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BOUNDEDNESS OF THE ANISOTROPIC FRACTIONAL MAXIMAL OPERATOR IN ANISOTROPIC LOCAL MORREY-TYPE SPACES

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Abstract. In this paper we study the boundedness of the anisotropic fractional maximal operator in anisotropic local Morrey-type spaces. We reduce this problem to the problem of boundedness of the supremal operator in weighted L_p-spaces on the cone of non-negative non-decreasing functions. This makes it possible to derive sharp sufficient conditions for boundedness for all admissible values of the numerical parameters, which, for a certain range of the numerical parameters, coincide with the necessary ones.

1 Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r. Let $d = (d_1, \dots, d_n)$, $d_i \geq 1$, $i = 1, \dots, n$, $|d| = \sum_{i=1}^n d_i$ and $t^d x \equiv \left(t^{d_1} x_1, \dots, t^{d_n} x_n\right)$. By [2, 14], the function $F(x,\rho) = \sum_{i=1}^n x_i^2 \rho^{-2d_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x,\rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. It is a simple matter to check that $\rho(x-y)$ defines a distance between any two points x, $y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([2, 3, 14]). The balls with respect to ρ, centered at x of radius r, are just the ellipsoids

$$\mathcal{E}_d(x,r) = \left\{ y \in \mathbb{R}^n : \frac{(y_1 - x_1)^2}{r^{2d_1}} + \cdots + \frac{(y_n - x_n)^2}{r^{2d_n}} < 1 \right\},$$

with the Lebesgue measure $|\mathcal{E}_d(x,r)| = v_n r^{|d|}$, where v_n is the volume of the unit ball in $\mathbb{R}^n$. If $d = \mathbf{1} \equiv (1, \dots, 1)$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_{\mathbf{1}}(x,r) = B(x,r)$. Note that in the standard parabolic case $d = (1, \dots, 1, 2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

If E is a non-empty measurable subset of $\mathbb{R}^n$ and f is a measurable function on E, then we use the following standard notation [1]

$$\|f\|_{L_p(E)} := \left(\int_E |f(y)|^p dy\right)^{\frac{1}{p}}, \ 0 < p < \infty,$$
$$\|f\|_{L_\infty(E)} := \sup\{\alpha : |\{y \in E : |f(y)| \geq \alpha\}| > 0\}.$$

Let $f \in L_1^{\rm loc}$. The anisotropic fractional maximal function $M_\alpha^d f$ is defined by

$$M_\alpha^d f(x) = \sup_{t>0} |\mathcal{E}_d(x,t)|^{-1+\frac{\alpha}{|d|}} \int_{\mathcal{E}_d(x,t)} |f(y)|dy, \quad 0 \leq \alpha < |d|.$$

If $\alpha = 0$, then $M^d \equiv M_0^d$ is the anisotropic maximal operator. If $d = \mathbf{1}$, then $M_\alpha \equiv M_\alpha^{\mathbf{1}}$ is the fractional maximal operator and $M \equiv M_0^{\mathbf{1}}$ is the Hardy-Littlewood maximal operator.

In the theory of partial differential equations, together with weighted $L_{p,w}$ spaces, Morrey spaces $\mathcal{M}_{p,\lambda}$ play an important role. They were introduced by C. Morrey in 1938 [6]. These spaces appeared to be quite useful in the study of a number of problems in the theory of partial differential equations, in particular in the study of local behaviour of solutions of parabolic or quasi-elliptic differential equations. The anisotropic Morrey space is defined as follows: for $1 \leq p \leq \infty$, $0 \leq \lambda \leq |d|$, a function $f \in \mathcal{M}_{p,\lambda,d}$ if $f \in L_p^{\rm loc}$ and

$$\|f\|_{\mathcal{M}_{p,\lambda,d}} \equiv \|f\|_{\mathcal{M}_{p,\lambda,d}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(\mathcal{E}_d(x,r))} < \infty.$$

Note that $\mathcal{M}_{p,\lambda} \equiv \mathcal{M}_{p,\lambda,\mathbf{1}}$. (If $\lambda = 0$, then $\mathcal{M}_{p,0,d} = L_p$; if $\lambda = |d|$, then $\mathcal{M}_{p,|d|,d} = L_\infty$; if $\lambda < 0$ or $\lambda > |d|$, then $\mathcal{M}_{p,\lambda,d} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.)

Also, by $W\mathcal{M}_{p,\lambda,d}$ we denote the weak Morrey space of all functions $f \in WL_p^{\rm loc}$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda,d}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda,d}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}_d(x,r))} < \infty,$$

where $WL_p(\mathcal{E}_d(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\begin{aligned} \|f\|_{WL_p(\mathcal{E}_d(x,r))} &\equiv \|f\chi_{\mathcal{E}_d(x,r)}\|_{WL_p(\mathbb{R}^n)} \\ &= \sup_{t>0} t\, |\{y \in \mathcal{E}_d(x,r) :\ |f(y)| > t\}|^{\frac{1}{p}} \\ &= \sup_{t>0} t^{\frac{1}{p}} \left(f\chi_{\mathcal{E}_d(x,r)}\right)^*(t) < \infty. \end{aligned} \tag{1.1}$$

Here g^* denotes the non-increasing rearrangement of the function g.

Spanne (see [25]) and Adams [1] studied boundedness of the fractional maximal operator M_α for $0 < \alpha < n$ in Morrey spaces $\mathcal{M}_{p,\lambda}$. Later on Chiarenza and Frasca [13] studied boundedness of the maximal operator M in these spaces. By more general results of Guliyev [17] (see also [18, 19]) one can obtain the following generalization of the results in [1, 13, 25] to the anisotropic case.

[1]Here and in the sequel we write just L_p for $L_p(\mathbb{R}^n)$, $0 < p \leq \infty$. If $E \neq \mathbb{R}^n$, then we preserve the full notation $L_p(E)$. The same refers to the cases of $L_p^{\rm loc}$ and of the weighted Lebesgue spaces $L_{p,v}$.

Theorem 1.1. (1) *Let $1 < p_1 < p_2 \le \infty$ and $0 < \alpha < |d|$. Then M^d_α is bounded from $\mathcal{M}_{p_1,\lambda,d}$ to $\mathcal{M}_{p_2,\lambda,d}$ if and only if*

$$\alpha \le |d| \left(\frac{1}{p_1} - \frac{1}{p_2}\right) \quad \textit{and} \quad \lambda = \left(|d| \left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \alpha\right) \left(\frac{1}{p_1} - \frac{1}{p_2}\right)^{-1}. \tag{1.2}$$

(2) *Let $1 < p_2 \le \infty$ and $0 < \alpha < |d|$. Then M^d_α is bounded from $\mathcal{M}_{1,\lambda,d}$ to $W\mathcal{M}_{p_2,\lambda,d}$ if and only if*

$$\alpha \le |d| \left(1 - \frac{1}{p_2}\right) \quad \textit{and} \quad \lambda = \left(|d| \left(1 - \frac{1}{p_2}\right) - \alpha\right) \left(1 - \frac{1}{p_2}\right)^{-1}. \tag{1.3}$$

(3) *Let $1 < p \le \infty$. Then M^d is bounded from $\mathcal{M}_{p,\lambda,d}$ to $\mathcal{M}_{p,\lambda,d}$ for all $0 \le \lambda \le |d|$.*
(4) *The operator M^d is bounded from $\mathcal{M}_{1,\lambda,d}$ to $W\mathcal{M}_{1,\lambda,d}$ for all $0 \le \lambda \le |d|$.*

If in the place of the power function $r^{-\frac{\lambda}{p}}$ in the definition of $\mathcal{M}_{p,\lambda,d}$ we consider any positive measurable weight function w defined on $(0, \infty)$, then it becomes the Morrey-type space $\mathcal{M}_{p,w,d}$. D. Fan, S. Lu, D. Yang [15] and V.S. Guliyev [17] (see also [18, 20, 21, 19]) generalized Theorem 1.1 and obtained sufficient conditions on weights w_1 and w_2 ensuring boundedness of the maximal operator M^d and the fractional maximal operator M^d_α for the limiting case $\alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$ from $\mathcal{M}_{p_1,w_1,d}$ to $\mathcal{M}_{p_2,w_2,d}$.

The following statement, containing the results in [15] was proved in [17] (see also [18, 20, 21, 19]).

Theorem 1.2. *Let $1 \le p_1 \le p_2 < \infty$ and $\alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$. Moreover, let w_1 and w_2 be positive measurable functions satisfying the following condition: there exists $c > 0$ such that for all $t > 0$*

$$\left\| w_1^{-1}(r)\, r^{\alpha - \frac{|d|}{p_1} - 1} \right\|_{L_1(t,\infty)} \le c w_2^{-1}(t)\, t^{\alpha - \frac{|d|}{p_1}}. \tag{1.4}$$

Then for $p_1 > 1$ M^d_α is bounded from $\mathcal{M}_{p_1,w_1,d}$ to $\mathcal{M}_{p_2,w_2,d}$, and for $p_1 = 1$ M^d_α is bounded from $\mathcal{M}_{1,w_1,d}$ to $W\mathcal{M}_{p_2,w,d}$.

Earlier, in [15] a weaker version of Theorem 1.2 was proved: it was assumed that $w_1 = w_2 = w$ and that w is a positive non-increasing function satisfying the pointwise doubling condition, namely that for some $c > 0$

$$c^{-1} w(r) \le w(t) \le c w(r)$$

for all t, $r > 0$ such that $0 < r \le t \le 2r$.

In [5]-[8] boundedness of the maximal and fractional maximal operators from one local Morrey-type space $LM_{p_1\theta_1,w_1}$ to another one $LM_{p_2\theta_2,w_2}$ have been investigated. (The definition and basic properties of these spaces in the anisotropic case are given in Section 2. In particular it is noted there that these spaces are non-trivial only if w_1, w_2 belong to classes Ω_{θ_1}, Ω_{θ_2} respectively, defined in that section.) Moreover, for some values of the parameters necessary and sufficient conditions for the operators M and M_α to be bounded from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ were obtained.

Theorem 1.3. *1) If* $1 < p_1 \le p_2 < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = n(\frac{1}{p_1} - \frac{1}{p_2})$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *then the Burenkov-Guliyevs condition*

$$\left\| w_2(r) \left(\frac{r}{t+r} \right)^{\frac{n}{p_2}} \right\|_{L_{\theta_2}(0,\infty)} \le c \|w_1\|_{L_{\theta_1}(t,\infty)} \tag{1.5}$$

for all $t > 0$, *where* $c > 0$ *is independent of* t, *is necessary and sufficient for the boundedness of* M_α *from* $LM_{p_1\theta_1,w_1}$ *to* $LM_{p_2\theta_2,w_2}$.

