_{|\alpha'|+2|\alpha''|=s} \|D^{(\alpha',2\alpha'')}_\gamma f\|_{L_{p,\gamma}} \\ &= \left\|F^{-1}_\gamma \left\{\xi^{2\alpha} F_\gamma f\right\}\right\|_{L_{p,\gamma}} \\ &= \left\|F^{-1}_\gamma \left\{\xi^{2\alpha}(1+|\xi|^2)^{-2s} F_\gamma J^s_\gamma f\right\}\right\|_{L_{p,\gamma}} \\ &\le C\|J^s_\gamma f\|_{L_{p,\gamma}} = C\|f\|_{H^s_{p,\gamma}}. \end{aligned}$$

Now we prove the inverse inequality. Once more we apply Theorem A. Further, let χ be an infinitely differentiable nonnegative function on $\mathbb{R}$, such that $\chi(x) = 1$ for $|x| > 2$ and $\chi(x) = 0$ for $|x| < 1$. Then we obtain

$$(1+|\xi|^2)^s \Big(1 + \sum_{j=1}^{n} \chi(\xi_j)|\xi_j|^{2s}\Big)^{-1} \in M_{p,\gamma}, \quad \chi(\xi_j)|\xi_j|^{2s}\xi_j^{-2s} \in M_{p,\gamma}, \quad 1 < p < \infty.$$

Thus,

$$\begin{aligned}\|f\|_{H^s_{p,\gamma}} = \left\|J^s_\gamma f\right\|_{L_{p,\gamma}} &\le C\left\|F^{-1}_\gamma\left\{\left(1+\sum_{i=1}^{n}\chi(\xi_i)\xi_i^{2s}\right)F_\gamma f\right\}\right\|_{L_{p,\gamma}}\\ &\le C\left(\|f\|_{L_{p,\gamma}}+\sum_{i=1}^{k}\left\|F^{-1}_\gamma\left\{\chi(\xi_i)\xi_i^{2s}\xi_i^{-2s}F_\gamma\left(B^s_i f\right)\right\}\right\|_{L_{p,\gamma}}\right.\\ &\left.+\sum_{i=k+1}^{n}\left\|F^{-1}_\gamma\left\{\chi(\xi_i)\xi_i^{2s}\xi_i^{-2s}F_\gamma\left(D^{2s}_i f\right)\right\}\right\|_{L_{p,\gamma}}\right)\\ &\le C\left(\|f\|_{L_{p,\gamma}}+\sum_{i=1}^{k}\|B^s_i f\|_{L_{p,\gamma}}+\sum_{i=k+1}^{n}\left\|D^{2s}_i f\right\|_{L_{p,\gamma}}\right).\end{aligned}$$

Finally we prove that the space $\mathbb{S}_{k,+}$ is dense in the space $H^s_{p,\gamma}$. Let $f\in H^s_{p,\gamma}$, i.e., $J^s_\gamma f\in L_{p,\gamma}$. Since $\mathbb{S}_{k,+}$ is dense in $L_{p,\gamma}$ $(1\le p<\infty)$ (see [7]), we can find a sequence $g_m\in\mathbb{S}_{k,+}$ satisfying $\|J^s_\gamma f-g_m\|_{L_{p,\gamma}}<\frac{1}{m}$ for all $m\in\mathbb{N}$. Then

$$\left\|f-J^{-s}_\gamma g_m\right\|_{H^s_{p,\gamma}}=\left\|J^s_\gamma f-g_m\right\|_{L_{p,\gamma}}<\frac{1}{m}.$$

Since $J^{-s}_\gamma g\in\mathbb{S}_{k,+}$, then we conclude, that $\mathbb{S}_{k,+}$ is dense in $H^s_{p,\gamma}$. □

The results obtained for the B-Besov spaces $B^s_{p,q,\gamma}$ correspond in part to the previous theorem for the B-Bessel potential spaces $H^s_{p,\gamma}$.

Theorem 4. *If $s_1<s_2$ we have*

$$B^{s_2}_{p,q_2,\gamma}\subset B^{s_1}_{p,q_1,\gamma}\quad(1\le p,q_1,q_2\le\infty). \tag{17}$$

If $1\le q_1<q_2\le\infty$ we have

$$B^{s}_{p,q_1,\gamma}\subset B^{s}_{p,q_2,\gamma}\quad(s\in\mathbb{R},1\le p\le\infty). \tag{18}$$

Moreover,

$$B^{s}_{p,1,\gamma}\subset H^{s}_{p,\gamma}\subset B^{s}_{p,\infty,\gamma}\quad(s\in\mathbb{R},1\le p\le\infty). \tag{19}$$

If $s_0\ne s_1$ we also have

$$\left(H^{s_0}_{p,\gamma},\,H^{s_1}_{p,\gamma}\right)_{\theta,q}=B^{s}_{p,q,\gamma}\quad(1\le p,q\le\infty,\;0<\theta<1), \tag{20}$$

where $s=(1-\theta)s_0+\theta s_1$. Finally, if $1\le p,q<\infty$ then $\mathbb{S}_{k,+}$ is dense in $B^s_{p,q,\gamma}$.

Proof. Formulas (17) and (18) follow immediately by the definition of $B^s_{p,q,\gamma}$. The density statement is a consequence of (20), Theorem 3, and Theorem 3.4.2 in [4]. Inclusions (19) are obviously implied by the inequalities in Lemma 4.

