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SOME SEMICLASSICAL ORTHOGONAL POLYNOMIALS OF CLASS ONE

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Abstract. In this work, we will solve the Laguerre-Freud equations for the recurrence coefficients of the semiclassical orthogonal polynomials of class one in a particular case. The integral representation and, as a consequence, the moments of the corresponding form are obtained. Furthermore, both the characteristic elements of the structure relation and of the second-order differential equation are explicitly given.

1 Introduction

Since the first work on semiclassical orthogonal polynomials by J. Shohat [19], many authors have dealt with this subject. Most of them have especially treated the case of semiclassical polynomials of class one [1, 2, 3, 4, 16].

Yet the description of all sequences of semiclassical orthogonal polynomials of class one has remained an open problem. However, when the sequences are symmetric ($\beta_n = 0$), the problem is solved in [1]. In this paper, we assume that $\beta_n = (-1)^n\tau$, $\tau \neq 0$ and we will examine the corresponding sequences of class one through solving the Laguerre-Freud system satisfied by (β_n, γ_{n+1}) the coefficients of the standard three-term recurrence relation, characterizing monic semiclassical orthogonal polynomials of class one. We will show that the solution is unique up to an affine transformation. It is worth mentioning that this case has already been studied by T. S. Chihara without reference to the semiclassical character [5]. Recently, this case was studied using the quadratic decomposition in [18].

The first section consists of material of introductory character. The second section deals with solving Laguerre-Freud equations for the semiclassical orthogonal polynomials of class one in the particular case mentioned above. In the third section, we will deal with the moments of the canonical case found in the preceding section. Incidentally, an identity satisfied by the Gamma function is obtained (see Proposition 3). In the fourth section, we will provide the integral representation of the canonical case. In the last section, we will establish the structure relation and the second-order differential equation satisfied by any polynomial of the sequence given in the second section.

2 Preliminary results

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and $\mathcal{P}'$ its algebraic dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, for any $f \in \mathcal{P}$, any $a \in \mathbb{C} \setminus \{0\}$, any $c \in \mathbb{C}$, and $b \in \mathbb{C}$, let u', fu, $h_a u$, $\tau_{-b}u$, $x^{-1}u$ and σu be the forms defined by duality

$$\langle u', p\rangle := -\langle u, p'\rangle;\ \ \langle fu, p\rangle := \langle u, fp\rangle;\ \ \langle h_a u, p\rangle := \langle u, h_a p\rangle;\ \ \langle \tau_{-b}u, p\rangle := \langle u, \tau_b p\rangle;$$

$$\langle (x-c)^{-1}u, p\rangle := \langle u, \theta_c p\rangle;\ \ \langle \sigma u, p\rangle := \langle u, \sigma p\rangle,\ \ p \in \mathcal{P},$$

where

$$(h_a p)(x) = p(ax),\ \ (\tau_b p)(x) = p(x-b),\ \ (\theta_c p)(x) = \frac{p(x)-p(c)}{x-c}\ \ (\sigma p)(x) = p(x^2).$$

We define the right multiplication of form u by a polynomial p as

$$(up)(x) := \langle u, \frac{xp(x)-\xi p(\xi)}{x-\xi}\rangle,\ \ u \in \mathcal{P}',\ \ p \in \mathcal{P}.$$

Let $\{W_n\}_{n\geq 0}$ be a monic polynomial sequence (MPS), i.e. a sequence of polynomials with the leading coefficients equal to 1, $\deg W_n = n,\ n \geq 0$ and $\{w_n\}_{n\geq 0}$ be its dual sequence, with $w_n \in \mathcal{P}'$ defined by $\langle w_n, W_m\rangle = \delta_{n,m},\ n, m \geq 0$. The sequence $\{W_n\}_{n\geq 0}$ is called orthogonal (MOPS) if there exists a form $w \in \mathcal{P}'$ such that

$$\langle w, W_n W_m\rangle = r_n \delta_{n,m},\ \ n, m \geq 0;\ \ r_n \neq 0,\ \ n \geq 0.$$

The form w is said to be normalized if $(w)_0 = 1$ where in general $(w)_n = \langle w, x^n\rangle$, $n \geq 0$, are the moments of w. In this paper, we suppose that the forms are be normalized. Thus, $w = w_0$ and $\{W_n\}_{n\geq 0}$ satisfies the standard recurrence relation

$$W_0(x) = 1,\ \ W_1(x) = x - \beta_0,$$

$$W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \gamma_{n+1}W_n(x),\ \ n \geq 0. \tag{1}$$

When $\gamma_{n+1} > 0$, $n \geq 0$, a form w_0 is said to be positive definite.

Definition. A MOPS $\{W_n\}_{n\geq 0}$ is called semiclassical if w_0 satisfies the equation

$$(\phi w_0)' + \psi w_0 = 0, \tag{2}$$

where ϕ, ψ are polynomials, ϕ is monic and $\deg \psi \geq 1$. In this case, the form w_0 is called semiclassical.

Let us introduce the integer $s(\phi, \psi) = \max\big(\deg\phi - 2, \deg\psi - 1\big)$. Then, $s = \min s(\phi, \psi)$, where the minimum taken all over all pairs (ϕ, ψ) occurring in (2), is called the class of w_0. By extension, the integer s is also the class of $\{W_n\}_{n\geq 0}$ [13].

We have the following result:

Proposition 1 ([12]). *The form w_0 satisfying* (2) *is of class $s = s(\phi, \psi)$ if and only if*

$$\prod_{c\in Z(\phi)} \big(\mid \psi(c) + \phi'(c) \mid + \mid \langle w_0, \theta_c\psi + \theta_c^2\phi\rangle \mid\big) \neq 0,$$

where $Z(\phi) = \{c,\, \phi(c) = 0\}$.

Lemma 1 ([10, 15]). *Let $\{W_n\}_{n\geq 0}$ be a semiclassical MOPS satisfying* (2)*. Then the sequence $\{\tilde{W}_n\}_{n\geq 0}$ where $\tilde{W}_n(x) = a^{-n}W_n(ax+b) = a^{-n}(h_a \circ \tau_{-b}W_n)(x)$ is a MOPS with respect to $\tilde{w}_0 = (h_{a^{-1}} \circ \tau_{-b})w_0$. Moreover, $\tilde{w}_0$ satisfies the equation*

$$(\tilde{\phi}\tilde{w}_0)' + \tilde{\psi}\tilde{w}_0 = 0,$$

where

$$\tilde{\phi}(x) = a^{-\deg\phi}\phi(ax+b),\;\; \tilde{\psi}(x) = a^{1-\deg\phi}\psi(ax+b).$$

In the sequel, we assume that $\{W_n\}_{n\geq 0}$ is a semiclassical MOPS of class one. This means that

$$\phi(x) = c_3x^3 + c_2x^2 + c_1x + c_0,\;\; \psi(x) = a_2x^2 + a_1x + a_0. \tag{3}$$

