# Letter to the editors

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## GREEN FUNCTIONS, REFLECTIONS, AND PLANE PARQUETING REVISITED

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The argumentation in the paper [1] needs some modification. Moreover, the presentation of the Green functions can be significantly simplified.

In [1] the Green function of the open triangle $T$ with the corners $-1$, $1$, $i\sqrt{3}$ is expressed via the doubly infinite product

$$P(z,\zeta) = \prod_{m,n\in 2\mathbb{Z}} \frac{\zeta - \overline{z} - \omega_{m,n}}{\zeta - z - \omega_{m,n}} \frac{\zeta - z_1 - \omega_{m,n}}{\zeta - \overline{z_1} - \omega_{m,n}} \frac{\zeta - \overline{z_2} - \omega_{m,n}}{\zeta - z_2 - \omega_{m,n}} \tag{1}$$

where

$$z_1 = -\frac{1}{2}(1 + i\sqrt{3})\overline{z} + \frac{\sqrt{3}}{2}(\sqrt{3} + i),$$

$$z_2 = -\frac{1}{2}(1 + i\sqrt{3})z + \frac{\sqrt{3}}{2}(\sqrt{3} + i),$$

and

$$\omega_{m,n} = 3m + i\sqrt{3}n$$

(Formula (6) in [1]).

Observing that

$$z + \overline{z_1} + z_2 = 3,\ z(\overline{z_1} + z_2) + \overline{z_1}z_2 = 3,\ z\overline{z_1}z_2 = z^3 - 3z^2 + 3z$$

$P(z,\zeta)$ can be rewritten as

$$P(z,\zeta) = \prod_{m,n\in 2\mathbb{Z}} \frac{(\zeta - \omega_{m,n} - 1)^3 - (\overline{z} - 1)^3}{(\zeta - \omega_{m,n} - 1)^3 - (z - 1)^3}. \tag{2}$$

Obviously this product converges as the sum

$$\sum_{m,n\in 2\mathbb{Z}} \frac{1}{(\zeta - \omega_{m,n} - 1)^3 - (z - 1)^3}$$

$$= \frac{1}{3(z-1)^2} \sum_{m,n\in 2\mathbb{Z}} \left[ \frac{1}{\zeta - z - \omega_{m,n}} - \frac{1 - i\sqrt{3}}{2} \frac{1}{\zeta - \overline{z_1} - \omega_{m,n}} - \frac{1 + i\sqrt{3}}{2} \frac{1}{\zeta - z_2 - \omega_{m,n}} \right]$$

is uniformly and absolutely convergent for $z$ and $\zeta$ in $T$, see e.g. [2], p. 261; [3], p. 268.

Reordering the factors the product $P$ can be expressed through the shifted points

$$\hat{z}_1 = -\frac{1}{2}(1 - i\sqrt{3})\overline{z} - \frac{\sqrt{3}}{2}(\sqrt{3} - i) = \overline{z_2} - 3 + i\sqrt{3},$$

$$\hat{z}_2 = -\frac{1}{2}(1 - i\sqrt{3})z - \frac{\sqrt{3}}{2}(\sqrt{3} - i) = \overline{z_1} - 3 + i\sqrt{3},$$

as

$$P(z,\zeta) = \prod_{m,n\in 2\mathbb{Z}} \frac{\zeta - \overline{z} - \omega_{m,n}}{\zeta - z - \omega_{m,n}} \frac{\zeta - \hat{z}_1 - \omega_{m,n}}{\zeta - \overline{\hat{z}_1} - \omega_{m,n}} \frac{\zeta - \overline{\hat{z}_2} - \omega_{m,n}}{\zeta - \hat{z}_2 - \omega_{m,n}}. \tag{3}$$

From

$$z + \overline{\hat{z}_1} + \hat{z}_2 = -3,\ z(\overline{\hat{z}_1} + \hat{z}_2) + \overline{\hat{z}_1}\hat{z}_2 = 3,\ z\overline{\hat{z}_1}\hat{z}_2 = z^3 + 3z^2 + 3z$$

$P$ is seen to be

$$P(z,\zeta) = \prod_{m,n\in 2\mathbb{Z}} \frac{(\zeta - \omega_{m,n} + 1)^3 - (\overline{z} + 1)^3}{(\zeta - \omega_{m,n} + 1)^3 - (z + 1)^3}. \tag{4}$$