It remains to prove (20). Let $f\in\left(H^{s_0}_{p,\gamma},\,H^{s_1}_{p,\gamma}\right)_{\theta,q}$, and put $f=f_0+f_1$, $f_i\in H^{s_i}_{p,\gamma}$, $i=0,1$. By Lemma 4 we obtain

$$\|\varphi_k\otimes f\|_{L_{p,\gamma}}\le\|\varphi_k\otimes f_0\|_{L_{p,\gamma}}+\|\varphi_k\otimes f_1\|_{L_{p,\gamma}}$$

$$\leq C\left(2^{-s_0k}\|J^{s_0}_\gamma f_0\|_{L_{p,\gamma}} + 2^{-s_1k}\|J^{s_1}_\gamma f_1\|_{L_{p,\gamma}}\right),$$

and, taking the infimum,

$$\|\varphi_k \otimes f\|_{L_{p,\gamma}} \leq C2^{-s_0k}K(2^{k(s_0-s_1)}, f; H^{s_0}_{p,\gamma}, H^{s_1}_{p,\gamma}),$$

where $K(t,f;H^{s_0}_{p,\gamma}, H^{s_1}_{p,\gamma}) = \inf\limits_{f=f_0+f_1}\left(\|f_0\|_{H^{s_0}_{p,\gamma}} + t\|f_1\|_{H^{s_1}_{p,\gamma}}\right)$.
This gives

$$\left(\sum_{k=1}^{\infty}\left(2^{sk}\|\varphi_k \otimes f\|_{L_{p,\gamma}}\right)^q\right)^{1/q} \leq C\|f\|_{\left(H^{s_0}_{p,\gamma},\, H^{s_1}_{p,\gamma}\right)_{\theta,q}}.$$

Similarly, we see that

$$\|\psi \otimes f\|_{L_{p,\gamma}} \leq CK(1,f;H^{s_0}_{p,\gamma}, H^{s_1}_{p,\gamma}) \leq C\,\|f\|_{\left(H^{s_0}_{p,\gamma},\, H^{s_1}_{p,\gamma}\right)_{\theta,q}}$$

and thus

$$\|f\|_{B^s_{p,q,\gamma}} \leq C\,\|f\|_{\left(H^{s_0}_{p,\gamma},\, H^{s_1}_{p,\gamma}\right)_{\theta,q}}.$$

Let $f \in B^s_{p,q,\gamma}$. The converse inequality follows easily by the inequalities in Lemma 4

$$2^{k(s-s_0)}K(2^{k(s_1-s_0)}, \varphi_k \otimes f; H^{s_0}_{p,\gamma}, H^{s_1}_{p,\gamma}) \leq C\,2^{ks}\|\varphi_k \otimes f\|_{L_{p,\gamma}},$$

$$K(1,\psi \otimes f; H^{s_0}_{p,\gamma}, H^{s_1}_{p,\gamma}) \leq C\,\|\psi \otimes f\|_{L_{p,\gamma}}.$$

It remains to show that

$$f = \psi \otimes f + \sum_{k=1}^{\infty}\varphi_k \otimes f \quad \text{in} \quad H^{s_0}_{p,\gamma} + H^{s_1}_{p,\gamma}.$$

But if, say, $s_0 < s_1$ then $H^{s_0}_{p,\gamma} + H^{s_1}_{p,\gamma} = H^{s_0}_{p,\gamma}$, and

$$\|\psi \otimes f\|_{H^{s_0}_{p,\gamma}} + \sum_{k=1}^{\infty}\|\varphi_k \otimes f\|_{H^{s_0}_{p,\gamma}}$$

$$\leq C\left(\|\psi \otimes f\|_{L_{p,\gamma}} + \sum_{k=1}^{\infty}\|\varphi_k \otimes f\|_{L_{p,\gamma}}\right) \leq C\,\|f\|_{B^s_{p,q,\gamma}},$$

by Hölder's inequality, since $s_0 < s_1$. □

For $0 < \theta < 1$, $s_0, s_1 \in \mathbb{R}$, $1 \leq p, q, q_0, q_1 \leq \infty$ the real interpolation B-Nikol'skii-Besov space denoted by $(B^{s_0}_{p,q_0,\gamma}, B^{s_1}_{p,q_1,\gamma})_{\theta,q}$ is a subspace of functions $f \in B^{s_0}_{p,q_0,\gamma} + B^{s_1}_{p,q_1,\gamma}$ satisfying

$$\left(\int_0^\infty \left(t^{-\theta}K\left(t,f;B^{s_0}_{p,q_0,\gamma}, B^{s_1}_{p,q_1,\gamma}\right)\right)^q \frac{dt}{t}\right)^{1/q} < \infty \quad \text{if} \quad q < \infty,$$

and

$$\sup_{t\in(0,\infty)} t^{-\theta}K\left(t,f;B^{s_0}_{p,q_0,\gamma}, B^{s_1}_{p,q_1,\gamma}\right) < \infty \quad \text{if} \quad q = \infty,$$

with K being the Peetre K-functional given by

$$K\left(t,f;B^{s_0}_{p,q_0,\gamma},B^{s_1}_{p,q_1,\gamma}\right)=\inf\left\{\|f_0\|_{B^{s_0}_{p,q_0,\gamma}}+t\|f_1\|_{B^{s_1}_{p,q_1,\gamma}}\right\},$$

where the infinimum is taken over all representations of f of the form

$$f=f_0+f_1,\qquad f_0\in B^{s_0}_{p,q_0,\gamma},\qquad f_1\in B^{s_1}_{p,q_1,\gamma}.$$