In accordance with [3, 4] we have, the so-called Laguerre-Freud system of equations satisfied by the coefficients of the three term recurrence relation (1)

$$a_2\gamma_1 = -(a_2\beta_0^2 + a_1\beta_0 + a_0), \tag{4}$$

$$(a_2 - 2c_3n)(\gamma_n + \gamma_{n+1}) - 4c_3\sum_{\nu=0}^{n-2}\gamma_{\nu+1} = (2c_3n - a_2)\beta_n^2 + (2c_2n - a_1)\beta_n +$$

$$+2c_1n - a_0 + 2c_3\sum_{\nu=0}^{n-1}\beta_\nu^2 + (2c_3\beta_n + 2c_2)\sum_{\nu=0}^{n-1}\beta_\nu,\;\; n \geq 1, \tag{5}$$

where $\sum\limits_{\nu=0}^{-1} = 0$,

$$\Big\{a_1 - c_2 + (a_2 - c_3)(\beta_0 + \beta_1) - c_3\beta_0\Big\}\gamma_1 = \phi(\beta_0), \tag{6}$$

$$\Xi_1(n)\gamma_{n+1} - 2c_2\sum_{\nu=0}^{n-1}\gamma_{\nu+1} - 3c_3\sum_{\nu=0}^{n-1}\gamma_{\nu+1}(\beta_\nu + \beta_{\nu+1}) = \sum_{\nu=0}^{n}\phi(\beta_\nu)\;,\;\; n \geq 1, \tag{7}$$

with

$$\Xi_1(n) = a_1 - (2n+1)c_2 + (a_2 - 2c_3n)(\beta_n + \beta_{n+1}) - 2c_3\sum_{\nu=0}^{n}\beta_\nu - c_3\beta_{n+1},\;\; n \geq 1. \tag{8}$$

Remark 1. 1) (4) – (8) appear as a limit case when ω tends to zero in formulas (2.40)-(2.43) given in [17].
2) In [4, p. 272], the right hand side of the first equation in (5.13), $-\psi(\beta_k)$ must be read as $-\psi(\beta_n)$; in the right hand side of the second equation in (5.13), $-\psi(\beta_0)$ must be read as $-\psi(\beta_1)$. In that paper, the system has not been solved.

3 Search for solutions of the Laguerre-Freud system when $\beta_{\mathbf{n}} = (-1)^{\mathbf{n}}\tau$, $\mathbf{n} \geq 0$, $\tau \neq 0$.

In order to solve (4) – (8) when $\beta_n = (-1)^n\tau$, $n \geq 0$, $\tau \neq 0$, from Lemma 1 and without loss of generality, we can assume that

$$\beta_n = (-1)^n,\ n \geq 0. \tag{9}$$

In this case, (4) – (8) become

$$a_2\gamma_1 = -(a_2 + a_1 + a_0), \tag{10}$$

$$(a_2 - 2c_3n)(\gamma_n + \gamma_{n+1}) - 4c_3\sum_{\nu=0}^{n-2}\gamma_{\nu+1} = (4n - 1 + (-1)^n)c_3 - a_2+$$
$$+\big((2n-1)(-1)^n + 1\big)c_2 - a_1(-1)^n + 2c_1n - a_0,\ \ n \geq 1, \tag{11}$$

$$(a_1 - c_2 - c_3)\gamma_1 = c_3 + c_2 + c_1 + c_0, \tag{12}$$

$$(a_1 - c_3 - (2n+1)c_2)\gamma_{n+1} - 2c_2\sum_{\nu=0}^{n-1}\gamma_{\nu+1} = \frac{1}{2}(c_3 + c_1)(1 + (-1)^n)+$$
$$+(c_2 + c_0)(n+1),\ \ n \geq 1. \tag{13}$$

System (10) – (13) can be written as follows

$$a_2\gamma_1 = A(0), \tag{14}$$

$$(a_2 - 2c_3n)(\gamma_n + \gamma_{n+1}) - 4c_3\sum_{\nu=0}^{n-2}\gamma_{\nu+1} = A(n),\ \ n \geq 1,. \tag{15}$$

$$(a_1 - c_2 - c_3)\gamma_1 = B(0), \tag{16}$$

$$(a_1 - c_3 - (2n+1)c_2)\gamma_{n+1} - 2c_2\sum_{\nu=0}^{n-1}\gamma_{\nu+1} = B(n),\ \ n \geq 1, \tag{17}$$

where

$$A(n) = \big(4n - 1 + (-1)^n\big)c_3 - a_2 + \big((2n-1)(-1)^n + 1\big)c_2 - a_1(-1)^n+$$
$$+2c_1n - a_0,\ \ n \geq 0, \tag{18}$$

$$B(n) = \frac{1}{2}(c_3 + c_1)(1 + (-1)^n) + (c_2 + c_0)(n+1),\ \ n \geq 0. \tag{19}$$

Let

$$T_n = \sum_{\nu=0}^{n}\gamma_{\nu+1},\ \ n \geq 0. \tag{20}$$

Then

$$T_n - T_{n-2} = \gamma_n + \gamma_{n+1},\ \ n \geq 1,\ \ T_{-1} = 0, \tag{21}$$

$$T_n - T_{n-1} = \gamma_{n+1},\ n \geq 0. \tag{22}$$

Taking into account relations (21) and (22), (14) – (17) become

$$a_2 T_0 = A(0), \tag{23}$$

$$(a_2 - 2nc_3)T_n - (a_2 - 2(n-2)c_3)T_{n-2} = A(n),\ n \geq 1, \tag{24}$$

$$(a_1 - c_3 - c_2)T_0 = B(0), \tag{25}$$

$$(a_1 - c_3 - (2n+1)c_2)T_n - (a_1 - c_3 - (2n-1)c_2)T_{n-1} = B(n),\ n \geq 1. \tag{26}$$

Note that (25) can be obtained from (26) with $n = 0$. Similarly, equation (23) can be obtained from (24) assuming that $T_{-2} = 0$.

Lemma 2. *We have*

$$(a_1 - c_3 - (2n+1)c_2)T_n = \frac{1}{2}(c_3 + c_1)\big\{n + 1 + \frac{1}{2}(1 + (-1)^n)\big\} +$$

$$+\frac{1}{2}(c_2 + c_0)(n+1)(n+2),\ n \geq 0, \tag{27}$$

$$(a_2 - 4nc_3)T_{2n} = (n+1)\big\{2n(2c_3 + c_2 + c_1) - a_2 - a_1 - a_0\big\},\ n \geq 0, \tag{28}$$

$$(a_2-(4n+2)c_3)T_{2n+1} = (n+1)\big\{2n(2c_3-c_2+c_1)+2c_3+2c_1-a_2+a_1-a_0\big\},\ n \geq 0, \tag{29}$$

$$c_2(c_3 + c_2 + c_1) - c_0c_3 = 0, \tag{30}$$

$$c_2\big\{a_2 - c_3 + 3a_1 - c_2 + 2a_0 - c_1\big\} + 2a_1c_3 + c_1c_3 + 2c_0c_3 + a_1c_1 - a_2c_0 = 0, \tag{31}$$

$$c_2(a_1 + a_0) + a_1(c_3 - a_2) + a_0c_3 - a_1^2 - a_2c_1 - a_0a_1 - a_2c_0 = 0, \tag{32}$$

$$c_2(c_3 - c_2 + c_1) - c_0c_3 = 0, \tag{33}$$

$$c_2\big\{c_3 - a_2 + 3(a_1 - c_2) + 7c_1 - 2a_0\big\} - 2a_1c_3 - c_1c_3 + a_2c_0 - a_1c_1 - 8c_0c_3 = 0, \tag{34}$$

$$3c_2(a_1+2c_1-a_0)+a_1(a_2-c_3)+(a_2c_1-a_1^2-a_0c_3)+3(a_2-2c_3)c_0-a_1(2c_1-a_0) = 0. \tag{35}$$

Proof. From equation (26), we get

$$(a_1 - c_3 - (2n+1)c_2)T_n = \sum_{\nu=0}^{n} B(\nu),\ n \geq 0,$$

and using (19) we can deduce (27).