2) If $1 \le p_1 \le p_2 < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = n(\frac{1}{p_1} - \frac{1}{p_2})$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *then Burenkov-Guliyevs condition* (1.5) *is necessary and sufficient for the boundedness of* M_α *from* $LM_{p_1\theta_1,w_1}$ *to* $WLM_{p_2\theta_2,w_2}$.

Condition (1.5) for the first time was introduced in [5, 3] for the case of the maximal operator and in [7, 8] for the case of the fractional maximal operator. It appeared to be rather 'stable': for $\theta_1 \le \theta_2$ it serves as necessary and sufficient condition not only for the maximal and the fractional maximal operator, but also, under the appropriate assumptions on the numerical parameters, for the Riesz potential [9, 10] and genuine singular integral operators [11, 12].

Theorem 1.3 in the case $\theta_1 \le p_1$ was proved in [7, 8] and in the case $\theta_1 > p_1$ in [4]. In [7, 8] the proof was based on a certain estimate for L_p-norms of $M_\alpha f$ over balls $B(x,r)$, which allowed reducing the problem of boundedness of M_α in local Morrey-type spaces to the problem of boundedness of the Hardy operator on the cone of non-negative non-decreasing functions. In [4] the problem of boundedness of M_α from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ was reduced to the problem of boundedness of the so-called supremal operator on the cone of non-negative non-decreasing functions. Also for the case $p_1 = 1$, $0 < p_2 < \infty$, and $n(1 - \frac{1}{p_2})_+ < \alpha < n$ necessary and sufficient conditions ensuring boundedness of M_α from $LM_{1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ were obtained in [4] for all $0 < \theta_1, \theta_2 \le \infty$ and $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$.

In this paper we reduce as in [4] the problem of boundedness of M_α^d in anisotropic local Morrey-type spaces to the problem of boundedness of the supremal operator on the cone of non-negative non-decreasing functions and obtain sharp sufficient conditions for boundedness for all admissible values of the parameters, which for certain range of the parameters similar to that in [4] coincide with the necessary ones.

2 Definitions and basic properties of anisotropic local Morrey-type spaces

Definition 2.1. Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,w,d}$, $GM_{p\theta,w,d}$, the anisotropic local Morrey-type spaces, the global Morrey-type spaces respectively, the spaces of all functions $f \in L_p^{\rm loc}$ with finite quasinorms

$$\|f\|_{LM_{p\theta,w,d}} \equiv \|f\|_{LM_{p\theta,w,d}(\mathbb{R}^n)} = \left\| w(r) \|f\|_{L_p(\mathcal{E}_d(0,r))} \right\|_{L_\theta(0,\infty)},$$

$$\|f\|_{GM_{p\theta,w,d}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,w,d}}$$

respectively.

Note that $GM_{p\theta,w,\mathbf{1}} = GM_{p\theta,w}$, $LM_{p\theta,w,\mathbf{1}} = LM_{p\theta,w}$ and

$$\|f\|_{LM_{p\infty,1,d}} = \|f\|_{GM_{p\infty,1,d}} = \|f\|_{L_p}.$$

Furthermore, $GM_{p\infty,r^{-\lambda/p},d} \equiv \mathcal{M}_{p,\lambda,d}$, $0 \le \lambda \le |d|$.

Lemma 2.1. *Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$.*

1.If for all $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} = \infty, \tag{2.1}$$

then $LM_{p\theta,w,d} = GM_{p\theta,w,d} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

2. If for all $t > 0$

$$\|w(r) r^{\frac{|d|}{p}}\|_{L_\theta(0,t)} = \infty, \tag{2.2}$$

then for all functions $f \in LM_{p\theta,w,d}$, continuous at 0, $f(0) = 0$, and for $0 < p < \infty$ $GM_{p\theta,w,d} = \Theta$.

Proof. 1. Let (2.1) be satisfied and f be not equivalent to zero. Then for some $t_0 > 0$

$$A = \|f\|_{L_p(\mathcal{E}(0,t_0))} > 0.$$

Hence

$$\|f\|_{GM_{p\theta,w,d}} \ \ge \ \|f\|_{LM_{p\theta,w,d}} \ \ge \ \Big\|w(r)\|f\|_{L_p(\mathcal{E}_d(0,r))}\Big\|_{L_\theta(t_0,\infty)} \ \ge \ A\|w(r)\|_{L_\theta(t_0,\infty)}.$$

Therefore $\|f\|_{GM_{p\theta,w,d}} = \|f\|_{LM_{p\theta,w,d}} = \infty$.

2. Let (2.2) be satisfied. If $f \in LM_{p\theta,w,d}$ and there exists

$$\lim_{r\to 0} |\mathcal{E}_d(0,r)|^{-\frac{1}{p}} \|f\|_{L_p(\mathcal{E}_d(0,r))} = B, \tag{2.3}$$

then $B = 0$.

Indeed, if $B > 0$, then there exists $t_0 > 0$ such that

$$|\mathcal{E}_d(0,r)|^{-\frac{1}{p}} \|f\|_{L_p(\mathcal{E}_d(0,r))} \ge \frac{B}{2} \tag{2.4}$$

for all $0 < r \le t_0$. Consequently,

$$\|f\|_{LM_{p\theta,w,d}} \ge \Big\|w(r)\|f\|_{L_p(\mathcal{E}_d(0,r))}\Big\|_{L_\theta(0,t_0)} \ge \frac{B}{2} v_n^{\frac{1}{p}} \Big\|w(r) r^{\frac{|d|}{p}}\Big\|_{L_\theta(0,t_0)},$$

Hence $\|f\|_{LM_{p\theta,w,d}} = \infty$, $f \notin LM_{p\theta,w,d}$ and we have arrived at a contradiction.

If $f \in LM_{p\theta,w,d}$ and it is continuous at 0, then (2.3) holds with $B = |f(0)|$. Hence $f(0) = 0$.

Next let $0 < p < \infty$ and let $f \in GM_{p\theta,w,d}$, then by the generalized Lebesgue theorem on differentiation of integrals (see, for example, [26]) for almost all $x \in \mathbb{R}^n$

$$\lim_{r\to 0} |\mathcal{E}_d(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(\mathcal{E}_d(x,r))} = |f(x)|.$$

By the above argument for all those x we have $f(x) = 0$. Hence f is equivalent to zero. ☐

Definition 2.2. Let $0 < p, \theta \le \infty$. We denote by Ω_θ the set of all functions w which are non-negative, measurable on $(0,\infty)$, not equivalent to 0 and such that for some $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty.$$

Moreover, we denote by $\Omega_{p,\theta,d}$ the set of all functions w which are non-negative, measurable on $(0,\infty)$, not equivalent to 0 and such that for some $t_1, t_2 > 0$

$$\|w(r)\|_{L_\theta(t_1,\infty)} < \infty, \qquad \|w(r) r^{\frac{|d|}{p}}\|_{L_\theta(0,t_2)} < \infty.$$

Keeping in mind Lemma 2.1, when considering the spaces $LM_{p\theta,w,d}$ we always assume that $w \in \Omega_\theta$, and when considering the spaces $GM_{p\theta,w,d}$ we always assume that $w \in \Omega_{p,\theta,d}$.

Lemma 2.2. *Let $0 < p < \infty$, $r > 0$. Then for $\beta > -\frac{|d|}{p}$*

$$\left\|\rho(x)^\beta\right\|_{L_p(\mathcal{E}_d(0,r))} = (|d| + \beta p)^{-\frac{1}{p}}\, C_0\, r^{\frac{|d|}{p}+\beta},$$

and for $\beta < -\frac{|d|}{p}$

$$\left\|\rho(x)^\beta\right\|_{L_p(^{\complement}\mathcal{E}_d(0,r))} = \left||d| + \beta p\right|^{-\frac{1}{p}}\, C_0\, r^{\frac{|d|}{p}+\beta},$$

where $^{\complement}\mathcal{E}_d(0,r)$ is the complement of $\mathcal{E}_d(0,r)$, and

$$\begin{aligned} C_0 &= \left(\int_{S^{n-1}} d\sigma(x')\right)^{\frac{1}{p}} \\ &= \left(\int_0^{\pi}\int_0^{\pi}\cdots\int_0^{2\pi} J(\varphi_1,\ldots,\varphi_{n-1}) d\varphi_1 d\varphi_2\cdots d\varphi_{n-1}\right)^{\frac{1}{p}} < \infty. \end{aligned}$$

Proof. For any $x = (x_1, ..., x_n) \in \mathbb{R}^n$, set

$$\begin{aligned} x_1 &= \rho^{d_1}\cos\varphi_1\ldots\cos\varphi_{n-2}\cos\varphi_{n-1}, \\ x_2 &= \rho^{d_2}\cos\varphi_1\ldots\cos\varphi_{n-2}\sin\varphi_{n-1}, \\ &\cdots \\ x_{n-1} &= \rho^{d_{n-1}}\cos\varphi_1\sin\varphi_2, \\ x_n &= \rho^{d_n}\sin\varphi_1. \end{aligned} \tag{2.5}$$

Thus, $dx = \rho^{|d|-1}J(\varphi_1,\ldots,\varphi_{n-1})d\rho d\sigma(x)$, where $d\sigma$ is the element of the area of S^{n-1} and $\rho^{|d|-1}J(\varphi_1,\ldots,\varphi_{n-1})$ is the Jacobian of this transform. In [2, 14], it was shown there exists a constant $M \ge 1$ such that $1 \le J(\varphi_1,\ldots,\varphi_{n-1}) \le M$ and $J(\varphi_1,\ldots,\varphi_{n-1}) \in C^\infty((0,2\pi)^{n-2}\times(0,\pi))$. Then by the properties of ρ and of the above transform, we have

$$\|\rho(x)^\beta\|_{L_p(\mathcal{E}_d(0,r))} = \left(\int_{\mathcal{E}_d(0,r)} \rho(x)^{\beta p} dx\right)^{\frac{1}{p}}$$

$$= \left(\int_0^r \int_{S^{n-1}} t^{|d|-1+\beta p} dt d\sigma(x') \right)^{\frac{1}{p}}$$
$$= C_0 \left(\int_0^r t^{|d|-1+\beta p} dt \right)^{\frac{1}{p}} = (|d| + \beta p)^{-\frac{1}{p}} C_0 \, r^{\frac{|d|}{p}+\beta}$$

and

$$\|\rho(x)^\beta\|_{L_p(\complement\mathcal{E}_d(0,r))} = C_0 \left(\int_r^\infty t^{|d|-1+\beta p} dt \right)^{\frac{1}{p}} = (|d| + \beta p)^{-\frac{1}{p}} C_0 \, r^{\frac{|d|}{p}+\beta}.$$