Theorem 5. *Let $0<\theta<1$ and $1\le p,q,q_0,q_1\le\infty$. Furthermore, let $s_0,s_1\in\mathbb{R}$, $s_0\neq s_1$ and $s=(1-\theta)s_0+\theta s_1$.*

(i) If $1\le p\le\infty$, then

$$(B^{s_0}_{p,q_0,\gamma},B^{s_1}_{p,q_1,\gamma})_{\theta,q}=B^{s}_{p,q,\gamma}. \tag{21}$$

(ii) If $1\le p<\infty$, then

$$(F^{s_0}_{p,q_0,\gamma},F^{s_1}_{p,q_1,\gamma})_{\theta,q}=B^{s}_{p,q,\gamma}. \tag{22}$$

Proof. We start with the proof of the inclusion $(B^{s_0}_{p,\infty,\gamma},B^{s_1}_{p,\infty,\gamma})_{\theta,q}\subset B^{s}_{p,q,\gamma}$. We may assume that $s_0>s_1$. Let $q<\infty$, for $f=f_0+f_1$ with $f_0\in B^{s_0}_{p,\infty,\gamma}$ and $f_1\in B^{s_1}_{p,\infty,\gamma}$ we get by Definition 4

$$\sum_{l=0}^{\infty}2^{qlsq}\|\varphi_j\otimes f\|^q_{L_{p,\gamma}}\le C\sum_{l=0}^{\infty}2^{-sql(s_0-s_1)}\left(2^{ls_0}\|\varphi_j\otimes f_0\|_{L_{p,\gamma}}+2^{l(s_0-s_1)}2^{s_1}\|\varphi_l\otimes f_1\|_{L_{p,\gamma}}\right)^q$$
$$\le C\sum_{l=0}^{\infty}2^{-sql(s_0-s_1)}\left(\|f_0\|_{B^{s_0}_{p,\infty,\gamma}}+2^{l(s_0-s_1)}\|f_0\|_{B^{s_1}_{p,\infty,\gamma}}\right)^q.$$

Then we deduce that

$$\sum_{l=0}^{\infty}2^{qlsq}\|\varphi_l\otimes f\|^q_{L_{p,\gamma}}\le C\sum_{l=0}^{\infty}2^{-sql(s_0-s_1)}\left(K\left(2^{l(s_0-s_0)},f;B^{s_0}_{p,\infty,\gamma},B^{s_1}_{p,\infty,\gamma}\right)\right)^q$$
$$\le C\int_0^\infty\left(t^\theta K\left(t,f;B^{s_0}_{p,\infty,\gamma},B^{s_1}_{p,\infty,\gamma}\right)\right)^q\frac{dt}{t}<\infty.$$

which proves the result. When $q=\infty$, we make the usual modification. For $1\le r\le q_0,q_1$ Theorem 4 gives

$$(B^{s_0}_{p,r,\gamma},B^{s_1}_{p,r,\gamma})_{\theta,q}\subset(B^{s_0}_{p,q_0,\gamma},B^{s_1}_{p,q_1,\gamma})_{\theta,q}\subset(B^{s_0}_{p,\infty,\gamma},B^{s_1}_{p,\infty,\gamma})_{\theta,q}\subset B^{s}_{p,r,\gamma}.$$

Then in order to complete the proof of the theorem we have to show only that

$$B^{s}_{p,r,\gamma}\subset(B^{s_0}_{p,r,\gamma},B^{s_1}_{p,r,\gamma})_{\theta,q}\quad\text{for}\quad 1\le r\le q.$$

Suppose that again $s_0>s_1$. Let $q<\infty$, we have

$$\left(\int_0^\infty\left(t^{-\theta}K\left(t,f;B^{s_0}_{p,r,\gamma},B^{s_1}_{p,r,\gamma}\right)\right)^q\frac{dt}{t}\right)^{1/q}\le\left(\int_0^1\dots\frac{dt}{t}\right)^{1/q}+\left(\int_1^\infty\dots\frac{dt}{t}\right)^{1/q}$$
$$=I_1+I_2.$$

Since $s > s_1$, by Theorem 4 we get

$$K\left(t, f; B^{s_0}_{p,r,\gamma}, B^{s_1}_{p,r,\gamma}\right) \le C\ t\|f\|_{B^{s_1}_{p,q,\gamma}} \le C\ t\|f\|_{B^{s}_{p,q,\gamma}},$$

hence we deduce

$$I_1 \le C\|f\|_{B^{s,\gamma}_{p,q}} \left(\int_0^1 t^{(1-\theta)q}\frac{dt}{t}\right)^{1/q} = \frac{c}{((1-\theta)q)^{1/q}}\|f\|_{B^{s,\gamma}_{p,q}}.$$

To estimate I_2 take $f_0 = \sum\limits_{j=0}^{l} a_j$ and $f_1 = \sum\limits_{j=l+1}^{\infty} a_j$, where $a_j = \varphi_j \otimes f$. Using the properties of the sequence $(\varphi_j)_{j\in\mathbb{N}}$ we obtain

$$\|f_0\|^r_{B^{s_0}_{p,r,\gamma}} \le \sum_{j=0}^{l+1} 2^{js_0r}\|a_j\|^r_{L_{p,\gamma}} \qquad \text{and} \qquad \|f_1\|^r_{B^{s_1}_{p,r,\gamma}} \le \sum_{j=l}^{\infty} 2^{js_1r}\|a_j\|^r_{L_{p,\gamma}}.$$