This fact reflects the symmetry of the domain when $z$ and $\zeta$ are replaced by $-\overline{z}$ and $-\overline{\zeta}$, respectively. That $P$ is identically 1 on the base, $z = \overline{z}$, of the boundary $\partial T$, is directly seen from (2) or from (4). For the right-hand side of $\partial T$, on the line between the points 1 and $i\sqrt{3}$, the product $P$ is identically 1. This follows from (2) because there

$$z = z_1 = -\frac{1}{2}(1 + i\sqrt{3})\overline{z} + \frac{\sqrt{3}}{2}(\sqrt{3} + i).$$

Thus the relation

$$z - 1 = -\frac{1}{2}(1 + i\sqrt{3})(\overline{z} - 1)$$

implies

$$(z - 1)^3 = (\overline{z} - 1)^3.$$

The form (4) for $P$ is used for the left-hand side of $\partial T$ on the line between the points $i\sqrt{3}$ and $-1$. As there

$$z = \hat{z}_1 = -\frac{1}{2}(1 - i\sqrt{3})\overline{z} - \frac{\sqrt{3}}{2}(\sqrt{3} - i)$$

it follows that

$$z + 1 = -\frac{1}{2}(1 - i\sqrt{3})(\overline{z} + 1)$$

and

$$(z + 1)^3 = (\overline{z} + 1)^3.$$

This manifests

$$G_1(z,\zeta) = \log|P(z,\zeta)|^2$$

as the harmonic Green function of the triangle $T$.

Denoting

$$S(z,\zeta)=\sum_{m,n\in 2\mathbb{Z}}\frac{(z-\omega_{m,n}-1)^3-(\overline{z-\omega_{m,n}-1})^3}{|(z-\omega_{m,n}-1)^3-(\zeta-1)^3|^2},$$

the Poisson kernel of $T$ for $\zeta$ on the base of $T$ is

$$-\frac{1}{2}\partial_{\nu_\zeta}G_1(z,\zeta)=-3i(\zeta-1)^2S(z,\zeta),$$

for $\zeta$ on the right-hand side of $T$ is

$$-\frac{1}{2}\partial_{\nu_\zeta}G_1(z,\zeta)=-3(\sqrt{3}+i)(\zeta-1)^2S(z,\zeta),$$

and for $\zeta$ on the left-hand side of $T$ is

$$-\frac{1}{2}\partial_{\nu_\zeta}G_1(z,\zeta)=-3(\sqrt{3}-i)(\zeta+1)^2\sum_{m,n\in 2\mathbb{Z}}\frac{(z-\omega_{m,n}+1)^3-(\overline{z-\omega_{m,n}+1})^3}{|(z-\omega_{m,n}+1)^3-(\zeta+1)^3|^2}.$$

A candidate for a Neumann function is

$$N_1(z,\zeta)=-\log|Q(z,\zeta)Q(\zeta,z)|, \tag{5}$$

where

$$Q(z,\zeta)=\prod_{m,n\in 2\mathbb{Z}}\left[1-\frac{(z-1)^3}{(\zeta-\omega_{m,n}-1)^3}\right]\left[1-\frac{(\overline{z}-1)^3}{(\zeta-\omega_{m,n}-1)^3}\right].$$

The products forming $Q$ are convergent. And $Q$ can also be written in the form

$$Q(z,\zeta)=\prod_{m,n\in 2\mathbb{Z}}\left[1-\frac{(z+1)^3}{(\zeta-\omega_{m,n}+1)^3}\right]\left[1-\frac{(\overline{z}+1)^3}{(\zeta-\omega_{m,n}+1)^3}\right].$$

The normal derivative of the function $N_1$ on the boundary is identically zero up to the three corner points. To show this the relations

$$(\sqrt{3}+i)(z-1)^2=-(\sqrt{3}-i)(\overline{z}-1)^2$$

on the right-hand side of $T$ and

$$(\sqrt{3}-i)(z+1)^2=-(\sqrt{3}+i)(\overline{z}+1)^2$$

on its left-hand side are helpful.

## References


[1] H. Begehr, T. Vaitekhovich, *Green functions, reflections, and plane parqueting.* Eurasian Math. Journal, 1, no. 1 (2010), 17 – 31.

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[3] E.M. Stein, R. Shakarchi, *Complex Analysis.* Princeton University Press, Princeton, Oxford, 2003.



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