Making the changes $n \to 2n$, and $n \to 2n+1$ respectively, in (24) we obtain

$$(a_2 - 4nc_3)T_{2n} - (a_2 - 4(n-1)c_3)T_{2n-2} = A(2n),\ n \geq 1, \tag{36}$$

$$(a_2 - (4n+2)c_3)T_{2n+1} - (a_2 - (4n-2)c_3)T_{2n-1} = A(2n+1),\ n \geq 0. \tag{37}$$

From (36) we get

$$(a_2 - 4nc_3)T_{2n} - a_2T_0 = \sum_{\nu=1}^{n} A(2\nu),\ n \geq 1,$$

and according to (23) it follows that

$$(a_2 - 4nc_3)T_{2n} = \sum_{\nu=0}^{n} A(2\nu),\ n \geq 0.$$

Taking into account (18), we get (28).

Likewise, from relations (37) and (18), we establish (29).

Making the changes $n \to 2n$, and $n \to 2n+1$ respectively, in (27) we obtain

$$(a_1 - c_3 - (4n+1)c_2)T_{2n} = (n+1)\big\{(2n+1)(c_2+c_0) + c_3 + c_1\big\},\ n \geq 0, \tag{38}$$

$$(a_1 - c_3 - (4n+3)c_2)T_{2n+1} = (n+1)\big\{(2n+3)(c_2+c_0) + c_3 + c_1\big\},\ n \geq 0. \tag{39}$$

Multiplying (38) by $(a_2 - 4nc_3)$ and taking into account formula (28), it follows that

$$-8n^2\big\{c_2(c_3+c_2+c_1) - c_0c_3\big\}+$$
$$+2n\Big\{c_2\big\{a_2 - c_3 + 3a_1 - c_2 + 2a_0 - c_1\big\} + 2a_1c_3 + c_1c_3 + 2c_0c_3 + a_1c_1 - a_2c_0\Big\}+$$
$$+c_2(a_1+a_0) + a_1(c_3 - a_2) + a_0c_3 - a_1^2 - a_2c_1 - a_0a_1 - a_2c_0 = 0,\ n \geq 0,$$

then we can deduce (30), (31) and (32).

In a similar way, multiplying (39) by $(a_2 - (4n+2)c_3)$ and taking into account (29), we get

$$-8n^2\big\{c_2(c_3 - c_2\tau + c_1) - c_0c_3\big\}-$$
$$-2n\Big\{c_2\big\{c_3 - a_2 + 3(a_1 - c_2) + 7c_1 - 2a_0\big\} - 2a_1c_3 - c_1c_3 + a_2c_0 - a_1c_1 - 8c_0c_3\Big\}-$$
$$-3c_2(a_1+2c_1-a_0) - a_1(a_2-c_3) - (a_2c_1 - a_1^2 - a_0c_3) - 3(a_2-2c_3)c_0 + a_1(2c_1 - a_0) = 0,\ n \geq 0.$$

Then, we can evidently deduce (33), (34), and (35) respectively. □

Theorem 1. *When the form w_0 is regular the system* (10) – (13) *has a unique solution*

$$\begin{cases} \gamma_{2n+1} = -\dfrac{\{n - \frac{1}{2}(a_2+2)\}\{n - \frac{1}{2}(1+a_2+a_0)\}}{(2n - \frac{a_2}{2})\{2n - \frac{1}{2}(2+a_2)\}}, & n \geq 0 \\ \gamma_{2n+2} = -\dfrac{(n+1)\{n + \frac{1}{2}(1+a_0)\}}{(2n - \frac{a_2}{2})\{2n + \frac{1}{2}(2-a_2)\}}, & n \geq 0 \end{cases}. \tag{40}$$

Proof. From (30) and (33) we find

$$c_2 = 0, \tag{41}$$

$$c_0c_3 = 0. \tag{42}$$

Taking into account (41) and (42), relations (31), (32) and (35) become respectively

$$2a_1c_3 + c_1c_3 + a_1c_1 - a_2c_0 = 0, \tag{43}$$

$$a_1(c_3 - a_2) + a_0c_3 - a_1^2 - a_2c_1 - a_0a_1 - a_2c_0 = 0, \tag{44}$$

$$a_1(c_3 - a_2) + a_0c_3 + a_1^2 - a_2c_1 - 3a_2c_0 + 2a_1c_1 - a_0a_1 = 0. \tag{45}$$

Assuming that $c_3 = 0$, system (43) – (45) becomes

$$a_1c_1 - a_2c_0 = 0, \tag{46}$$

$$a_1a_2 + a_2c_1 + a_1^2 + a_0a_1 + a_2c_0 = 0, \tag{47}$$

$$a_1a_2 + a_2c_1 - a_1^2 + 3a_2c_0 - 2a_1c_1 + a_0a_1 = 0. \tag{48}$$

Subtracting identities (47) and (48), we obtain

$$a_1^2 + a_1c_1 - a_2c_0 = 0.$$

According to (46) we get

$$a_1 = 0. \tag{49}$$

Taking into account (49), (46) and (47), we get the following relations

$$a_2c_0 = 0, \;\; a_2c_1 = 0. \tag{50}$$

If $a_2 = 0$, by (49) we obtain $\deg \psi \leq 0$, which is a contradiction. Hence, $a_2 \neq 0$ and we have

$$c_0 = 0, \;\; c_1 = 0.$$

By the previous relation and (41), we get $\phi(x) = 0$ which yields a contradiction. Thus, we find that $c_3 \neq 0$, so we can choose

$$c_3 = 1. \tag{51}$$

Indeed, by virtue of (51) system (42) – (45) can be written as

$$c_0 = 0, \tag{52}$$

$$2a_1 + c_1 + a_1c_1 = 0, \tag{53}$$

$$a_1(1 - a_2) + a_0 - a_1^2 - a_2c_1 - a_0a_1 = 0, \tag{54}$$

$$a_1(1 - a_2) + a_0 + a_1^2 - a_2c_1 + 2a_1c_1 - a_0a_1 = 0. \tag{55}$$

Subtracting identities (55) and (54) we get

$$a_1(a_1 + c_1) = 0. \tag{56}$$

If $a_1 = 0$, equation (53) gives $c_1 = 0$.