Hence we can write

$$\begin{aligned}
I_2 &\le C\left(\sum_{l=0}^{\infty} 2^{-sql(s_0-s_1)}\left(K\left(2^{l(s_0-s_0)}, f; B^{s_0}_{p,r,\gamma}, B^{s_1}_{p,r,\gamma}\right)\right)^q\right)^{1/q}\\
&\le C\Bigg(\sum_{l=0}^{\infty} 2^{-sql(s_0-s_1)}\Bigg[\left(\sum_{j=0}^{l+1} 2^{js_0s}\|a_j\|^r_{L_{p,\gamma}}\right)^{1/r} + 2^{l(s_0-s_1)}\left(\sum_{j=l}^{\infty} 2^{js_1r}\|a_j\|^r_{L_{p,\gamma}}\right)^{1/r}\Bigg]^q\Bigg)^{1/q}\\
&\le C\left(\sum_{l=0}^{\infty} 2^{qls}\left[\sum_{j=0}^{l+1} 2^{(j-l)s_0r}\|a_j\|^r_{L_{p,\gamma}} + \sum_{j=l}^{\infty} 2^{(j-1)s_1r}\|a_j\|^r_{L_{p,\gamma}}\right]^{q/r}\right)^{1/q}.
\end{aligned}$$

For $r = q$ it is easy to see that $I_2 \le C\|f\|_{B^s_{p,q,\gamma}}$. For $r < q$ we take $u > r$ such that $\frac{r}{q} + \frac{r}{u} = 1$ and $s_1 < \alpha_1 < s < \alpha_0 < s_0$, then by Hölder's inequality we have

$$\begin{aligned}
I_2 &\le C\left(\sum_{l=0}^{\infty} 2^{ql(s-s_0)}\left(\sum_{j=0}^{l+1} 2^{(s_0-\alpha_0)ju}\right)^{q/u}\left(\sum_{j=0}^{l+1} 2^{\alpha_0 jq}\|a_j\|^q_{L_{p,\gamma}}\right)\right)^{1/q}\\
&\quad + c\left(\sum_{l=0}^{\infty} 2^{ql(s-s_1)}\left(\sum_{j=l}^{\infty} 2^{(s_1-\alpha_1)ju}\right)^{q/u}\left(\sum_{j=l}^{\infty} 2^{\alpha_1 jq}\|a_j\|^q_{L_{p,\gamma}}\right)\right)^{1/q}\\
&\le C\left(\sum_{l=0}^{\infty} 2^{ql(s-\alpha_0)}\sum_{j=0}^{l+1} 2^{(\alpha_0)jq}\|a_j\|^q_{L_{p,\gamma}}\right)^{1/q} + c\left(\sum_{j=0}^{\infty} 2^{jl(s-\alpha_1)}\sum_{j=l}^{\infty} 2^{\alpha_1 jq}\|a_j\|^q_{L_{p,\gamma}}\right)^{1/q}\\
&\le C\left(\sum_{j=0}^{\infty} 2^{\alpha_0 jq}\|a_j\|^q_{L_{p,\gamma}}\sum_{l=j-1}^{\infty} 2^{ql(s-\alpha_0)}\right)^{1/q} + c\left(\sum_{j=0}^{\infty} 2^{\alpha_1 jq}\|a_j\|^q_{L_{p,\gamma}}\sum_{l=0}^{j} 2^{ql(s-\alpha_1)}\right)^{1/q}\\
&\le C\|f\|_{B^s_{p,q,\gamma}}.
\end{aligned}$$

Hence it follows that

$$\left(\int_0^{\infty}\left(t^{-\theta}K\left(t, f; B^{s_0}_{p,r,\gamma}, B^{s_1}_{p,r,\gamma}\right)\right)^q\frac{dt}{t}\right)^{1/q} \le C\|f\|_{B^s_{p,q,\gamma}}.$$

When $q = \infty$ we make the usual modification.

This proves (21). Finally, (22) follows in the same way by (21) and (3). □

In the following theorem we state a lifting property.

Theorem 6. *Let $\sigma, s \in \mathbb{R}$ and $1 \le q \le \infty$. Then J_γ^σ is a bounded one-to-one linear operator from $B^s_{p,q,\gamma}$ onto $B^{s+\sigma}_{p,q,\gamma}$, if $1 \le p \le \infty$ and from $F^s_{p,q,\gamma}$ onto $F^{s+\sigma}_{p,q,\gamma}$, if $1 \le p < \infty$.*

Proof. Consider $\{\varphi_j\}_{j=0}^\infty \in \Phi$. We define $\{\psi_j\}_{j=0}^\infty$ as follows

$$\psi_j = (\varphi_j \otimes F_\gamma)\left((1+|x|^2)^{\sigma/2} 2^{j\sigma}\right).$$

A straightforward argument leads to $\{\psi_j\}_{j=0}^\infty \in \Phi$. Thus

$$\begin{aligned} \left((J_\gamma^\sigma f) \otimes \psi_j\right) &= F_\gamma^{-1}\left(F_\gamma \psi_j \cdot F_\gamma (J_\gamma^\sigma f)\right) \\ &= F_\gamma^{-1}\left(F_\gamma \psi_j \cdot (1+|\xi|^2)^{-\sigma/2} F_\gamma f\right) \\ &= F_\gamma^{-1}\left(2^{j\sigma} F_\gamma \varphi_j \cdot F_\gamma f\right) = 2^{j\sigma}\left(f \otimes \varphi_j\right). \end{aligned}$$

Now, the proof follows immediately as in [15], pp. 180 – 181. □

By using the methods given in [10], Theorem 6.2.4, and by Theorems 1 and 2 we can get the following statement.