□

Corollary 2.1. *Let $0 < p, \theta, t < \infty$ and $w \in \Omega_\theta$. Then*

$$1) \;\; \rho(x)^\beta \in LM_{p\theta,w,d} \Longleftrightarrow \beta > -\frac{|d|}{p} \quad and \;\; \|w(r) r^{\frac{|d|}{p}+\beta}\|_{L_\theta(0,\infty)} < \infty;$$
$$2) \;\; \rho(x)^\beta \chi_{\mathcal{E}_d(0,t)} \in LM_{p\theta,w,d} \Longleftrightarrow \beta > -\frac{|d|}{p} \quad and$$
$$\|w(r) r^{\frac{|d|}{p}+\beta}\|_{L_\theta(0,t)} < \infty, \|w(r)\|_{L_\theta(t,\infty)} < \infty;$$
$$3a) \rho(x)^\beta \chi_{\complement\mathcal{E}_d(0,t)} \in LM_{p\theta,w,d} \;\; for \;\; \beta > -\frac{|d|}{p}$$
$$\Longleftrightarrow \|(r^{\frac{|d|}{p}+\beta} - t^{\frac{|d|}{p}+\beta}) w(r)\|_{L_\theta(t,\infty)} < \infty;$$
$$3b) \rho(x)^\beta \chi_{\complement\mathcal{E}_d(0,t)} \in LM_{p\theta,w,d} \;\; for \;\; \beta = -\frac{|d|}{p} \Longleftrightarrow \|w(r) \Big(\ln \frac{r}{t} \Big)^{\frac{1}{p}} \|_{L_\theta(t,\infty)} < \infty;$$
$$3c) \rho(x)^\beta \chi_{\complement\mathcal{E}_d(0,t)} \in LM_{p\theta,w,d} \;\; for \;\; \beta < -\frac{|d|}{p}$$
$$\Longleftrightarrow \|(t^{\frac{|d|}{p}+\beta} - r^{\frac{|d|}{p}+\beta}) w(r)\|_{L_\theta(t,\infty)} < \infty.$$

Lemma 2.3. *Let $1 < p_1 \le \infty$, $0 < p_2 \le \infty$, $0 \le \alpha < |d|$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$. Then the condition*

$$\alpha \le \frac{|d|}{p_1}$$

is necessary for the boundedness of M^d_α from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$.

Proof. Assume that $\alpha > \frac{|d|}{p_1}$ and M^d_α is bounded from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$. Since $w_1 \in \Omega_{\theta_1}$ for some $t > 0$ $\|w_1\|_{L_\theta(t,\infty)} < \infty$. Let $f(x) = \rho(x)^\beta \chi_{\complement B(0,t)}$ where $-\alpha < \beta < -\frac{|d|}{p_1}$. By Corollary 2.1, part 3c) $f \in LM_{p_1\theta_1,w_1,d}$. On the other hand for all $x \in \mathbb{R}^n$

$$M^d_\alpha f(x) \ge \lim_{t\to\infty} |\mathcal{E}_d(x,t)|^{-1+\frac{\alpha}{|d|}} \int\limits_{\mathcal{E}_d(x,t)\setminus\mathcal{E}_d(x,\rho(x)+2)} \rho(y)^\beta dy \ge c \lim_{t\to\infty} t^{\alpha+\beta} = \infty,$$

where c depends only on n, α and β, hence $f \notin LM_{p_2\theta_2,w_2,d}$. □

For the isotropic case $d = \mathbf{1}$ Lemma 2.1 was proved in [3] and Lemma 2.3 was proved in [8].

Throughout this paper $a \lesssim b$, $(b \gtrsim a)$, means that $a \le \lambda b$, where $\lambda > 0$ depends on unessential parameters. If $b \lesssim a \lesssim b$, then we write $a \approx b$.

3 L_p-estimates of anisotropic fractional maximal function over ellipsoids

We consider the following ‘partial’ anisotropic fractional maximal functions

$$\underline{M}^d_{\alpha,r} f(x) = \sup_{0<t\le r} |\mathcal{E}_d(x,t)|^{-1+\frac{\alpha}{|d|}} \int_{\mathcal{E}_d(x,t)} |f(y)|dy,$$
$$\overline{M}^d_{\alpha,r} f(x) = \sup_{t>r} |\mathcal{E}_d(x,t)|^{-1+\frac{\alpha}{|d|}} \int_{\mathcal{E}_d(x,t)} |f(y)|dy.$$

Lemma 3.1. *Let $0 < p \le \infty$, $0 \le \alpha < |d|$ and $f \in L_1^{\rm loc}$. Then for any ellipsoid $\mathcal{E}_d(x,r)$ in $\mathbb{R}^n$*

$$\|M^d_\alpha f\|_{WL_p(\mathcal{E}_d(x,r))} \gtrsim r^{\frac{|d|}{p}}\, \overline{M}^d_{\alpha,r} f(x). \tag{3.1}$$

Proof. If $y \in \mathcal{E}_d(x,r)$ and $t > 2r$, then $\mathcal{E}_d(x,\frac{t}{2}) \subset \mathcal{E}_d(y,t)$ and

$$M^d_\alpha f(y) \ge 2^{\alpha-|d|} \sup_{t>2r} \frac{1}{|\mathcal{E}_d(x,\frac{t}{2})|^{1-\frac{\alpha}{|d|}}} \int_{\mathcal{E}_d(x,\frac{t}{2})} |f(z)|dz = 2^{\alpha-|d|}\overline{M}^d_{\alpha,r} f(x).$$

Hence, if f is not equivalent to 0 on $\mathbb{R}^n$, then

$$\begin{aligned}\|M^d_\alpha f\|_{WL_p(\mathcal{E}_d(x,r))} &\ge \sup_{0<t<2^{\alpha-|d|}\overline{M}^d_{\alpha,r}f(x)} t\,\left|\{y \in \mathcal{E}_d(x,r) : M^d_\alpha f(y) > t\}\right|^{\frac{1}{p}} \\ &\ge \sup_{0<t<2^{\alpha-|d|}\overline{M}^d_{\alpha,r}f(x)} t(v_n r^{|d|})^{\frac{1}{p}} = 2^{\alpha-|d|} v_n^{\frac{1}{p}}\, r^{\frac{|d|}{p}}\, \overline{M}^d_{\alpha,r} f(x).\end{aligned}$$

(If f is equivalent to 0 inequality (3.1) is trivial.) □

Lemma 3.2. *Let $0 < p \le \infty$, $0 \le \alpha < |d|$ and $f \in L_1^{\rm loc}$. Then for any ellipsoid $\mathcal{E}_d(x,r)$ in $\mathbb{R}^n$*

$$\|M^d_\alpha f\|_{L_p(\mathcal{E}_d(x,r))} \approx \|M^d_\alpha (f\chi_{\mathcal{E}_d(x,2r)})\|_{L_p(\mathcal{E}_d(x,r))} + r^{\frac{|d|}{p}}\, \overline{M}^d_{\alpha,2r} f(x). \tag{3.2}$$

Proof. It is obvious that for any ellipsoid $\mathcal{E}_d(x,r)$

$$\|M^d_\alpha f\|_{L_p(\mathcal{E}_d(x,r))} \lesssim \|M^d_\alpha (f\chi_{\mathcal{E}_d(x,2r)})\|_{L_p(\mathcal{E}_d(x,r))} + \|M^d_\alpha (f\chi_{\complement_{\mathcal{E}_d(x,2r)}})\|_{L_p(\mathcal{E}_d(x,r))}.$$

Let y be an arbitrary point in $\mathcal{E}_d(x,r)$. If $\mathcal{E}_d(y,t) \cap {}^\complement\mathcal{E}_d(x,2r) \ne \emptyset$, then $t > r$. Indeed, if $z \in \mathcal{E}_d(y,t) \cap {}^\complement\mathcal{E}_d(x,2r)$, then $t > \rho(z-y) \ge \rho(z-x) - \rho(x-y) > 2r - r = r$.

On the other hand $\mathcal{E}_d(y,t) \cap {}^\complement\mathcal{E}_d(x,2r) \subset \mathcal{E}_d(x,2t)$. Indeed, if $z \in \mathcal{E}_d(y,t) \cap {}^\complement\mathcal{E}_d(x,2r)$, then we get $\rho(z-x) \le \rho(z-y) + \rho(y-x) < t + r < 2t$.

Hence

$$M_\alpha^d(f\chi_{\complement\mathcal{E}_d(x,2r)})(y) = \sup_{t>0}\frac{1}{|\mathcal{E}_d(y,t)|^{1-\frac{\alpha}{|d|}}}\int_{\mathcal{E}_d(y,t)\cap\complement\mathcal{E}_d(x,2r)}|f(z)|dz$$
$$\lesssim \sup_{t\ge r}\frac{1}{|\mathcal{E}_d(x,2t)|^{1-\frac{\alpha}{|d|}}}\int_{\mathcal{E}_d(x,2t)}|f(y)|dy = \overline{M}_{\alpha,2r}^d f(x)$$

and the right-hand side inequality in (3.2) follows.

The left-hand side inequality in (3.2) follows by Lemma 3.1 and obvious inequality

$$\|M_\alpha^d f\|_{L_p(\mathcal{E}_d(x,r))} \ge \|M_\alpha^d(f\chi_{\mathcal{E}_d(x,2r)})\|_{L_p(\mathcal{E}_d(x,r))}.$$

□

Lemma 3.3. *Let* $1 \le p_1 \le p_2 \le \infty$ *and* $0 \le \alpha < |d|$. *The inequality*

$$\|M_\alpha^d(f\chi_{\mathcal{E}_d(x,2r)})\|_{L_{p_2}(\mathcal{E}_d(x,r))} \lesssim r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))} \tag{3.3}$$

holds for all $f \in L_{p_1}^{\mathrm{loc}}$ *if and only if in the case* $p_1 > 1$

$$\alpha \ge |d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right), \tag{3.4}$$

and in the case $p_1 = 1$

$$p_2 < \infty \qquad \text{and} \qquad \alpha > |d|\left(1-\frac{1}{p_2}\right). \tag{3.5}$$

Moreover for $1 \le p_2 < \infty$ *and* $\alpha = |d|\left(1-\frac{1}{p_2}\right)$ *the inequality*

$$\|M_\alpha^d(f\chi_{\mathcal{E}_d(x,2r)})\|_{WL_{p_2}(\mathcal{E}_d(x,r))} \lesssim \|f\|_{L_1(\mathcal{E}_d(x,2r))} \tag{3.6}$$

holds for all $f \in L_1^{\mathrm{loc}}$.