From the previous relation, (41) and (52), equation (27) becomes

$$T_n = -\frac{1}{2}\left\{n + 1 + \frac{1}{2}(1 + (-1)^n)\right\}, \;\; n \geq 0.$$

From (22) we can deduce that $\gamma_{2n+2} = 0$, $n \geq 0$, which means the form is not regular. Therefore, $a_1 \neq 0$ hence (56) becomes

$$a_1 + c_1 = 0.$$

By the previous relation, (53) can be written as

$$a_1(1+c_1)=0.$$

Therefore,

$$c_1=-1,\ \ a_1=1. \tag{57}$$

Taking into account (41), (51), (52) and (57), we successively get for (3), (28) and (29)

$$\phi(x)=x(x^2-1),\ \ \psi(x)=a_2x^2+x+a_0, \tag{58}$$

$$(a_2-4n)T_{2n}=(n+1)\{2n-1-a_2-a_0\},\ \ n\geq 0, \tag{59}$$

$$(a_2-2-4n)T_{2n+1}=(n+1)\{2n+1-a_2-a_0\},\ \ n\geq 0. \tag{60}$$

As a consequence, (59), (60) and (22) yield (40). □

Corollary 1. *The following canonical case arises*

$$\begin{cases} \beta_n=(-1)^n, & n\geq 0 \\ \gamma_{2n+1}=-\dfrac{(n+\alpha+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, & n\geq 0 \\ \gamma_{2n+2}=-\dfrac{(n+1)(n+\beta+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)}, & n\geq 0 \\ \Big(x(x^2-1)w_0(\alpha,\beta)\Big)'+\Big(-2(\alpha+\beta+2)x^2+x+2\beta+1\Big)w_0(\alpha,\beta)=0 & \end{cases}. \tag{61}$$

Proof. From Theorem 1, as well as from relations (9) and (58) we get that in general case

$$\begin{cases} \beta_n=(-1)^n, & n\geq 0 \\ \gamma_{2n+1}=-\dfrac{\{n-\frac{1}{2}(a_2+2)\}\{n-\frac{1}{2}(1+a_2+a_0)\}}{(2n-\frac{a_2}{2})\{2n-\frac{1}{2}(2+a_2)\}}, & n\geq 0 \\ \gamma_{2n+2}=-\dfrac{(n+1)\{n+\frac{1}{2}(1+a_0)\}}{(2n-\frac{a_2}{2})\{2n+\frac{1}{2}(2-a_2)\}}, & n\geq 0 \\ \big(x(x^2-1)w_0\big)'+\big(a_2x^2+x+a_0\big)w_0=0 & \end{cases}.$$

Putting $\beta=-\frac{1}{2}+\frac{1}{2}a_0$, $\alpha=-\frac{3}{2}-\frac{1}{2}(a_2+a_0)$, we obtain (61). □

Remark 2. 1) The form $w_0(\alpha,\beta)$ is regular if and only if $\alpha\neq -n-1$, $n\geq 0$, $\beta\neq -n-1$, $n\geq 0$ and $\alpha+\beta\neq -n-1$, $n\geq 0$. The form $w_0(\alpha,\beta)$ is not positive definite.
2) According to Proposition 1, $w_0(\alpha,\beta)$ is a semiclassical form of class one.
3) When $\alpha=-\frac{1}{2}$, $\beta=\frac{1}{2}$ we obtain $\gamma_{n+1}=-\dfrac{1}{4}$, $n\geq 0$, and the following functional equation [16]

$$\big(x(x^2-1)w_0(-\frac{1}{2},\frac{1}{2})\big)'+\big(-4x^2+x+2\big)w_0(-\frac{1}{2},\frac{1}{2})=0.$$

In this case, $w_0(-\frac{1}{2},\frac{1}{2})$ is a second order self-associated form: the MOPS $\{W_n\}_{n\geq 0}$ is identical to its associated MOPS of the second kind.

4) The recurrence coefficients of the form $h_{-1}w_0(\alpha, p+\frac{1}{2})$, where p takes nonnegative integer values, are given in [2].
5) The form $w_0(\alpha-1,\beta+1)$ is studied in [18]. Note that in [18, p. 532]

$$\gamma_{2n+2} = -\frac{2(n+1)(2n+2-g_0)}{(4n-3+e_0)(4n+5+e_0)}$$

must be read as

$$\gamma_{2n+2} = -\frac{2(n+1)(2n+2-g_0)}{(4n+3+e_0)(4n+5+e_0)}.$$

4 The moments

Proposition 2. *The sequence of moments* $\{(w_0(\alpha,\beta))_n\}_{n\geq 0}$ *is given by*

$$(w_0(\alpha,\beta))_n = \frac{(\beta+1)_{[\frac{n}{2}]}}{(\alpha+\beta+2)_{[\frac{n}{2}]}} = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+1)}\frac{\Gamma(\beta+1+[\frac{n}{2}])}{\Gamma(\alpha+\beta+2+[\frac{n}{2}])},\ n\geq 0, \tag{62}$$

where $(a)_n$, $a\in\mathbb{C}$ *is the Pochhammer symbol*

$$\begin{cases} (a)_0 = 1 \\ (a)_n = a(a+1)\cdots(a+n-1), \quad n\geq 1 \end{cases}.$$

Proof. Notice that $(w_0(\alpha,\beta))_1 = \beta_0$. By (61) we obtain $(w_0(\alpha,\beta))_1 = 1$, then

$$(w_0(\alpha,\beta))_1 = (w_0(\alpha,\beta))_0 = 1. \tag{63}$$

Taking into account the functional equation in (61) we get

$$\langle (x(x^2-1)w_0(\alpha,\beta))' + \big(-2(\alpha+\beta+2)x^2+x+2\beta+1\big)w_0(\alpha,\beta), x^n\rangle = 0,\ n\geq 0,$$

which is equivalent to

$$-(n+2(\alpha+\beta+2))(w_0(\alpha,\beta))_{n+2} + (w_0(\alpha,\beta))_{n+1} + \\ +(n+2\beta+1)(w_0(\alpha,\beta))_n = 0,\ n\geq 0. \tag{64}$$

Making the change $n\to n+1$ in (64) we obtain

$$-(n+1+2(\alpha+\beta+2))(w_0(\alpha,\beta))_{n+3} + (w_0(\alpha,\beta))_{n+2} + \\ +(n+2\beta+2)(w_0(\alpha,\beta))_{n+1} = 0,\ n\geq 0.$$

Subtracting the previous relation from (64) leads to

$$(n+1+2(\alpha+\beta+2))((w_0(\alpha,\beta))_{n+3} - (w_0(\alpha,\beta))_{n+2}) = \\ = (n+2\beta+1)((w_0(\alpha,\beta))_{n+1} - (w_0(\alpha,\beta))_n),\ n\geq 0.$$

Let

$$v_n = (w_0(\alpha,\beta))_{n+1} - (w_0(\alpha,\beta))_n,\ n\geq 0. \tag{65}$$

Then, we get

$$(n+1+2(\alpha+\beta+2))v_{n+2} = (n+2\beta+1)v_n,\ n \geq 0. \tag{66}$$

From (63) we have $v_0 = 0$, hence we obtain

$$v_{2n} = (w_0(\alpha,\beta))_{2n+1} - (w_0(\alpha,\beta))_{2n} = 0,\ n \geq 0. \tag{67}$$

After changing $n \to 2n+1$ in (66) we get

$$(n+\alpha+\beta+3)v_{2n+3} = (n+\beta+1)v_{2n+1},\ n \geq 0.$$

Therefore,

$$v_{2n+1} = v_1 \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+1+n)}{\Gamma(\alpha+\beta+3+n)},\ n \geq 0,$$

but $v_1 = (w_0(\alpha,\beta))_2 - (w_0(\alpha,\beta))_1 = \gamma_1 = -\frac{\alpha+1}{\alpha+\beta+2}$, and, as a consequence

$$v_{2n+1} = -\frac{\alpha+1}{\alpha+\beta+2} \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+1+n)}{\Gamma(\alpha+\beta+3+n)},\ n \geq 0. \tag{68}$$