Theorem 7. *If $s_1 < s_2$ we have*

$$F^{s_2}_{p,q_2,\gamma} \subset F^{s_1}_{p,q_1,\gamma} \quad (1 \le p < \infty, \quad 1 \le q_1, q_2 \le \infty).$$

If $1 \le q_1 < q_2 \le \infty$ we have

$$F^{s}_{p,q_1,\gamma} \subset F^{s}_{p,q_2,\gamma} \quad (s \in \mathbb{R}, \quad 1 \le p < \infty).$$

Let $s \in \mathbb{R}$, $1 < p < \infty$ and $1 < q < \infty$. Then $\mathbb{S}_{k,+}$ is dense in $F^s_{p,q,\gamma}$.

We need the following lemma to obtain a new characterization of the B-Bessel potential spaces. Note that the Rademacher functions defined as follows: $r_j(t) = r_0(2^j t)$, where $r_0(t) = 1$ for $t \in [0, 1/2]$, and $r_0(t) = -1$, for $t \in (1/2, 1]$; r_0 is extended outside the unit by periodicity, that is $r_0(t+1) = r_0(t)$.

Lemma 5. *Let $s \in \mathbb{R}$, let $\{\varphi_j\}_{j=0}^\infty \in \Phi$ and $\{r_j\}_{j=0}^\infty$ be the Rademacher functions (see [17], p. 104 or [18], Chapter V, Theorem 8.4, p. 213). Then, for every p with $1 < p < \infty$ and for all $t \in (0,1)$, there are some constants C_i, $i = 1, 2$, such that*

$$\left\| F_\gamma^{-1}\left(m_i F_\gamma f\right)\right\|_{L_{p,\gamma}} \le C_i \left\| f \right\|_{L_{p,\gamma}},$$

where

$$m_1(x) = \sum_{j=0}^\infty 2^{js} r_j(t) (1+|x|^2)^{-s/2} F_\gamma \varphi_j(x) \qquad \textit{and} \qquad m_2(x) = \left(\sum_{j=0}^\infty \left(F_\gamma \varphi_j\right)^2(x)\right)^{-1}.$$

Proof. We can see without difficulty that m_i satisfies the inequalities

$$\left| D_\gamma^{(\alpha',2\alpha'')} m_i(x) \right| \leq C_i |x|^{-|\alpha|},$$

for $k = 0, 1, ...[(|\gamma| - 1)/2] + 2$ and $i = 1, 2$. Then applying the methods in [12] we obtain the desired result. □

Next, we prove that for $q = 2$ the B-Lizorkin-Triebel spaces are reduced to the B-Bessel potentials spaces.

Theorem 8. *If $s \in \mathbb{R}$ and $1 < p < \infty$, then we have*

$$F_{p,2,\gamma}^{2s} = H_{p,\gamma}^{s},$$

and $\|f\|_{H_{p,\gamma}^s}$ is an equivalent norm in $F_{p,2,\gamma}^s$.

Proof. We will show that there exist c_1, c_2 positive constants such that

$$c_1 \|f\|_{H_{p,\gamma}^s} \leq \left\| \left(\sum_{j=0}^{\infty} 2^{2js} \|\varphi_j \otimes f\|^2 \right)^{1/2} \right\|_{L_{p,\gamma}} \leq c_2 \|f\|_{H_{p,\gamma}^s}. \tag{23}$$

By Theorem 7 we know that $\mathbb{S}_{k,+}$ is dense in $F_{p,2,\gamma}^s$ for $1 < p < \infty$. Then, it is not difficult to see that functions $f \in L_{p,\gamma}$ with compact *supp* $F_\gamma f$ are dense both in $H_{p,\gamma}^s$ and in $F_{p,2,\gamma}^s$, for $1 < p < \infty$. Therefore it is enough to prove (23) for functions of this type. Note that in this case the infinite sum in (23) is actually finite.

First, we shall prove the estimate on the right-hand side. Let $f \in H_{p,\gamma}^s$, then $f = J_{s,\gamma} \otimes g$, $g \in L_{p,\gamma}$, i.e., $F_\gamma f(\xi) = (1 + |\xi|^2)^{-s/2} F_\gamma g(\xi)$. Applying Lemma 5 we have for all $t \in (0, 1)$

$$\left\| \sum_{j=0}^{\infty} r_j(t) 2^{js} \varphi_j \otimes f \right\|_{L_{p,\gamma}} \leq C_1 \|g\|_{L_{p,\gamma}} = C_1 \|f\|_{H_{p,\gamma}^s}.$$

Thus, it follows that

$$\int_0^1 \left\| \sum_{j=0}^{\infty} r_j(t) 2^{js} \varphi_j \otimes f \right\|_{L_{p,\gamma}} dt \leq C_1 \|f\|_{H_{p,\gamma}^s}. \tag{24}$$