Proof. Recall the well-known inequalities for the fractional maximal operator [26]. If $1 < p_1 \le p_2 \le \infty$, then

$$\|M^d_{|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}f\|_{L_{p_2}(\mathbb{R}^n)} \lesssim \|f\|_{L_{p_1}(\mathbb{R}^n)}. \tag{3.7}$$

Also if $1 \le p_2 < \infty$, then

$$\|M^d_{|d|\left(1-\frac{1}{p_2}\right)}f\|_{WL_{p_2}(\mathbb{R}^n)} \lesssim \|f\|_{L_1(\mathbb{R}^n)}. \tag{3.8}$$

If $1 < p_1 \le p_2 \le \infty$, inequality (3.4) holds and $z \in \mathcal{E}_d(x,r)$, then

$$M_\alpha^d(f\chi_{\mathcal{E}_d(x,2r)})(z) = \sup_{0<t\le 3r}|\mathcal{E}_d(z,t)|^{\frac{\alpha}{|d|}-1}\int_{\mathcal{E}_d(z,t)}\left|f(y)\chi_{\mathcal{E}_d(x,2r)}(y)\right|dy,$$

because for $t > 3r$ $\mathcal{E}_d(z,t) \supset \mathcal{E}_d(x,2r)$ hence

$$|\mathcal{E}_d(z,t)|^{\frac{\alpha}{|d|}-1} \int_{\mathcal{E}_d(z,t)} \left| f(y)\chi_{\mathcal{E}_d(x,2r)}(y) \right| dy$$
$$\leq |\mathcal{E}_d(z,3r)|^{\frac{\alpha}{|d|}-1} \int_{\mathcal{E}_d(z,3r)} \left| f(y)\chi_{\mathcal{E}_d(x,2r)}(y) \right| dy.$$

Therefore

$$M^d_\alpha(f\chi_{\mathcal{E}_d(x,2r)})(z) \lesssim r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} M_{|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \left(f\chi_{\mathcal{E}_d(x,2r)}\right)(z)$$

and by (3.7)

$$\|M^d_\alpha(f\chi_{\mathcal{E}_d(x,2r)})\|_{L_{p_2}(\mathcal{E}_d(x,r))} \lesssim r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \left\| M_{|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \left(f\chi_{\mathcal{E}_d(x,2r)}\right) \right\|_{L_{p_2}(\mathbb{R}^n)}$$
$$\lesssim r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))}.$$

If $1 \leq p_2 < \infty$ and inequality (3.6) holds, then by (3.8) and (1.1)

$$\|M^d_\alpha\left(f\chi_{\mathcal{E}_d(x,2r)}\right)\|_{L_{p_2}(\mathcal{E}_d(x,r))} \leq \| \left(M^d_\alpha\left(f\chi_{\mathcal{E}_d(x,2r)}\right)\right)^* \|_{L_{p_2}(0,|\mathcal{E}_d(x,r)|)}$$
$$\leq \sup_{0<t\leq|\mathcal{E}_d(x,r)|} t^{1-\frac{\alpha}{|d|}} \left(M^d_\alpha\left(f\chi_{\mathcal{E}_d(x,2r)}\right)\right)^*(t) \|t^{\frac{\alpha}{|d|}-1}\|_{L_{p_2}(0,|\mathcal{E}_d(x,r)|)}$$
$$\lesssim r^{\alpha-|d|\left(1-\frac{1}{p_2}\right)} \left\|M^d_\alpha\left(f\chi_{\mathcal{E}_d(x,2r)}\right)\right\|_{WL_{\frac{|d|}{|d|-\alpha}}(\mathbb{R}^n)}$$
$$\lesssim r^{\alpha-|d|\left(1-\frac{1}{p_2}\right)} \|f\|_{L_1(\mathcal{E}_d(x,2r))}.$$

Inequality (3.6) follows directly from (3.8).

If $p_1 > 1$ and $\alpha < |d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)$, then inequality (3.7) cannot hold for all $f \in L^{\rm loc}_{p_1}$. Indeed if $f \in L_{p_1}(\mathbb{R}^n)$ and $f \nsim 0$ then by passing in (3.3) to the limit as $r \to \infty$ we arrive at a contradiction.

Assume that $p_1 = 1$, $1 \leq p_2 < \infty$ and $\alpha = |d|\left(1-\frac{1}{p_2}\right)$. Then by passing to the limit in (3.3) we get

$$\|M^d_\alpha f\|_{L_{p_2}(\mathbb{R}^n)} \lesssim \|f\|_{L_1(\mathbb{R}^n)}$$

which, according to known results [26], is not possible. □

Corollary 3.1. *Let* $1 \leq p_1 \leq \infty$, $0 < p_2 \leq \infty$, $|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ \leq \alpha < |d|$ *if* $p_1 > 1$, *and* $|d|\left(1-\frac{1}{p_2}\right)_+ < \alpha < |d|$ *if* $p_1 = 1$. *Then the inequality*

$$\|M^d_\alpha(f\chi_{\mathcal{E}_d(x,2r)})\|_{L_{p_2}(\mathcal{E}_d(x,r))} \lesssim r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))}$$

holds for all $f \in L^{\rm loc}_{p_1}$.

Moreover for $0 < p_2 < \infty$ *and* $\alpha = |d|\left(1 - \frac{1}{p_2}\right)_+$ *the inequality*

$$\|M^d_\alpha(f\chi_{\mathcal{E}_d(x,2r)})\|_{WL_{p_2}(\mathcal{E}_d(x,r))} \lesssim r^{\alpha-|d|\left(1-\frac{1}{p_2}\right)} \|f\|_{L_1(\mathcal{E}_d(x,2r))} \tag{3.9}$$

holds for all $f \in L_1^{\mathrm{loc}}$.

Proof. If $p_2 \geq p_1$, the statement follows by Lemma 3.3. If $p_2 < p_1$, then by applying Hölder's inequality and statement of Lemma 3.3 we have

$$\begin{aligned}\|M^d_\alpha(f\chi_{\mathcal{E}_d(x,2r)})\|_{L_{p_2}(\mathcal{E}_d(x,r))} &\lesssim r^{\frac{|d|}{p_2}-\frac{|d|}{p_1}}\|M^d_\alpha(f\chi_{\mathcal{E}_d(x,2r)})\|_{L_{p_1}(\mathcal{E}_d(x,r))} \\ &\lesssim r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))}\,.\end{aligned}$$

Inequality (3.9) similarly follows by Hölder's inequality for weak L_p-spaces.

□

Lemmas 3.2, 3.3 and Corollary 3.1 imply the following statement.

Lemma 3.4. *Let* $1 \leq p_1 \leq \infty$, $0 < p_2 \leq \infty$, $|d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \leq \alpha < |d|$ *if* $p_1 > 1$, *and* $|d|\left(1 - \frac{1}{p_2}\right)_+ < \alpha < |d|$ *if* $p_1 = 1$. *Then for any ellipsoid* $\mathcal{E}_d(x,r) \subset \mathbb{R}^n$ *the inequality*

$$\|M^d_\alpha f\|_{L_{p_2}(\mathcal{E}_d(x,r))} \lesssim r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))} + r^{\frac{|d|}{p_2}}\,\overline{M}^d_{\alpha,2r}f(x) \tag{3.10}$$

holds for all $f \in L_{p_1}^{\mathrm{loc}}$.

Moreover for $0 < p_2 < \infty$ *and* $\alpha = |d|\left(1 - \frac{1}{p_2}\right)_+$ *the inequality*

$$\|M^d_\alpha f\|_{WL_{p_2}(\mathcal{E}_d(x,r))} \lesssim r^{\alpha-|d|\left(1-\frac{1}{p_2}\right)}\|f\|_{L_1(\mathcal{E}_d(x,2r))} + r^{\frac{|d|}{p_2}}\,\overline{M}^d_{\alpha,2r}f(x) \tag{3.11}$$

holds for all $f \in L_1^{\mathrm{loc}}$.

Lemma 3.5. *Let* $0 < p < \infty$.

1. *If* $|d|\left(1 - \frac{1}{p}\right)_+ < \alpha < |d|$, *then for any ellipsoid* $\mathcal{E}_d(x,r) \subset \mathbb{R}^n$ *the equivalences*

$$\|M^d_\alpha f\|_{L_p(\mathcal{E}_d(x,r))} \approx \|M^d_\alpha f\|_{WL_p(\mathcal{E}_d(x,r))} \approx r^{\frac{|d|}{p}}\,\overline{M}^d_{\alpha,r}f(x) \tag{3.12}$$

hold for all $f \in L_1^{\mathrm{loc}}$.

2. *If* $\alpha = |d|\left(1 - \frac{1}{p}\right)_+$, *then for any ellipsoid* $\mathcal{E}_d(x,r) \subset \mathbb{R}^n$ *the equivalence*

$$\|M^d_\alpha f\|_{WL_p(\mathcal{E}_d(x,r))} \approx r^{\frac{|d|}{p}}\,\overline{M}^d_{\alpha,r}f(x) \tag{3.13}$$

holds for all $f \in L_1^{\mathrm{loc}}$.

3. *If* $1 < p_1 < \infty$, $|d|\left(\frac{1}{p_1} - \frac{1}{p}\right)_+ \leq \alpha < \frac{|d|}{p_1}$, *then for any ellipsoid* $\mathcal{E}_d(x,r) \subset \mathbb{R}^n$ *the inequalities*

$$r^{\frac{|d|}{p}}\,\overline{M}^d_{\alpha,r}f(x) \lesssim \|M^d_\alpha f\|_{L_p(\mathcal{E}_d(x,r))} \lesssim r^{\frac{|d|}{p}}\,\left(\overline{M}^d_{\alpha p_1,r}(|f|^{p_1})(x)\right)^{\frac{1}{p_1}} \tag{3.14}$$

hold for all $f \in L_1^{\mathrm{loc}}$.