From (67), we have $(w_0(\alpha,\beta))_{2n+1} = (w_0(\alpha,\beta))_{2n}$, then, from (64), we get

$$(n+\alpha+\beta+2)(w_0(\alpha,\beta))_{2n+2} = (n+\beta+1)(w_0(\alpha,\beta))_{2n},\ n \geq 0,$$

therefore,

$$(w_0(\alpha,\beta))_{2n} = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+1+n)}{\Gamma(\alpha+\beta+2+n)},\ n \geq 0.$$

Finally, the previous relation provides (61). □

On the other hand, by (68) and (65) we have

$$(w_0(\alpha,\beta))_{2n+2} - (w_0(\alpha,\beta))_{2n} = -\frac{\alpha+1}{\alpha+\beta+2} \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+1+n)}{\Gamma(\alpha+\beta+3+n)},\ n \geq 0.$$

Therefore it follows that

$$(w_0(\alpha,\beta))_{2n+2} - 1 = -\frac{\alpha+1}{\alpha+\beta+2} \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\beta+1)} \sum_{\nu=0}^{n} \frac{\Gamma(\beta+1+\nu)}{\Gamma(\alpha+\beta+3+\nu)},\ n \geq 0.$$

Taking into account (62), we get

$$\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+2+n))}{\Gamma(\alpha+\beta+3+n)} - 1 =$$

$$= -\frac{\alpha+1}{\alpha+\beta+2} \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\beta+1)} \sum_{\nu=0}^{n} \frac{\Gamma(\beta+1+\nu)}{\Gamma(\alpha+\beta+3+\nu)},\ n \geq 0. \tag{69}$$

Clearly, we obtain:

Proposition 3. *The following identity holds*

$$\sum_{\nu=0}^{n} \frac{\Gamma(t+\nu)}{\Gamma(s+1+\nu)} = \frac{1}{s-t}\left\{\frac{\Gamma(t)}{\Gamma(s)} - \frac{\Gamma(t+n+1)}{\Gamma(s+n+1)}\right\},\ n \geq 0. \tag{70}$$

Proof. Making $\beta+1=t$ and $\alpha+\beta+2=s$ in equation (69), we obtain (70). □

Remark 3. Consequently, the sequence $\left\{\frac{\Gamma(t+n)}{\Gamma(s+n)}\right\}_{n\geq 0}$ is decreasing when $s>t>0$ and increasing when $0<s<t$. In the first case, when $n\longrightarrow\infty$ we get the well known Gauss sum ${}_2F_1(a,b;c;1)=\frac{s}{s-t}$ where $a=t$, $b=1$ and $c=s+1$.

5 Integral representation

Theorem 2. *The form $w_0(\alpha,\beta)$ has the following integral representation (see* [5]*)*

$$\langle w_0(\alpha,\beta), f\rangle = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{+1} sgn\, x \mid x\mid^{2\beta} (1-x^2)^{\alpha}(1+x)f(x)dx,\ f\in\mathcal{P}, \tag{71}$$

with $\Re\beta > -\frac{1}{2}$, $\Re\alpha > -1$.

Proof. We will look for a weight function U such that

$$\langle w_0(\alpha,\beta), f\rangle = \int_{-\infty}^{+\infty} U(x)f(x)dx\ ,\ f\in\mathcal{P}, \tag{72}$$

where we suppose that U is regular as far as it is necessary.

The relation

$$\langle (x(x^2-1)w_0(\alpha,\beta))' + (-2(\alpha+\beta+2)x^2+x+2\beta+1)w_0(\alpha,\beta), f\rangle = 0,\ f\in\mathcal{P},$$

reads

$$\int_{-\infty}^{+\infty} -x(x^2-1)U(x)f'(x) + (-2(\alpha+\beta+2)x^2+x+2\beta+1)U(x)f(x)dx = 0,\ f\in\mathcal{P}.$$

The previous equation can be written as

$$-x(x^2-1)U(x)f(x)]_{-\infty}^{+\infty} +$$

$$+\int_{-\infty}^{+\infty}\{x(x^2-1)U'(x) + (-(2\alpha+2\beta+1)x^2+x+2\beta)U(x)\}f(x)dx = 0,\ f\in\mathcal{P},$$

which is equivalent to

$$-x(x^2-1)U(x)f(x)]_{-\infty}^{+\infty} = 0,\ f\in\mathcal{P}. \tag{73}$$

$$x(x^2-1)U'(x) + \big(-(2\alpha+2\beta+1)x^2+x+2\beta\big)U(x) = \lambda S(x), \tag{74}$$

where $\lambda \in \mathbb{C}$ and S is the so-called ghost function locally integrable with rapid decay representing the null form. The following formula

$$S(x) = \begin{cases} 0, & x \le 0 \\ \exp(-x^{1/4})\sin(x^{1/4}), & x > 0 \end{cases},$$

was given by Stieltjes [20]. When $\lambda = 0$, equation (74) becomes

$$U'(x) = (\frac{2\beta}{x} + \frac{\alpha}{x-1} + \frac{\alpha+1}{x+1})U(x),$$

therefore,

$$U(x) = \begin{cases} 0, & x \le -1 \\ c_1 \mid x \mid^{2\beta} (1+x)(1-x^2)^{\alpha}, & -1 < x < 0 \\ c_2 \mid x \mid^{2\beta} (1+x)(1-x^2)^{\alpha}, & 0 < x < 1 \\ 0, & x \ge 1 \end{cases}, \tag{75}$$

where $\Re\beta > -1/2$, $\Re\alpha > -1$.

From (62) we have $(w_0(\alpha,\beta))_0 = 1$ and $(w_0(\alpha,\beta))_1 = 1$, by virtue of (75) we get

$$\begin{cases} (w_0(\alpha,\beta))_0 = c_1 \int\limits_{-1}^{0} \mid x \mid^{2\beta} (1-x^2)^{\alpha}(1+x)dx + c_2 \int\limits_{0}^{1} x^{2\beta}(1-x^2)^{\alpha}(1+x)dx = 1 \\ (w_0(\alpha,\beta))_1 = c_1 \int\limits_{-1}^{0} \mid x \mid^{2\beta} x(1-x^2)^{\alpha}(1+x)dx + c_2 \int\limits_{0}^{1} x^{2\beta+1}(1-x^2)^{\alpha}(1+x)dx = 1 \end{cases}.$$

The last system can be written as

$$\begin{cases} c_1 \int\limits_{0}^{1} x^{2\beta}(1-x^2)^{\alpha}(1-x)dx + c_2 \int\limits_{0}^{1} x^{2\beta}(1-x^2)^{\alpha}(1+x)dx = 1 \\ -c_1 \int\limits_{0}^{1} x^{2\beta+1}(1-x^2)^{\alpha}(1-x)dx + c_2 \int\limits_{0}^{1} x^{2\beta+1}(1-x^2)^{\alpha}(1+x)dx = 1 \end{cases}.$$

Therefore, we obtain

$$c_1 = -c_2 = \frac{-1}{B(\beta+1,\alpha+1)} = -\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)},$$

by virtue of the previous relation, (75) gives (71) and we can conclude that condition (73) is satisfied. □

Remark 4. The weight function given in (71) does not appear in [4]. Thus, the classification given in [4] is not exhaustive.