Using the right-hand inequality of ([17], p. 104 or [18], Chapter V, Theorem 8.4, p. 213) with $p = 1$ and the Minkowski's inequality we obtain

$$\begin{aligned} \left\| \left(\sum_{j=0}^{\infty} \|2^{js} \varphi_j \otimes f\|^2 \right)^{1/2} \right\|_{L_{p,\gamma}} &\leq C \left\| \int_0^1 \left\| \sum_{j=0}^{\infty} r_j(t) 2^{js} \varphi_j \otimes f \right\| dt \right\|_{L_{p,\gamma}} \\ &\leq C \int_0^1 \left\| \sum_{j=0}^{\infty} r_j(t) 2^{js} \varphi_j \otimes f \right\|_{L_{p,\gamma}} dt. \end{aligned}$$

Now, by (24) we have

$$\Big\|\Big(\sum_{j=0}^{\infty}\left\|2^{js}\varphi_j\otimes f\right\|^2\Big)^{1/2}\Big\|_{L_{p,\gamma}}\leq C\,\|f\|_{H^s_{p,\gamma}}\,,$$

where C a suitable positive constant. Therefore we obtain $f\in F^{2s}_{p,2,\gamma}$.

Next, we shall prove the converse inequality. For this we use duality. Let $f\in F^{2s}_{p,2,\gamma}$ and

$$K=F_\gamma^{-1}\Big(\sum_{j=0}^{\infty}\Big(F_\gamma\varphi_j\Big)^2\cdot F_\gamma g\Big). \tag{25}$$

Applying Lemma 5 with $m_2(x)=\Big(\sum\limits_{j=0}^{\infty}(F_\gamma\varphi_j)^2\Big)^{-1}$ we obtain

$$\|g\|_{L_{p,\gamma}}=\Big\|F_\gamma^{-1}\Big(\Big(\sum_{j=0}^{\infty}(F_\gamma\varphi_j)^2\Big)^{-1}F_\gamma F_\gamma^{-1}\Big(\sum_{j=0}^{\infty}(F_\gamma\varphi_j)^2\cdot F_\gamma g\Big)\Big)\Big\|_{L_{p,\gamma}}$$

$$\leq C_2\Big\|F_\gamma^{-1}\Big(\sum_{j=0}^{\infty}(F_\gamma\varphi_j)^2\cdot F_\gamma g\Big)\Big\|_{L_{p,\gamma}}=C_2\,\|K\|_{L_{p,\gamma}}\,. \tag{26}$$

Consider $u\in L_{p',\gamma}$ to be a function such that $\|u\|_{L_{p',\gamma}}=1$, $suppF_\gamma u$ is compact $(1/p+1/p'=1)$, and

$$\int_{\mathbb{R}^n_{k,+}}u(x)K(x)(x')^\gamma dx\geq\frac{1}{2}\,\|K\|_{L_{p,\gamma}}\,. \tag{27}$$

Let w be the function defined by $F_\gamma w(\xi)=(1+|\xi|^2)^{s/2}F_\gamma u(\xi)$, i.e., $u=J_{s,\gamma}\otimes w$ and $f=J_{s,\gamma}\otimes g$, as above, so that $F_\gamma f\cdot F_\gamma w=F_\gamma g\cdot F_\gamma u$. Then by (25) – (27) we obtain

$$\begin{aligned}\|f\|_{H^s_{p,\gamma}}&=\|g\|_{L_{p,\gamma}}\leq C_2\,\|K\|_{L_{p,\gamma}}\leq 2C_2\int_{\mathbb{R}^n_{k,+}}u(x)K(x)(x')^\gamma dx\\&=2C_2\int_{\mathbb{R}^n_{k,+}}F_\gamma u(\xi)F_\gamma K(\xi)\xi^\gamma d\xi=2C_2\int_{\mathbb{R}^n_{k,+}}F_\gamma u(\xi)\sum_{j=0}^{\infty}\left(F_\gamma\varphi_j\right)^2F_\gamma g(\xi)\xi^\gamma d\xi\\&=2C_2\int_{\mathbb{R}^n_{k,+}}\sum_{j=0}^{\infty}\left(2^{js}F_\gamma f(\xi)\left(F_\gamma\varphi_j\right)(\xi)\right)\left(2^{-js}(F_\gamma w)(\xi)\left(F_\gamma\varphi_j\right)(\xi)\right)\xi^\gamma d\xi.\end{aligned}$$

Hence, by Plancharel's formula and the Cauchy and Hölder inequalities we get

$$\|f\|_{H^s_{p,\gamma}}\leq 2C_2\int_{\mathbb{R}^n_{k,+}}\sum_{j=0}^{\infty}\left(2^{js}\left(F_\gamma\varphi_j\right)(x)\right)\left(2^{-js}\left(\varphi_j\otimes w\right)(x)\right)(x')^\gamma dx$$

$$\leq 2C_2\int_{\mathbb{R}^n_{k,+}}\left(\sum_{j=0}^{\infty}\left(2^{js}\left(\varphi_j\otimes f\right)(x)\right)^2\right)^{1/2}\left(\sum_{j=0}^{\infty}\left(2^{-js}\left(\varphi_j\otimes w\right)(x)\right)^2\right)^{1/2}(x')^\gamma dx$$

$$\leq 2C_2 \left\| \left(\sum_{j=0}^{\infty} 2^{2js} \left\| \varphi_j \otimes f \right\|^2 \right)^{1/2} \right\|_{L_{p,\gamma}} \left\| \left(\sum_{j=0}^{\infty} 2^{-2js} \left\| \varphi_j \otimes w \right\|^2 \right)^{1/2} \right\|_{L_{p',\gamma}} . \tag{28}$$

Then by the right-hand side of inequality (23) we get

$$\left\| \left(\sum_{j=0}^{\infty} 2^{-2js} \left\| \varphi_j \otimes w \right\|^2 \right)^{1/2} \right\|_{L_{p',\gamma}} \leq c_2 \left\| w \right\|_{H^{-s}_{p',\gamma}} = c_2 \left\| u \right\|_{L_{p',\gamma}} = c_2 \tag{29}$$

Therefore, combining (28) and (29) the proof is completed. □

As a consequence of Theorem 8 we obtain the following results.