Proof. Denote

$$A_1 := r^{\frac{|d|}{p}} \sup_{t\ge 2r} \frac{1}{|\mathcal{E}_d(x,t)|^{1-\frac{\alpha}{|d|}}} \int_{\mathcal{E}_d(x,t)} |f(y)|dy,$$
$$A_2 := r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p}\right)} \|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))}.$$

By Lemma 3.4

$$\|M^d_\alpha f\|_{L_p(\mathcal{E}_d(x,r))} \le A_1 + A_2.$$

By applying Hölder's inequality we get

$$A_1 \lesssim r^{\frac{|d|}{p}} \sup_{t\ge 2r} \frac{1}{|\mathcal{E}_d(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{|d|}}} \left(\int_{\mathcal{E}_d(x,t)} |f(y)|^{p_1} dy\right)^{\frac{1}{p_1}}$$
$$= r^{\frac{|d|}{p}} \left(\overline{M}^d_{\alpha p_1,2r}(|f|^{p_1})(x)\right)^{\frac{1}{p_1}}.$$

On the other hand, since $\alpha < \frac{|d|}{p_1}$ it follows that

$$A_2 \approx r^{\frac{|d|}{p}} \left(\sup_{t\ge 2r} |\mathcal{E}_d(x,t)|^{\frac{\alpha}{|d|}-\frac{1}{p_1}}\right) \|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))}$$
$$\lesssim r^{\frac{|d|}{p}} \left(\sup_{t\ge 2r} \frac{1}{|\mathcal{E}_d(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{|d|}}} \left(\int_{\mathcal{E}_d(x,t)} |f(y)|^{p_1} dy\right)^{\frac{1}{p_1}}\right)$$
$$\lesssim r^{\frac{|d|}{p}} \left(\overline{M}^d_{\alpha p_1,r}(|f|^{p_1})(x)\right)^{\frac{1}{p_1}}.$$

Estimates from below follow by Lemma 3.1. □

Remark 3.1. We note that the right-hand side inequality in (3.14) implies the inequality

$$\|M^d_\alpha f\|_{L_p(\mathcal{E}_d(x,r))} \lesssim r^{\frac{|d|}{p}} \left(\int_r^\infty \left(\int_{\mathcal{E}_d(x,t)} |f(y)|^{p_1} dy\right) \frac{dt}{t^{|d|-\alpha p_1+1}}\right)^{\frac{1}{p_1}}.$$

This follows since

$$\left(\overline{M}^d_{\alpha p_1,r}(|f|^{p_1})(x)\right)^{\frac{1}{p_1}} \lesssim \left(\int_r^\infty \left(\int_{\mathcal{E}_d(x,t)} |f(y)|^{p_1} dy\right) \frac{dt}{t^{|d|-\alpha p_1+1}}\right)^{\frac{1}{p_1}}.$$

In fact

$$\begin{aligned}\left(\overline{M}^d_{\alpha p_1,r}(|f|^{p_1})(x)\right)^{\frac{1}{p_1}} &= \sup_{t\geq r} \frac{1}{|\mathcal{E}_d(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{|d|}}} \left(\int_{\mathcal{E}_d(x,t)} |f(y)|^{p_1} dy\right)^{\frac{1}{p_1}} \\ &\leq \sup_{t\geq r} \frac{1}{|\mathcal{E}_d(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{|d|}}} \left(\int_{\mathcal{E}_d(x,r)} |f(y)|^{p_1} dy\right)^{\frac{1}{p_1}} \\ &+ \sup_{t\geq r} \frac{1}{|\mathcal{E}_d(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{|d|}}} \left(\int_{\mathcal{E}_d(x,t)\setminus\mathcal{E}_d(x,r)} |f|^{p_1} dy\right)^{\frac{1}{p_1}} \\ &\lesssim \frac{1}{|\mathcal{E}_d(x,r)|^{\frac{1}{p_1}-\frac{\alpha}{|d|}}} \left(\int_{\mathcal{E}_d(x,r)} |f(y)|^{p_1} dy\right)^{\frac{1}{p_1}} \\ &+ \sup_{t\geq r} \left(\int_{\mathcal{E}_d(x,t)\setminus\mathcal{E}_d(x,r)} \frac{|f(y)|^{p_1}}{\rho(y)^{|d|-\alpha p_1}} dy\right)^{\frac{1}{p_1}}.\end{aligned}$$

By using the equality

$$\frac{1}{\rho^{|d|-\alpha p_1}} = \frac{1}{|d|-\alpha p_1} \int_\rho^\infty \frac{d\tau}{\tau^{|d|-\alpha p_1+1}}$$

with $\rho = r$ or $\rho = \rho(y)$ and the Fubini theorem we get

$$\begin{aligned}\left(\overline{M}^d_{\alpha p_1,r}(|f|^{p_1})(x)\right)^{\frac{1}{p_1}} &\lesssim \left(\int_r^\infty \left(\int_{\mathcal{E}_d(x,r)} |f(y)|^{p_1} dy\right) \frac{d\tau}{\tau^{|d|-\alpha p_1+1}}\right)^{\frac{1}{p_1}} \\ &+ \sup_{t\geq r} \left(\int_r^t \left(\int_{\mathcal{E}_d(x,\tau)\setminus\mathcal{E}_d(x,r)} |f(y)|^{p_1} dy\right) \frac{d\tau}{\tau^{|d|-\alpha p_1+1}}\right)^{\frac{1}{p_1}} \\ &\lesssim \left(\int_r^\infty \left(\int_{\mathcal{E}_d(x,\tau)} |f(y)|^{p_1} dy\right) \frac{d\tau}{\tau^{|d|-\alpha p_1+1}}\right)^{\frac{1}{p_1}}.\end{aligned}$$

Remark 3.2. Statement 3 of Lemma 3.5 also makes sense if $\alpha = \frac{|d|}{p_1}$ in which case the right-hand side inequality in (3.14) takes the form

$$\|M^d_{\frac{|d|}{p_1}} f\|_{L_p(\mathcal{E}_d(x,r))} \lesssim r^{\frac{|d|}{p}} \|f\|_{L_{p_1}(\mathbb{R}^n)}.$$

This inequality easily follows directly by the definition of $M^d_{\frac{|d|}{p_1}} f$ and Hölder's inequality.

Remark 3.3. All statements of this section in the isotropic case $d = \mathbf{1}$ were proved in [4].

4 Anisotropic fractional maximal operator and supremal operator

For a measurable set $E \subset \mathbb{R}^n$ and a function v non-negative and measurable on E, let $L_{p,v}(E)$ be the weighted L_p-space of all functions f measurable on E for which

$$\|f\|_{L_{p,v}(E)} = \|vf\|_{L_p(E)} < \infty.$$

Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all non-negative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and we set

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let u be continuous and non-negative on $(0,\infty)$. We define the supremal operators $\underline{S}_u$ and $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\underline{S}_u g)(t) := \|u\, g\|_{L_\infty(0,t)}, \quad t \in (0,\infty),$$
$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

In the case $u(r) = r^\beta$, $\beta \in \mathbb{R}$

$$(\underline{S}_\beta g)(t) := \|r^\beta\, g(r)\|_{L_\infty(0,t)}, \quad t \in (0,\infty),$$
$$(\overline{S}_\beta g)(t) := \|r^\beta\, g(r)\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

Also let $\underline{S} \equiv \underline{S}_0$ and $\overline{S} \equiv \overline{S}_0$.

If in Lemma 3.5 $x = 0$, then in the above notation it reduces to the following statement.

Lemma 4.1. *Let $0 < p < \infty$.*

1. *If $|d|\left(1 - \frac{1}{p}\right)_+ < \alpha < |d|$, then for any $r > 0$ the inequalities*

$$\|M_\alpha^d f\|_{L_p(\mathcal{E}_d(0,r))} \approx \|M_\alpha^d f\|_{WL_p(\mathcal{E}_d(0,r))} \approx r^{\frac{|d|}{p}}\, \overline{S}_{\alpha-|d|}\left(\|f\|_{L_1(\mathcal{E}_d(0,\cdot))}\right)(r) \tag{4.1}$$

holds for all $f \in L_1^{\mathrm{loc}}$.

2. *If $\alpha = |d|\left(1 - \frac{1}{p}\right)_+$, then for any $r > 0$ the inequality*

$$\|M_\alpha^d f\|_{WL_p(\mathcal{E}_d(0,r))} \approx r^{\frac{|d|}{p}}\, \overline{S}_{\alpha-|d|}\left(\|f\|_{L_1(\mathcal{E}_d(0,\cdot))}\right)(r) \tag{4.2}$$

holds for all $f \in L_1^{\mathrm{loc}}$.

3. *If $1 < p_1 < \infty$, $|d|\left(\frac{1}{p_1} - \frac{1}{p}\right)_+ \le \alpha < \frac{|d|}{p_1}$, then for any $r > 0$ the inequality*

$$\begin{aligned} r^{\frac{|d|}{p}}\, \overline{S}_{\alpha-|d|}\left(\|f\|_{L_1(\mathcal{E}_d(0,\cdot))}\right)(r) &\lesssim \|M_\alpha^d f\|_{L_p(\mathcal{E}_d(0,r))} \\ &\lesssim r^{\frac{|d|}{p}}\, \overline{S}_{\alpha-\frac{|d|}{p_1}}\left(\|f\|_{L_{p_1}(\mathcal{E}_d(0,\cdot))}\right)(r) \end{aligned} \tag{4.3}$$

holds for all $f \in L_1^{\mathrm{loc}}$.

4. *If $1 \le p_1 < \infty$, $|d|\left(\frac{1}{p_1} - \frac{1}{p}\right)_+ \le \alpha < \frac{|d|}{p_1}$, then for any $r > 0$ the inequality*

$$\begin{aligned} r^{\frac{|d|}{p}}\, \overline{S}_{\alpha-|d|}\left(\|f\|_{L_1(\mathcal{E}_d(0,\cdot))}\right)(r) &\lesssim \|M_\alpha^d f\|_{WL_p(\mathcal{E}_d(0,r))} \\ &\lesssim r^{\frac{|d|}{p}}\, \overline{S}_{\alpha-\frac{|d|}{p_1}}\left(\|f\|_{L_{p_1}(\mathcal{E}_d(0,\cdot))}\right)(r) \end{aligned} \tag{4.4}$$

holds for all $f \in L_1^{\mathrm{loc}}$.

Lemma 4.2. *Let $1 \le p_1 < \infty$, $0 < p_2 < \infty$, $|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ \le \alpha < \frac{|d|}{p_1}$ if $p_1 > 1$, and $|d|\left(1-\frac{1}{p_2}\right)_+ < \alpha < |d|$ if $p_1 = 1$. Let also $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$.*

Then the operator M_α^d is bounded from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$ if, and in the case $p_1 = 1$ only if, the operator $\overline{S}_{\alpha-\frac{|d|}{p_1}}$ is bounded from $L_{\theta_1,w_1(r)}(0,\infty)$ to $L_{\theta_2,w_2(r)r^{\frac{|d|}{p_2}}}(0,\infty)$ on the cone $\mathbb{A}$.