6 Structure relation and differential equation

6.1 Structure relation

The sequence $\{W_n\}_{n\ge 0}$ satisfies the following structure relation [12]

$$\phi(x)W'_{n+1}(x) = \frac{1}{2}(C_{n+1}(x) - C_0(x))W_{n+1}(x) - \gamma_{n+1}D_{n+1}(x)W_n(x),\ n \ge 0, \tag{76}$$

where

$$C_{n+1}(x) = -C_n(x) + 2(x-(-1)^n)D_n(x),\ n \geq 0,\ \deg C_n \leq 2, \tag{77}$$

$$\gamma_{n+1}D_{n+1}(x) = -\phi(x) + \gamma_n D_{n-1}(x) - (x-(-1)^n)C_n(x)+$$
$$+(x-(-1)^n)^2 D_n(x),\ n \geq 0,\ \deg D_n \leq 1, \tag{78}$$

with

$$D_{-1}(x) = 0,\ \ C_0(x) = -\psi(x) - \phi'(x),\ \ D_0(x) = -(w_0\theta_0\phi)'(x) - (w_0\theta_0\psi)(x), \tag{79}$$

$$\phi(x) = x(x^2-1),\ \ \psi(x) = -2(\alpha+\beta+2)x^2 + x + 2\beta + 1. \tag{80}$$

From (79) and (80) we get

$$C_0(x) = (2\alpha+2\beta+1)x^2 - x - 2\beta, \tag{81}$$

$$D_0(x) = 2(\alpha+\beta+1)(x+1). \tag{82}$$

In order to determine the coefficients of the previous structure relation of $\{W_n\}_{n\geq 0}$, we use, the quadratic decomposition of $\{W_n\}_{n\geq 0}$ [11]

$$W_{2n}(x) = P_n(x^2),\ n \geq 0, \tag{83}$$

$$W_{2n+1}(x) = (x-1)R_n(x^2),\ n \geq 0. \tag{84}$$

The sequences $\{P_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$, are orthogonal with respect to u_0 and v_0, satisfying the following relations

$$P_0(x) = 1,\ \ P_1(x) = x - \gamma_1 - 1,$$
$$P_{n+2}(x) = (x - \gamma_{2n+2} - \gamma_{2n+3} - 1)P_{n+1}(x) - \gamma_{2n+1}\gamma_{2n+2}P_n(x),\ n \geq 0, \tag{85}$$

$$R_0(x) = 1,\ \ R_1(x) = x - \gamma_1 - \gamma_2 - 1,$$
$$R_{n+2}(x) = (x - \gamma_{2n+3} - \gamma_{2n+4} - 1)R_{n+1}(x) - \gamma_{2n+2}\gamma_{2n+3}R_n(x),\ n \geq 0, \tag{86}$$

$$P_{n+1}(x) = R_{n+1}(x) + \gamma_{2n+2}R_n(x),\ n \geq 0, \tag{87}$$

$$(x-1)R_n(x) = P_{n+1}(x) + \gamma_{2n+1}P_n(x),\ n \geq 0. \tag{88}$$

Moreover, the forms u_0, v_0 and $w_0(\alpha,\beta)$ satisfy

$$u_0 = \sigma(w_0(\alpha,\beta)), \tag{89}$$

$$\sigma(xw_0(\alpha,\beta)) = \sigma(w_0(\alpha,\beta)), \tag{90}$$

$$v_0 = \gamma_1^{-1}(x-1)\sigma(w_0(\alpha,\beta)). \tag{91}$$

Proposition 4. *The following functional relations hold*

$$(h_{-2} \circ \tau_{-1/2})(u_0) = \mathcal{J}_{(\alpha,\beta)}, \tag{92}$$

$$(h_{-2} \circ \tau_{-1/2})(v_0) = \mathcal{J}_{(\alpha+1,\beta)}, \tag{93}$$

where $\mathcal{J}_{(\alpha,\beta)}$ is the Jacobi form ([5, 6, 13, 14]).

Proof. Following the integral representation (71) of the linear functional $w_0(\alpha,\beta)$, we can easily deduce (92)-(93). □

Corollary 2. *We have*

$$\tilde{P}_n^{(\alpha,\beta)}(x) = (-1)^n 2^n P_n(\frac{1-x}{2}),\ n \geq 0, \tag{94}$$

$$\tilde{P}_n^{(\alpha+1,\beta)}(x) = (-1)^n 2^n R_n(\frac{1-x}{2}),\ n \geq 0, \tag{95}$$

where $\{\tilde{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$ *is the sequence of monic Jacobi polynomials, orthogonal with respect to* $\tilde{u}_0 = \mathcal{J}(\alpha,\beta)$.

Proof. We get the desired results as a consequence of Proposition 4 and Lemma 1. □

Corollary 3. *(Compare with [18]) We have the following hypergeometric representation*

$$W_{2n}(x) = (-1)^n \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}\, {}_2F_1(-n, n+\alpha+\beta+1; \beta+1, x^2),\ n \geq 0,$$

$$W_{2n+1}(x) = (-1)^n (x-1) \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} \times$$
$$\times {}_2F_1(-n, n+\alpha+\beta+2; \beta+1, x^2),\ n \geq 0,$$

where

$${}_2F_1(a,b,c,x) = \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k.$$

Proof. We know that the monic Jacobi polynomials $\tilde{P}_n^{(\alpha,\beta)}$ are represented by [21]

$$\tilde{P}_n^{(\alpha,\beta)}(x) = 2^n \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}\, {}_2F_1(-n, n+\alpha+\beta+1; \beta+1, \frac{1-x}{2}).$$

From the previous relation, Corollary 2, (83) and (84), we obtain the desired results. □

In the sequel we need the following result:

Lemma 3 ([1]). *Let* $\{B_n\}_{n\geq 0}$ *be a MOPS, and* $M(x;n)$, $N(x;n)$ *two polynomials such that*

$$M(x;n) B_{n+1}(x) = N(x;n) B_n(x),\ n \geq 0.$$

Then, for any n *for which* $\deg N(x;n) \leq n$, *we have*

$$M(x;n) = 0 \quad \text{and} \quad N(x;n) = 0.$$

Proposition 5. *The sequences* $\{C_n\}_{n\geq 0}$ *and* $\{D_n\}_{n\geq 0}$ *are given by*

$$C_n(x) = \big(2n+2\alpha+2\beta+1\big)x^2 + (-1)^{n+1}x - 2\beta - 2n + (2\alpha+1)\big((-1)^n - 1\big),\ n \geq 0, \tag{96}$$

$$D_n(x) = 2\big(n+\alpha+\beta+1\big)\big(x+(-1)^n\big),\ n \geq 0. \tag{97}$$

Proof. We take into account that the Jacobi polynomials satisfy (see [13] and [15])

$$(x^2-1)\tilde{P}_{n+1}^{(\alpha,\beta)\prime}(x)=\frac{1}{2}\big(C_{n+1}(x;\alpha,\beta)-C_0(x;\alpha,\beta)\big)\tilde{P}_{n+1}^{(\alpha,\beta)}(x)-$$