Corollary 1. *If $s \in \mathbb{N}$ and $1 < p < \infty$, then*

$$F^{2s}_{p,2,\gamma} = W^s_{p,\gamma},$$

and the corresponding norms are equivalent.

Proof. By Theorems 3 and 8, we have $H^s_{p,\gamma} \equiv W^s_{p,\gamma}$ and the corresponding norms are equivalent. □

Moreover, by Theorem 8 and Theorem 2 we can obtain results similar to those previously obtained in [16], Theorem 15 and [17], p. 155, Theorem 5 for the Fourier transform. Namely, for $s \in \mathbb{R}$, we have

$$B^{2s}_{p,2,\gamma} \subset H^s_{p,\gamma} \subset B^{2s}_{p,p,\gamma}, \quad 2 \leq p < \infty,$$

$$B^{2s}_{p,p,\gamma} \subset H^s_{p,\gamma} \subset B^{2s}_{p,2,\gamma}, \quad 1 < p \leq 2.$$

5 Some applications

First, we give a global regularity result.

Theorem 9. *Let $P(\Delta_\gamma) = \sum_{j=0}^{m} a_j \, \Delta^j_\gamma$, $m \in \mathbb{N}$, be a differential operator with constant coefficients a_j, and symbol*

$$P(\tau) = \sum_{j=0}^{m} a_j \, \tau^j \neq 0 \quad \forall \tau \in (0, \infty).$$

If $u \in L_{2,\gamma}$, $P(-\Delta_\gamma)\, u = f$, and $f \in L_{2,\gamma}$, then $u \in H^m_{2,\gamma}$.

Proof. First we show that there exists $C > 0$ such that $|P(\tau)| \geq C\tau^m$ for all $\tau \in (0, \infty)$ (see [14]). We have

$$|P(\tau)| = |\sum_{j=0}^{m} a_j \tau^j| \geq |a_m|\tau^m - |a_{m-1}|\tau^{m-1} - \ldots - |a_0|$$
$$\geq |a_m|\tau^m - C_1(\tau^{m-1} + \ldots + 1),$$

where $C_1 = \max\{|a_0|, |a_1|, \ldots, |a_m|\}$.

If $\tau > R \geq 1$ and $k = 0, 1, \ldots, m-1$, then we get $\tau^k \leq (1/R)\tau^m$, and so

$$|P(\tau)| \geq (|a_m| - mC_1/R)\tau^m.$$

Therefore, by choosing sufficiently large R one can find $C > 0$ such that for all $\tau > R$

$$|P(\tau)| \geq C\tau^m. \tag{30}$$

Now let $v \in \mathbb{S}_{k,+}$. Then

$$\|v\|_{H^m_{2,\gamma}} = \|J^{-m}_\gamma v\|_{L_{2,\gamma}(\mathbb{R}^n_{k,+})} = \|(1+|\xi|^2)^m F_\gamma v(\xi)\|_{L_{2,\gamma}(\mathbb{R}^n_{k,+})}$$
$$\leq \|(1+|\xi|^2)^m F_\gamma v(\xi)\|_{L_{2,\gamma}(B(0,R))} + \|(1+|\xi|^2)^m F_\gamma v(\xi)\|_{L_{2,\gamma}(^\complement B(0,R))},$$

where $R \geq 1$. For $\xi \in B(0,R)$, $(1+|\xi|^2)^m \leq (1+R^2)^m$ and for $\xi \in {}^\complement B(0,R)$, we have $(1+|\xi|^2)^m \leq 2^m|\xi|^{2m}$. Thus

$$\|(1+|\xi|^2)^m F_\gamma v(\xi)\|_{L_{2,\gamma}(\mathbb{R}^n_{k,+})}$$
$$\leq (1+R^2)^{m/2}\|F_\gamma v\|_{L_{2,\gamma}(B(0,R))} + 2^{m/2}\||\xi|^m F_\gamma v(\xi)\|_{L_{2,\gamma}(^\complement B(0,R))}.$$

Using the Parseval formula (12) and inequality (30) we obtain

$$\|(1+|\xi|^2)^m F_\gamma v(\xi)\|_{L_{2,\gamma}(\mathbb{R}^n_{k,+})}$$
$$\leq (1+R^2)^{m/2}\|v\|_{L_{2,\gamma}(\mathbb{R}^n_{k,+})} + 2^{m/2}C^{-1}\||P(\xi^2)F_\gamma v(\xi)\|_{L_{2,\gamma}(\mathbb{R}^n_{k,+})}.$$

Since

$$F_\gamma\left(P(\Delta_\gamma)v\right)(\xi) = P(-|\xi|^2)F_\gamma v(\xi),$$

the Parseval formula (12) yields

$$\|v\|_{H^m_{2,\gamma}(\mathbb{R}^n_{k,+})} \leq C\left(\|P(-\Delta_\gamma)v\|_{L_{2,\gamma}(\mathbb{R}^n_{k,+})} + \|v\|_{L_{2,\gamma}(\mathbb{R}^n_{k,+})}\right),$$

where $C > 0$. Since $\mathbb{S}_{k,+}$ dense in $H^m_{2,\gamma}(\mathbb{R}^n_{k,+})$, the assertion of the theorem follows. □

Now, we give an application to solving a differential equation.