Proof. Sufficiency. Since $\overline{S}_{\alpha-\frac{|d|}{p_1}}$ is bounded from $L_{\theta_1,w_1(r)}(0,\infty)$ to $L_{\theta_2,w_2(r)r^{\frac{|d|}{p_2}}}(0,\infty)$ on the cone $\mathbb{A}$, by Lemma 4.1 we have

$$\begin{aligned}\|M_\alpha^d f\|_{LM_{p_2\theta_2,w_2,d}} &\lesssim \|\overline{S}_{\alpha-\frac{|d|}{p_1}}\left(\|f\|_{L_{p_1}(\mathcal{E}_d(0,\cdot))}\right)\|_{L_{\theta_2,w_2(r)r^{\frac{|d|}{p_2}}}} \\ &\lesssim \|w_1(r)\|f\|_{L_{p_1}(B(0,r))}\|_{L_{\theta_1}(0,\infty)} = \|f\|_{LM_{p_1\theta_1,w_1,d}}.\end{aligned} \tag{4.5}$$

Necessity. Let $p_1 = 1$ and the inequality

$$\|M_\alpha^d f\|_{LM_{p_2\theta_2,w_2,d}} \lesssim \|f\|_{LM_{1\theta_1,w_1,d}}$$

be satisfied. Then by (4.2)

$$\|\overline{S}_{\alpha-|d|}\left(\|f\|_{L_1(\mathcal{E}_d(0,\cdot))}\right)\|_{L_{\theta_2,w_2(r)r^{\frac{|d|}{p_2}}}} \lesssim \|\|f\|_{L_1(\mathcal{E}_d(0,\cdot))}\|_{L_{\theta_1,w_1}}. \tag{4.6}$$

Let $g \in \mathbb{A}$. Then there exists a sequence of non-negative functions $f_n \in L_1^{\rm loc}$ such that

$$g_n(r) = \|f_n\|_{L_1(\mathcal{E}_d(0,r))} \nearrow g(r), \quad r \in (0,\infty).$$

By (4.6) and the Fatou lemma

$$\|\overline{S}_{\alpha-|d|}\, g\|_{L_{\theta_2,w_2(r)r^{\frac{|d|}{p_2}}}} \lesssim \|g\|_{L_{\theta_1,w_1}}.$$

□

5 Necessary and sufficient conditions

By Lemma 4.2 and Theorem 5.4 in [4] we get

Theorem 5.1. *Let $1 \le p_1 < \infty$, $0 < p_2 < \infty$, $|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ \le \alpha < \frac{|d|}{p_1}$ if $p_1 > 1$, and $|d|\left(1-\frac{1}{p_2}\right)_+ < \alpha < |d|$ if $p_1 = 1$. Let also $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$.*

Then the operator M_α^d is bounded from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$ if, and in the case $p_1 = 1$ only if,

(i) if $\theta_1 \le \theta_2$ and $\theta_1 < \infty$, then

$$\sup_{t>0}\left(t^{\alpha-\frac{|d|}{p_1}}\|w_2(r)r^{\frac{|d|}{p_2}}\|_{L_{\theta_2}(0,t)} + \|w_2(r)r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|_{L_{\theta_2}(t,\infty)}\right)\|w_1\|_{L_{\theta_1}(t,\infty)}^{-1} < \infty; \tag{5.1}$$

(ii) if $\theta_2 < \theta_1 < \infty$, then

$$\left\| w_2(t) t^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|w_2(r) r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|_{L_{\theta_2}(t,\infty)}^{\frac{\theta_2}{\theta_1-\theta_2}} \|w_1\|_{L_{\theta_1}(t,\infty)}^{-\frac{\theta_1}{\theta_1-\theta_2}} \right\|_{L_{\theta_2}(0,\infty)} < \infty \tag{5.2}$$

and

$$\left\| w_2(t) t^{\frac{|d|}{p_2}} \|w_2(r) r^{\frac{|d|}{p_2}}\|_{L_{\theta_2}(0,t)}^{\frac{\theta_2}{\theta_1-\theta_2}} \ \overline{S}\big(r^{\alpha-\frac{|d|}{p_1}} \|w_1\|_{L_{\theta_1}(r,\infty)}^{-1}\big)(t)^{\frac{\theta_1}{\theta_1-\theta_2}} \right\|_{L_{\theta_2}(0,\infty)} < \infty; \tag{5.3}$$

(iii) if $\theta_1 = \infty$, then

$$\left\| w_2(t) t^{\frac{|d|}{p_2}} \, \overline{S}\left(r^{\alpha-\frac{|d|}{p_1}} \|w_1\|_{L_\infty(r,\infty)}^{-1} \right)(t) \right\|_{L_{\theta_2}(0,\infty)} < \infty. \tag{5.4}$$

Example 5.1 (power-type case). Let $1 \le p_1 < \infty$, $0 < p_2 < \infty$, $1 < \theta_1 \le \theta_2 < \infty$, $|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ \le \alpha < \frac{|d|}{p_1}$ if $p_1 > 1$, and $|d|\left(1-\frac{1}{p_2}\right)_+ < \alpha < |d|$ if $p_1 = 1$.

Moreover, let $w_1(t) = t^{-\frac{\lambda_1}{p_1}}$, $w_2(t) = t^{-\frac{\lambda_2}{p_2}}$, where $\lambda_1, \lambda_2 \in \mathbb{R}$. Then

$$w_i \in \Omega_{\theta_i} \Longleftrightarrow \frac{\lambda_i}{p_i} - \frac{1}{\theta_i} > 0 \quad (i = 1, 2).$$

By Theorems 5.1 and 1.3 it follows that the operator M_α^d is bounded from $LM_{p_1\theta_1, t^{-\frac{\lambda_1}{p_1}}, d}$ to $LM_{p_2\theta_2, t^{-\frac{\lambda_2}{p_2}}, d}$ if and only if

$$\alpha - |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right) + \frac{1}{\theta_2} < \frac{\lambda_2}{p_2} < \frac{|d|}{p_2} + \frac{1}{\theta_2}$$

and

$$\alpha = \frac{|d| - \lambda_1}{p_1} - \frac{|d| - \lambda_2}{p_2} + \frac{1}{\theta_1} - \frac{1}{\theta_2}. \tag{5.5}$$

The necessity of condition (5.5) also follows by the homogeneity argument.

Corollary 5.1. *Let $1 \le p_1 < \infty$, $0 < p_2 < \infty$, $|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ \le \alpha < \frac{|d|}{p_1}$ if $p_1 > 1$, and $|d|\left(1-\frac{1}{p_2}\right)_+ < \alpha < |d|$ if $p_1 = 1$. Let also w_1, w_2 be non-negative measurable functions satisfying $w_1 \in \Omega_{p_1\infty}$, $w_2 \in \Omega_{p_2\infty}$ and*

$$\operatorname*{ess\,sup}_{t>0} \left(w_2(t) t^{\frac{|d|}{p_2}} \operatorname*{ess\,sup}_{t<r<\infty} \frac{r^{\alpha-\frac{|d|}{p_1}}}{\|w_1\|_{L_\infty(r,\infty)}} \right) < \infty, \tag{5.6}$$

Then M_α^d is bounded from $\mathcal{M}_{p_1,w_1,d}$ to $\mathcal{M}_{p_2,w_2,d}$.

Proof. It is easy to see that boundedness of M_α^d from $LM_{p_1\infty,w_1,d}$ to $LM_{p_2\infty,w_2,d}$ implies boundedness of M_α^d from $GM_{p_1\infty,w_1,d} \equiv \mathcal{M}_{p_1,w_1,d}$ to $GM_{p_2\infty,w_2,d} \equiv \mathcal{M}_{p_2,w_2,d}$.

□

Remark 5.1. Note that condition (5.6) is weaker than condition (1.4) in Theorem 1.2. Indeed, if condition (1.4) holds, then for any r satisfying $t < r < \infty$ we get

$$\begin{aligned} \frac{1}{w_2(t)t^{\frac{|d|}{p_2}}} &\gtrsim \int_t^\infty \frac{ds}{w_1(s)s^{\frac{|d|}{p_1}-\alpha+1}} \geq \int_r^\infty \frac{ds}{w_1(s)s^{\frac{|d|}{p_1}-\alpha+1}} \\ &\geq \int_r^\infty \frac{ds}{\|w_1\|_{L_\infty(s,\infty)}s^{\frac{|d|}{p_1}-\alpha+1}} \geq \frac{1}{\|w_1\|_{L_\infty(r,\infty)}} \int_r^\infty \frac{ds}{s^{\frac{|d|}{p_1}-\alpha+1}} \\ &\approx \frac{1}{\|w_1\|_{L_\infty(r,\infty)}r^{\frac{|d|}{p_1}-\alpha}}. \end{aligned}$$

Thus

$$\operatorname*{ess\,sup}_{t<r<\infty} \frac{r^{\alpha-\frac{|d|}{p_1}}}{\|w_1\|_{L_\infty(r,\infty)}} \lesssim \frac{1}{w_2(t)t^{\frac{|d|}{p_2}}}, \quad t \in (0,\infty),$$

so condition (5.6) holds.

On the other hand the functions $w_1(t) = t^{\alpha-\frac{|d|}{p_1}}$, $w_2(t) = t^{-\frac{|d|}{p_2}}$ satisfy condition (5.6), but do not satisfy condition (1.4).

Theorem 5.1 contains necessary and sufficient conditions if $p_1 = 1$. If $p_1 > 1$ it contains sufficient conditions. However for $\theta_1 \leq \theta_2$ and the limiting case $\alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$ Theorem 5.1 together with the appropriate necessity condition implies necessary and sufficient conditions.

Theorem 5.2. *Let $1 < p_1 \leq p_2 < \infty$, $0 < \theta_1 \leq \theta_2 \leq \infty$, $\theta_1 < \infty$, $\alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, then the Burenkov-Guliyevs type condition*

$$\left\| w_2(r)\left(\frac{r}{t+r}\right)^{\frac{|d|}{p_2}} \right\|_{L_{\theta_2}(0,\infty)} \leq c\|w_1\|_{L_{\theta_1}(t,\infty)} \tag{5.7}$$

for all $t > 0$, where $c > 0$ is independent of t, is necessary and sufficient for the boundedness of M_α^d from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$.

Proof of sufficiency. Follows by Theorem 5.1 because condition (5.7) is equivalent to condition (5.1) if $\theta_1 < \infty$.

To prove necessity one should act like in paper [8] by using the test-function

$$f_t(y) = \chi_{\mathcal{E}_d(0,2t)\setminus\mathcal{E}_d(0,t)}(y), \quad y \in \mathbb{R}^n, \quad t > 0. \tag{5.8}$$

Note that,

$$\|f_t\|_{L_{p_1}(\mathcal{E}_d(0,r))} = 0, \;\; 0 < r \leq t, \quad \|f_t\|_{L_{p_1}(\mathcal{E}_d(0,r))} \leq c_1 t^{\frac{|d|}{p_1}}, \quad t < r < \infty, \tag{5.9}$$

where $c_1 > 0$ depends only on n, d and p_1.