$$-\gamma_{n+1}^{(\alpha,\beta)}D_{n+1}(x;\alpha,\beta)\tilde{P}_n^{(\alpha,\beta)}(x),\ n\geq 0, \tag{98}$$

where

$$C_n(x;\alpha,\beta)=(2n+\alpha+\beta)x-\frac{\alpha^2-\beta^2}{2n+\alpha+\beta},\ n\geq 0, \tag{99}$$

$$D_n(x;\alpha,\beta)=2n+\alpha+\beta+1,\ n\geq 0, \tag{100}$$

$$\gamma_{n+1}^{(\alpha,\beta)}=4\frac{(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)},\ n\geq 0, \tag{101}$$

as well as

$$(x^2-1)\tilde{P}_{n+1}^{(\alpha+1,\beta)\prime}(x)=\frac{1}{2}\big(C_{n+1}(x;\alpha+1,\beta)-C_0(x;\alpha+1,\beta)\big)\tilde{P}_{n+1}^{(\alpha+1,\beta)}(x)-$$

$$-\gamma_{n+1}^{(\alpha+1,\beta)}D_{n+1}(x;\alpha+1,\beta)\tilde{P}_n^{(\alpha+1,\beta)}(x),\ n\geq 0. \tag{102}$$

From (94), (98) and (83) we deduce

$$x(x^2-1)W'_{2n+2}(x)=-\frac{1}{2}\big(C_{n+1}(1-2x^2;\alpha,\beta)-C_0(1-2x^2;\alpha,\beta)\big)W_{2n+2}(x)-$$

$$-\frac{1}{2}\gamma_{n+1}^{(\alpha,\beta)}D_{n+1}(1-2x^2;\alpha,\beta)W_{2n}(x),\ n\geq 0.$$

Using (1), we obtain

$$x(x^2-1)W'_{2n+2}(x)=$$

$$=\frac{1}{2}\Big\{\frac{\gamma_{n+1}^{(\alpha,\beta)}}{\gamma_{2n+1}}D_{n+1}(1-2x^2;\alpha,\beta)-C_{n+1}(1-2x^2;\alpha,\beta)+C_0(1-2x^2;\alpha,\beta)\Big\}W_{2n+2}(x)-$$

$$-\frac{1}{2}\frac{\gamma_{n+1}^{(\alpha,\beta)}}{\gamma_{2n+1}}(x+1)D_{n+1}(1-2x^2;\alpha,\beta)W_{2n+1}(x),\ n\geq 0. \tag{103}$$

On the other hand, by (76) we have

$$x(x^2-1)W'_{2n+2}(x)=$$

$$=\frac{1}{2}\big(C_{2n+2}(x)-C_0(x)\big)W_{2n+2}(x)-\gamma_{2n+2}D_{2n+2}(x)W_{2n+1}(x),\ n\geq 0. \tag{104}$$

Comparing relations (103) with (104), by Lemma 3 we find

$$C_{2n+2}(x)=C_0(x)+\frac{\gamma_{n+1}^{(\alpha,\beta)}}{\gamma_{2n+1}}D_{n+1}(1-2x^2;\alpha,\beta)-C_{n+1}(1-2x^2;\alpha,\beta)+$$

$$+C_0(1-2x^2;\alpha,\beta),\ n\geq 0, \tag{105}$$

$$D_{2n+2}(x) = \frac{\gamma_{n+1}^{(\alpha,\beta)}}{2\gamma_{2n+1}\gamma_{2n+2}}(x+1)D_{n+1}(1-2x^2;\alpha,\beta),\ n \geq 0. \tag{106}$$

From relations (61), (81), (97) and (101), equation (105) becomes

$$C_{2n+2}(x) = (4n+2\alpha+2\beta+5)x^2 - x - 2\beta - 4(n+1),\ n \geq 0.$$

By (81) we get

$$C_{2n}(x) = (4n+2\alpha+2\beta+1)x^2 - x - 2\beta - 4n,\ n \geq 0. \tag{107}$$

Taking into account relations (61), (100) and (101), equation (106) can be written as

$$D_{2n+2}(x) = 2(2n+\alpha+\beta+3)(x+1),\ n \geq 0.$$

By (82), we obtain

$$D_{2n}(x) = 2(2n+\alpha+\beta+1)(x+1),\ n \geq 0. \tag{108}$$

Likewise, based on relations (95), (102), (84), (101) and (1) we get

$$x(x^2-1)W'_{2n+3}(x) = \frac{1}{2}\Big\{2x(x+1) + \frac{\gamma_{n+1}^{(\alpha+1,\beta)}}{\gamma_{2n+2}}D_{n+1}(1-2x^2;\alpha+1,\beta) -$$

$$-C_{n+1}(1-2x^2,\alpha+1,\beta) + C_0(1-2x^2,\alpha+1,\beta)\Big\}W_{2n+3}(x) -$$

$$-\frac{1}{2}\frac{\gamma_{n+1}^{(\alpha+1,\beta)}}{\gamma_{2n+2}}(x-1)D_{n+1}(1-2x^2;\alpha+1,\beta)W_{2n+2}(x),\ n \geq 0. \tag{109}$$

From (76), we have

$$x(x^2-1)W'_{2n+3}(x) = \frac{1}{2}\big(C_{2n+3}(x) - C_0(x)\big)W_{2n+3}(x) -$$

$$-\gamma_{2n+3}W_{2n+2}D_{2n+3}(x)(x),\ n \geq 0.$$

Comparing the previous equation with (109), from Lemma 3, we obtain

$$C_{2n+3}(x) = C_0(x) + 2x(x+1) + \frac{\gamma_{n+1}^{(\alpha+1,\beta)}}{\gamma_{2n+2}}D_{n+1}(1-2x^2;\alpha+1,\beta) -$$

$$-C_{n+1}(1-2x^2,\alpha+1,\beta) + C_0(1-2x^2,\alpha+1,\beta),\ n \geq 0, \tag{110}$$

$$D_{2n+3}(x) = \frac{1}{2}\frac{\gamma_{n+1}^{(\alpha+1,\beta)}}{\gamma_{2n+2}\gamma_{2n+3}}(x-1)D_{n+1}(1-2x^2;\alpha+1,\beta),\ n \geq 0. \tag{111}$$

Based on relations (61), (81), (99) and (101), equation (110) becomes

$$C_{2n+3}(x) = (4n+2\alpha+2\beta+7)x^2 + x - 2(\beta+2n+2\alpha+4),\ n \geq 0.$$

By (77) and (78), we have $C_1(x) = (2\alpha+2\beta+3)x^2 + x - 2\beta - 4\alpha - 4$, then

$$C_{2n+1}(x) = (4n+2\alpha+2\beta+3)x^2 + x - 2(\beta+2n+2\alpha+2),\ n \geq 0. \tag{112}$$

Taking into account relations (61), (82), (100) and (101), equation (111) becomes

$$D_{2n+3}(x) = 2(2n+\alpha+\beta+4)(x-1),\ n \geq 0.$$

By (78) and (81), we have $D_1(x) = 2(\alpha+\beta+2)(x-1)$, therefore

$$D_{2n+1}(x) = 2(2n+\alpha+\beta+2)(x-1),\ n \geq 0. \tag{113}$$

Relations (107), (108), (112) and (113) prove the Proposition. □

Corollary 4. *The zeros of W_n are simple for $n \geq 2$. Moreover, if $\alpha > -1$ and $\beta > -1$, then they are in the interval $]-1,+1]$.*

Proof. Let c be a zero of W_{n+1}, $n \geq 1$ of order $\mu \geq 2$, then $W_n(c) \neq 0$ and $W'_{n+1}(c) = 0$. Taking into account (76) we get $D_{n+1}(c) = 0$, by virtue of (96) we obtain $c = (-1)^n$. But $\phi((-1)^n) = 0$, hence from (76) it follows that $D'_{n+1}(c) = 0$, which is contradictory.