Theorem 10. *Let $f \in B^s_{p,q,\gamma}$, $s \in \mathbb{R}$, $1 \leq p \leq \infty$, and $1 \leq q \leq \infty$. Then there exists $u \in B^{s+2m}_{p,q,\gamma} \subset \mathbb{S}'_{k,+}$ such that*

$$(I - \Delta_\gamma)^m u = f, \tag{31}$$

where m is a positive integer.

Proof. Consider $f \in B^s_{p,q,\gamma}$. We need to find a distribution $u \in \mathbb{S}'_{k,+}$ satisfying (31). Applying the Fourier-Bessel transform and by (1) we obtain

$$\left(1+|\xi|^2\right)^m F_\gamma u = F_\gamma f \text{ in } \mathbb{S}'_{k,+}.$$

Then $u = F_\gamma^{-1}(1+|\xi|^2)^{-m}F_\gamma f = J_\gamma^{2m} f$ and by Theorem 6 we have that $u \in B^{s+2m}_{p,q,\gamma}$. □

We state also the following theorem which can be proved analogously.

Theorem 11. *Let $f \in F^s_{p,q,\gamma}$, $s \in \mathbb{R}$, $1 \le p < \infty$, and $1 \le q < \infty$. Then there exists $u \in F^{s+2m}_{p,q,\gamma} \subset \mathbb{S}'_{k,+}$ satisfying (31).*

Finally, we give another application in the context of the Poisson semigroup associated with Δ_γ. This semigroup is an integral operator of convolution type generated by the Fourier-Bessel transform. The kernel of this operator is defined as the Fourier-Bessel transform of the function $\exp(-s|x|)$, $s > 0$ and $x \in \mathbb{R}^n_{k,+}$. We define the Poisson kernel as

$$P_{t,\gamma}(x) = C_{n,k,\gamma}\, t\,(t^2+|x|^2)^{-\frac{n+|\gamma|+1}{2}}.$$

It can be easily verified that the following properties of $P_{t,\gamma}$ are valid:

1) $P_{t,\gamma} > 0$ is a radial function;
2) $F_\gamma(P_{t,\gamma}(\cdot))(x) = e^{-t|x|}$;
3) $P_{t,\gamma} \in L_{1,\gamma}$ and $\|P_{t,\gamma}\|_{L_{1,\gamma}} = 1$ for all $t > 0$;
4) $P_{t_1+t_2,\gamma}(x) = \left(P_{t_1,\gamma} \otimes P_{t_2,\gamma}\right)(x)$, $t_1, t_2 > 0$.

Now, we define the Poisson integral (semigroup) generated by the generalized shift operator as

$$u(x,t) = \left(P_{t,\gamma} \otimes f\right)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)\, T^y \left(P_{t,\gamma}(x)\right)\, (y')^\gamma dy.$$

By using the Fourier-Bessel transform, it is not difficult to verify (see [1, 19]) that the Poisson integral $u(x,t)$ is a solution of the following boundary value problem

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} + \Delta_\gamma\right) u(x,t) = 0 \\ u(x,0) = f(x) \end{cases} \tag{32}$$

for $f \in \mathbb{S}'_{k,+}$. Now we give the following applications to solving the boundary value problem (32) in the spaces $B^s_{p,q,\gamma}$ and $F^s_{p,q,\gamma}$.

Theorem 12. *Let $f \in B^s_{p,q,\gamma}$, $s \in \mathbb{R}$, $1 \le p \le \infty$, and $1 \le q \le \infty$. Then $u(\cdot,t) = \left(P_{t,\gamma} \otimes f\right)(\cdot) \in B^s_{p,q,\gamma} \subset \mathbb{S}'_{k,+}$ for all $t > 0$ is a solution to the boundary value problem (32).*

Proof. Consider $f \in B^s_{p,q,\gamma}$. We need to find distributions $u(\cdot,t) = \left(P_{t,\gamma} \otimes f\right)(\cdot) \in \mathbb{S}'_{k,+}$ which for all $t > 0$ are solutions to the boundary value problem (32). By applying the Fourier-Bessel transform and by using property 2) of the Poisson kernel we obtain

$$F_\gamma u(x,t) = e^{-t|x|} F_\gamma f(x)$$

in $\mathbb{S}'_{k,+}$. Then $u(x,t) = F_\gamma^{-1}\left(e^{-t|\cdot|}F_\gamma f\right)(x)$ and by Theorem 6 we have that $u \in B^s_{p,q,\gamma}$. □

Theorem 13. *Let $f \in F^s_{p,q,\gamma}$, $s \in \mathbb{R}$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$. Then $u(\cdot,t) = (P_{t,\gamma} \otimes f)(\cdot) \in F^s_{p,q,\gamma} \subset \mathbb{S}'_{k,+}$ for all $t > 0$ is a solution to the boundary value problem (32).*

The proof of this theorem is similar to the proof of Theorem 12.

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