Lemma 5.1. *If* $0 \le \alpha < |d|$, *then for all* $t > 0$ *and* $x \in \mathbb{R}^n$,

$$\frac{1}{2} v_n^{\frac{\alpha}{|d|}} \frac{t^{|d|}}{(\rho(x)+t)^{|d|-\alpha}} \le (M_\alpha^d f_t)(x) \le 8^n v_n^{\frac{\alpha}{|d|}} \frac{t^{|d|}}{(\rho(x)+t)^{|d|-\alpha}}.$$

Proof. The proof is similar to the proof of Lemma 8 in [3].

Lemma 5.2. *If* $0 \le \alpha < |d|$, $0 < p < \infty$, *then*

$$\left\| M_\alpha^d f_t \right\|_{L_p(\mathcal{E}_d(0,r))} \asymp t^\alpha r^{\frac{|d|}{p}} \begin{cases} \left(\frac{t}{r+t}\right)^{\min\{|d|-\alpha, \frac{|d|}{p}\}}, & p \ne \frac{|d|}{|d|-\alpha}, \\ \left(\frac{t}{r+t}\right)^{\frac{|d|}{p}} \ln\left(e + \frac{r}{t}\right), & p = \frac{|d|}{|d|-\alpha}. \end{cases}$$

Proof. By Lemma 5.1 we get

$$\left(\frac{1}{2}\right)^p v_n^{\frac{\alpha p}{|d|}} t^{|d|p} \int_{\mathcal{E}_d(0,r)} \frac{dy}{(\rho(y)+t)^{(|d|-\alpha)p}} \le \int_{\mathcal{E}_d(0,r)} (M_\alpha^d f_t)^p(y) dy$$
$$\le 8^{|d|p} v_n^{\frac{\alpha p}{|d|}} t^{|d|p} \int_{\mathcal{E}_d(0,r)} \frac{dy}{(\rho(y)+t)^{(|d|-\alpha)p}}.$$

Furthermore by passing to generalized spherical coordinates (2.5) we get

$$\int_{\mathcal{E}_d(0,r)} \frac{1}{(\rho(y)+t)^{(|d|-\alpha)p}} dy = |d| v_n \int_0^r \frac{\tau^{|d|-1}}{(\tau+t)^{(|d|-\alpha)p}} d\tau.$$

If $0 < r \le t$, then

$$\frac{(2t)^{(\alpha-|d|)p} r^{|d|}}{|d|} = (2t)^{(\alpha-|d|)p} \int_0^r \tau^{|d|-1} d\tau \le \int_0^r \frac{\tau^{|d|-1}}{(\tau+t)^{(|d|-\alpha)p}} d\tau \le$$
$$\le t^{(\alpha-|d|)p} \int_0^r \tau^{|d|-1} d\tau = \frac{t^{(\alpha-|d|)p} r^{|d|}}{|d|}. \tag{5.10}$$

Hence

$$2^{-p(|d|+1-\alpha)} v_n^{\frac{|d|+\alpha p}{|d|}} t^{\alpha p} r^{|d|} \le \int_{\mathcal{E}_d(0,r)} \left((M_\alpha^d f_t)(y)\right)^p dy \le 8^{|d|p} v_n^{\frac{|d|+\alpha p}{|d|}} t^{\alpha p} r^{|d|}.$$

If $r > t$, then we consider separately three cases.

1. If $p < \frac{|d|}{|d|-\alpha}$, then by applying (5.10) with $r = t$ we get

$$\frac{2^{(\alpha-|d|)p}}{|d|} r^{|d|-(|d|-\alpha)p} \le \int_0^r \frac{\tau^{|d|-1}}{(\tau+t)^{(|d|-\alpha)p}} d\tau$$
$$\le \int_0^r \tau^{|d|-1-(|d|-\alpha)p} d\tau \le \frac{r^{|d|-(|d|-\alpha)p}}{|d|-(|d|-\alpha)p},$$

hence

$$\frac{2^{(\alpha-|d|-1)p}}{|d|} r^{|d|-(|d|-\alpha)p} t^{|d|p} \le \frac{1}{v_n} \int_{\mathcal{E}_d(0,r)} \left((M_\alpha^d f_t)(y)\right)^p dy$$
$$\le \frac{8^{|d|p}}{|d|-(|d|-\alpha)p} r^{|d|-(|d|-\alpha)p} t^{|d|p}.$$

2. If $p = \frac{|d|}{|d|-\alpha}$, then

$$2^{-|d|}\left(\frac{1}{|d|} + \ln\frac{r}{t}\right) = (2t)^{-|d|}\int_0^t \tau^{|d|-1}d\tau + 2^{-|d|}\int_t^r \frac{d\tau}{\tau}$$
$$\le \int_0^r \frac{\tau^{|d|-1}}{(\tau+t)^{|d|}}d\tau$$
$$= \int_0^t \frac{\tau^{|d|-1}}{(\tau+t)^{|d|}}d\tau + \int_t^r \frac{\tau^{|d|-1}}{(\tau+t)^{|d|}}d\tau$$
$$\le t^{-|d|}\int_0^t \tau^{|d|-1}d\tau + \int_t^r \frac{d\tau}{\tau} = \frac{1}{|d|} + \ln\frac{r}{t},$$

hence

$$\frac{2^{(\alpha-|d|-1)p}}{|d|}\left(1+|d|\ln\frac{r}{t}\right)t^{|d|p} \le \upsilon_n^{-p}\int_{\mathcal{E}_d(0,r)}\left((M_\alpha^d f_t)(y)\right)^p dy \le \frac{8^{|d|p}}{|d|}\ln\left(e+\frac{r}{t}\right).$$

3. Finally, if $p > \frac{|d|}{|d|-\alpha}$, then

$$\frac{2^{(\alpha-|d|)p}}{|d|}t^{|d|-(|d|-\alpha)p} \le \int_0^t \frac{\tau^{|d|-1}}{(\tau+t)^{(|d|-\alpha)p}}d\tau \le \int_0^r \frac{\tau^{|d|-1}}{(\tau+t)^{(|d|-\alpha)p}}d\tau$$
$$= \int_0^t \frac{\tau^{|d|-1}}{(\tau+t)^{(|d|-\alpha)p}}d\tau + \int_t^r \frac{\tau^{|d|-1}}{(\tau+t)^{(|d|-\alpha)p}}d\tau$$
$$\le \frac{1}{|d|}t^{|d|-(|d|-\alpha)p} + \int_t^\infty \tau^{|d|-1-(|d|-\alpha)p}d\tau$$
$$= \left(\frac{1}{|d|} - \frac{1}{|d|-(|d|-\alpha)p}\right)t^{|d|-(|d|-\alpha)p},$$

hence

$$2^{(\alpha-|d|)p}\upsilon_n^{\frac{|d|+\alpha p}{|d|}}|d|^{-1}t^{|d|+\alpha p} \le \int_{\mathcal{E}_d(0,r)}\left((M_\alpha^d f_t)(y)\right)^p dy$$
$$\le 8^{|d|p}\upsilon_n^{\frac{|d|+\alpha p}{|d|}}\frac{(|d|-\alpha)p}{|d|((|d|-\alpha)p-|d|)}\,t^{|d|+\alpha p}.$$

These estimates imply the statement. □

Proof of necessity in Theorem 5.2. Assume that, for some $c > 0$ and for all $f \in LM_{p_1\theta_1,w_1}$

$$\|M_\alpha^d f\|_{LM_{p_2\theta_2,w_2}} \le c\|f\|_{LM_{p_1\theta_1,w_1}}. \tag{5.11}$$

In (5.11) take $f = f_t$, where f_t is defined by (5.8). Then by (5.9) the right-hand side of (5.11) does not exceed a constant multiplied by $t^{|d|/p_1}\|w_1\|_{L_{\theta_1}(t,\infty)}$. Furthermore by Lemma 5.2 for $p_2 \ne \frac{|d|}{|d|-\alpha}$ the left-hand side of inequality (5.11) is greater than or equal to a constant multiplied by

$$t^{\alpha+\min\{|d|-\alpha,|d|/p_2\}}\left\|w_2(r)\frac{r^{|d|/p_2}}{(t+r)^{\min\{|d|-\alpha,|d|/p_2\}}}\right\|_{L_{\theta_2}(0,\infty)},$$

which implies the necessity of (5.7).

This argument works also for the case $p_2 = \frac{|d|}{|d|-\alpha}$ since $\ln(e + \frac{r}{t}) \geq 1$. □

Example 5.2 (standard parabolic case). Recall that in this case $d = (1, \ldots, 1, 2)$ hence $|d| = n+1$. Thus if $1 < p_1 \leq p_2 < \infty$, $0 < \theta_1 \leq \theta_2 \leq \infty$, $\alpha = (n+1)\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, then condition (5.7) with $|d| = n+1$ is necessary and sufficient for the boundedness of M_α^d from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$.

Corollary 5.2. *Let $1 < p_1 \leq p_2 < \infty$, $0 < \theta_1 \leq \theta_2 \leq \infty$, $\alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $w_2 \in \Omega_{\theta_2}$ and*

$$\left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{|d|}{p_2}} \right\|_{L_{\theta_2}(0,\infty)} < \infty \tag{5.12}$$

for all $t > 0$. Moreover, if $\theta_2 = \infty$ and $\theta_1 < \infty$ it is also assumed that

$$\lim_{t\to\infty} \left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{|d|}{p_2}} \right\|_{L_\infty(0,\infty)} = 0\,. \tag{5.13}$$

Then

1) M_α^d is bounded from $LM_{p_1\theta_1,w_1^*,d}$ to $LM_{p_2\theta_2,w_2,d}$, where w_1^* is a non-increasing continuous function on $(0,\infty)$ defined by

$$\|w_1^*\|_{L_{\theta_1}(t,\infty)} = \left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{|d|}{p_2}} \right\|_{L_{\theta_2}(0,\infty)}, \qquad t \in (0,\infty). \tag{5.14}$$

2) If $w_1 \in \Omega_{\theta_1}$ and M_α^d is bounded from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$, then

$$LM_{p_1\theta_1,w_1,d} \subset LM_{p_1\theta_1,w_1^*,d}. \tag{5.15}$$

(Hence $LM_{p_1\theta_1,w_1^*,d}$ is the maximal among spaces $LM_{p_1\theta_1,w_1,d}$ for which M_α^d is bounded from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$.)