When $\alpha > -1$ and $\beta > -1$, taking into account relations (83), (84), (94) and (95), we see that that the zeros of $\{W_n\}_{n\geq 0}$ are in $]-1,+1]$. □

6.2 Differential equation

It is well- nown that any polynomial of a semiclassical MOPS satisfies [7, 8, 12] the second-order linear differential equation

$$J(x;n)W''_{n+1}(x) + K(x;n)W'_{n+1}(x) + L(x;n)W_{n+1}(x) = 0,\ n \geq 0, \tag{114}$$

with

$$\begin{gathered} J(x;n) = \phi(x)D_{n+1}(x),\ n \geq 0, \\ K(x;n) = \big(\phi'(x) + C_0(x)\big)D_{n+1}(x) - \phi(x)D'_{n+1}(x),\ n \geq 0, \\ L(x;n) = \frac{1}{2}\big(C_{n+1}(x) - C_0(x)\big)D'_{n+1}(x) - \frac{1}{2}\big(C'_{n+1}(x) - C'_0(x)\big)D_{n+1}(x) - \\ -D_{n+1}(x)\sum_{\nu=0}^{n} D_\nu(x),\ n \geq 0. \end{gathered} \tag{115}$$

Proposition 6. *For these polynomials the following formulas hold*

$$J(x;n) = 2\big(n+\alpha+\beta+2\big)x\big(x^2-1\big)\big(x-(-1)^n\big),\ n \geq 0, \tag{116}$$

$$\begin{gathered} K(x;n) = 2\big(n+\alpha+\beta+2\big)\times \\ \times\Big\{\big(x-(-1)^n\big)\Big(2(\alpha+\beta+2)x^2 - x - 2\beta - 1\Big) - x(x^2-1)\Big\},\ n \geq 0, \end{gathered} \tag{117}$$

$$L(x,n) = 2(n+\alpha+\beta+2)\big\{d_2(n)x^2 + d_1(n)x + d_0(n)\big\},\ n \geq 0, \tag{118}$$

where

$$\begin{aligned} &d_2(n) = -(n+1)(n+2\alpha+2\beta+3),\ n \geq 0 \\ &d_1(n) = (-1)^n\Big(n^2 + 2(\alpha+\beta+2)n + \alpha+\beta+\tfrac{5}{2}\Big) - \alpha - \beta - \tfrac{1}{2},\ n \geq 0\ . \\ &d_0(n) = (\beta+\tfrac{1}{2})\big(1+(-1)^n\big),\ n \geq 0 \end{aligned} \tag{119}$$

Proof. From (96) and (97) we obtain (116) and (117). By using the formulas

$$a_n = \sum_{\nu=0}^{n} \nu = \frac{n(n+1)}{2}, \;\; b_n = \sum_{\nu=0}^{n} (-1)^\nu = \frac{1+(-1)^n}{2},$$

$$d_n = \sum_{\nu=0}^{n} \nu(-1)^\nu = \frac{(2n+1)(-1)^n - 1}{4}, \;\; n \geq 0,$$

and (97) we have

$$\sum_{\nu=0}^{n} D_\nu(x) = 2x\big(a_n + (\alpha+\beta+1)(n+1)\big) + 2d_n + 2(\alpha+\beta+1)b_n =$$

$$= (n+1)\big(n+2(\alpha+\beta+1)\big)x + \frac{1}{2}\big((2n+1)(-1)^n - 1\big) + (\alpha+\beta+1)\Big((-1)^n + 1\Big), \;\; n \geq 0.$$

Relations (96), (97) and the previous yield (118) with (119). □

Remark 5. 1) We denote $\{x_{n+1,k}\}_{1\leq k \leq n+1}$ the zeros of W_{n+1}. Evaluating the linear differential equation (113) at $x_{n+1,k}$, $1 \leq k \leq n+1$, we obtain

$$\frac{W''_{n+1}(x_{n+1,k})}{W'_{n+1}(x_{n+1,k})} + \frac{K(x_{n+1,k};n)}{J(x_{n+1,k};n)} = 0, \;\; n \geq 0.$$

Using (116) and (117), we get

$$\frac{W''_{n+1}(x_{n+1,k})}{W'_{n+1}(x_{n+1,k})} + \frac{2\beta+1}{x_{n+1,k}} + \frac{\alpha+1}{x_{n+1,k}-1} + \frac{\alpha+2}{x_{n+1,k}+1} - \frac{1}{x_{n+1,k}-(-1)^n} = 0, \;\; n \geq 0.$$

Applying the following property (see [9] and [22])

$$\frac{W''_{n+1}(x_{n+1,k})}{W'_{n+1}(x_{n+1,k})} = -2 \sum_{j=1, j\neq k}^{n+1} \frac{1}{x_{n+1,j} - x_{n+1,k}}$$

we obtain

$$\sum_{j=1, j\neq k}^{n+1} \frac{1}{x_{n+1,j} - x_{n+1,k}} - \frac{1}{2}\frac{2\beta+1}{x_{n+1,k}} - \frac{1}{2}\frac{\alpha+1}{x_{n+1,k}-1} - \frac{1}{2}\frac{\alpha+2}{x_{n+1,k}+1} + \frac{1}{2}\frac{1}{x_{n+1,k}-(-1)^n} = 0,$$

$$n \geq 0, \;\; 1 \leq k \leq n+1, \;\; \alpha > -1, \;\; \beta > -1/2.$$

This relation can be interpreted in terms of a logarithmic potential interaction of positive unit charges under an external field. (For more details see [8]).
2) Notice that, the second order differential equation given in [18, p. 533] is not correct, it must be read as

$$X(x;n)B''_{n+1}(x) + Y(x;n)B'_{n+1}(x) + Z(x;n)B_{n+1}(x) = 0, \;\; n \geq 0,$$

with

$$X(x;n) = x(x^2-1)(x+(-1)^{n+1}),$$

$$Y(x;n)=(2\alpha+2\beta+3)x^3-(1+(-1)^n(2\alpha+2\beta+4))x^2-\big(2\beta+2-(-1)^n\big)x-(-1)^{n+1}(2\beta+3),$$

$$Z(x;n)=-(n+1)(n+2\alpha+2\beta+3)x^2-$$

$$-\Big((-1)^{n+1}\big(n^2+(2\alpha+2\beta+4)n+\alpha+\beta+\frac{5}{2}\big)+\alpha+\beta+\frac{1}{2}\Big)x+(\beta+\frac{3}{2})(1+(-1)^n).$$

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