Note that equality (5.14), under assumptions (5.12) and (if $\theta_2 = \infty$ and $\theta_1 < \infty$) (5.13), defines a non-increasing continuous function w_1^* uniquely. In particular, if $\theta_1 = \infty$, then

$$w_1^*(t) = \left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{|d|}{p_2}} \right\|_{L_{\theta_2}(0,\infty)}, \qquad t \in (0,\infty).$$

Proof. 1) Statement 1 of the theorem follows by Theorem 5.2.

2) Let $w_1 \in \Omega_{\theta_1}$ and M_α^d is bounded from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$. By Theorem 5.2 and equality (5.14) there exists $c > 0$ such that for all $t > 0$

$$\|w_1^*\|_{L_{\theta_1}(t,\infty)} \leq c\|w_1\|_{L_{\theta_1}(t,\infty)}.$$

Therefore, as in the proof of Lemma 2 in [8], inclusion (5.15) follows. □

Recall that, for $0 < p \leq \infty$

$$\|f\|_{LM_{pp,w}} = \|f\|_{L_{p,W}},$$

where for all $x \in \mathbb{R}^n$ $W(x) = \|w\|_{L_p(\rho(x),\infty)}$. For this reason Theorem 5.2 implies necessary and sufficient conditions for boundedness of M^d_α from one weighted Lebesgue spaces L_{p_1,W_1} to another one L_{p_2,W_2} for the case of radially non-increasing weights W_1 and W_2. It is interesting to note that these conditions have the form that differs form the known necessary and sufficient conditions discussed in detail, for example, in [16].

Corollary 5.3. *Let $1 < p_1 \leq p_2 < \infty$, $\alpha = |d| \left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, and W_1, W_2 be non-increasing radially symmetric functions with respect to the distance ρ. Then the condition*

$$\left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{|d|}{p_2}} \right\|_{L_{\theta_2}(0,\infty)} \leq c\|w_1\|_{L_{\theta_1}(t,\infty)} \tag{5.16}$$

for all $t > 0$, where functions w_1 and w_2 are defined by the equations

$$W_1(x) = \|w_1\|_{L_{p_1}(\rho(x),\infty)}, \qquad W_2(x) = \|w_2\|_{L_{p_2}(\rho(x),\infty)}, \quad x \in \mathbb{R}^n, \tag{5.17}$$

and $c > 0$ is independent of t, is necessary and sufficient for the boundedness of M^d_α from L_{p_1,W_1} to L_{p_2,W_2}.

Example 5.3 (standard parabolic case). Recall that in this case $d = (1, \dots, 1, 2)$. So if $1 < p_1 \leq p_2 < \infty$, $\alpha = (n+1)\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$ and W_1, W_2 are non-increasing radially symmetric functions with respect to the distance ρ, then condition (5.16) with $|d| = n+1$, where functions w_1 and w_2 are defined by the equations (5.17), is necessary and sufficient for the boundedness of M^d_α from L_{p_1,W_1} to L_{p_2,W_2}.

In the isotropic case Corollary 5.3 was proved in [4].

6 The case of anisotropic weak Morrey-type spaces

Next we consider anisotropic local and global weak Morrey-type spaces and formulate the results for boundedness of M^d_α in these spaces, which follows by the estimates of the previous sections.

Definition 6.1. Let $0 < p, \theta \leq \infty$ and let w be a non-negative measurable function on $(0, \infty)$. Denote by $LWM_{p\theta,w,d}$ $GWM_{p\theta,w,d}$, the anisotropic local weak Morrey-type spaces, the anisotropic global weak Morrey-type spaces respectively, the spaces of all functions $f \in L^{\text{loc}}_p$ with finite quasinorms

$$\|f\|_{LWM_{p\theta,w,d}} \equiv \|f\|_{LWM_{p\theta,w,d}(\mathbb{R}^n)} = \left\|w(r)\|f\|_{WL_p(\mathcal{E}_d(0,r))}\right\|_{L_\theta(0,\infty)},$$

$$\|f\|_{GWM_{p\theta,w,d}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LWM_{p\theta,w,d}}$$

respectively.

Lemma 6.1. *Let* $1 \le p_1 < \infty$, $0 < p_2 < \infty$, $|d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \le \alpha < \frac{|d|}{p_1}$. *Let also* $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$. *Then the operator* M_α^d *is bounded from* $LM_{p_1\theta_1,w_1,d}$ *to* $LWM_{p_2\theta_2,w_2,d}$ *if, and in the case* $p_1 = 1$ *only if, the operator* $\overline{S}_{\alpha-\frac{|d|}{p_1}}$ *is bounded from* $L_{\theta_1,w_1(r)}(0,\infty)$ *to* $L_{\theta_2,w_2(r)r^{\frac{|d|}{p_2}}}(0,\infty)$ *on the cone* $\mathbb{A}$.

Proof. The proof is similar to the proof of Lemma 4.2. □

By Lemma 6.1 and Theorem 5.4 in [4] we get

Theorem 6.1. *Let* $1 \le p_1 < \infty$, $0 < p_2 < \infty$, $|d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \le \alpha < \frac{|d|}{p_1}$. *Let also* $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$.

Then the operator M_α^d *is bounded from* $LM_{p_1\theta_1,w_1,d}$ *to* $LWM_{p_2\theta_2,w_2,d}$ *if, and in the case* $p_1 = 1$ *only if,*

(i) if $\theta_1 \le \theta_2$ *and* $\theta_1 < \infty$, *then condition* (5.1) *holds;*

(ii) if $\theta_2 < \theta_1 < \infty$, *then conditions* (5.2) *and* (5.3) *hold;*

(iii) if $\theta_1 = \infty$, *then condition* (5.4) *holds.*

Theorem 6.2. *Let* $1 \le p_1 \le p_2 < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *then condition* (5.7) *is necessary and sufficient for the boundedness of* M_α^d *from* $LM_{p_1\theta_1,w_1,d}$ *to* $WLM_{p_2\theta_2,w_2,d}$.

Example 6.1 (standard parabolic case). Recall that in this case $d = (1, \dots, 1, 2)$. So if $1 \le p_1 \le p_2 < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = (n+1)\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, then condition (5.7) with $|d| = n + 1$ is necessary and sufficient for the boundedness of M_α^d from $LM_{p_1\theta_1,w_1,d}$ to $WLM_{p_2\theta_2,w_2,d}$.

Finally we note that Corollary 5.2 holds for all $1 \le p_1 \le p_2 < \infty$ if the space $LM_{p_2\theta_2,w_2,d}$ is replaced by the space $LWM_{p_2\theta_2,w_2,d}$. So $LM_{p_1\theta_1,w_1^*,d}$ is the maximal among spaces $LM_{p_1\theta_1,w_1,d}$ for which M_α^d is bounded from $LM_{p_1\theta_1,w_1,d}$ to $LWM_{p_2\theta_2,w_2,d}$.

Remark 6.1. All statements of Sections 4-6 in the isotropic case $d = \mathbf{1}$ were proved in [4].

7 Concluding remarks

The assumption made at the beginning of the paper $d_i \ge 1$, $i = 1, \dots, n$, is not essential. One may just assume that $d_i > 0$, $i = 1, \dots, n$. (Under this assumption the function $\rho(x-y)$, $x, y \in \mathbb{R}^n$ is in general a quasi-distance, not a distance, which however does note cause any problems.) This follows since for all $\nu > 0$

$$M_{\nu\alpha}^{\nu d} = M_\alpha^d,$$

$$\|f\|_{L_p(\mathcal{E}_d(0,r))} = \|f\|_{L_p(\mathcal{E}_{\nu d}(0,r^{1/\nu}))}$$

and the following statement holds.

Lemma 7.1. *Let $1 < p_1 \le p_2 < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$. Then for $\nu > 0$*

$$\|M^d_\alpha f\|_{LM_{p_1\theta_1,w_1,d} \to LM_{p_2\theta_2,w_2,d}} = \|M^{\nu d}_{\nu\alpha} f\|_{LM_{p_1\theta_1,w_1(\rho^\nu)\rho^{\frac{\nu-1}{\theta_1}},\nu d} \to LM_{p_2\theta_2,w_2(\rho^\nu)\rho^{\frac{\nu-1}{\theta_2}},\nu d}}.$$

Proof.

$$\|M^d_\alpha f\|_{LM_{p_1\theta_1,w_1,d} \to LM_{p_2\theta_2,w_2,d}} = \sup_{f \nsim 0, f \in LM_{p_1\theta_1,w_1,d}} \frac{\|M^d_\alpha f\|_{LM_{p_2\theta_2,w_2,d}}}{\|f\|_{LM_{p_1\theta_1,w_1,d}}}$$

$$= \sup_{f \nsim 0, f \in LM_{p_1\theta_1,w_1,d}} \frac{\|w_2(r)\|M^d_\alpha f\|_{L_p(\mathcal{E}_d(0,r))}\|_{L_{\theta_2}(0,\infty)}}{\|w_1(r)\|f\|_{L_p(\mathcal{E}_d(0,r))}\|_{L_{\theta_1}(0,\infty)}}$$

$$= \sup_{f \nsim 0, f \in LM_{p_1\theta_1,w_1,d}} \frac{\|w_2(r)\|M^{\nu d}_{\nu\alpha} f\|_{L_p(\mathcal{E}_{\nu d}(0,r^{1/\nu}))}\|_{L_{\theta_2}(0,\infty)}}{\|w_1(r)\|f\|_{L_p(\mathcal{E}_{\nu d}(0,r^{1/\nu}))}\|_{L_{\theta_1}(0,\infty)}}$$

$$= \nu^{1/\theta_2 - 1/\theta_1} \sup_{f \nsim 0, f \in LM_{p_1\theta_1,w_1,d}} \frac{\|w_2(\rho^\nu)\rho^{\frac{\nu-1}{\theta_2}}\|M^{\nu d}_{\nu\alpha} f\|_{L_p(\mathcal{E}_{\nu d}(0,\rho))}\|_{L_{\theta_2}(0,\infty)}}{\|w_1(\rho^\nu)\rho^{\frac{\nu-1}{\theta_1}}\|f\|_{L_p(\mathcal{E}_{\nu d}(0,\rho))}\|_{L_{\theta_1}(0,\infty)}}$$

$$= \|M^{\nu d}_{\nu\alpha} f\|_{LM_{p_1\theta_1,w_1(\rho^\nu)\rho^{\frac{\nu-1}{\theta_1}},\nu d} \to LM_{p_2\theta_2,w_2(\rho^\nu)\rho^{\frac{\nu-1}{\theta_2}},\nu d}}.$$

□

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