

SOBOLEV'S EMBEDDING THEOREM
FOR ANISOTROPICALLY IRREGULAR DOMAINS

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Abstract. We establish a Sobolev-type embedding theorem, namely, an embedding of the Sobolev space $W_p^s(G)$ in the Lebesgue space $L_q(G)$, for anisotropically irregular domains $G \subset \mathbb{R}^n$ of various classes.

1 Introduction

The well-known Sobolev's embedding theorem $W_p^m(G) \subset L_q(G)$, which is characterized by the inequality

$$\|f\|_{L_q(G)} \leq C \|f\|_{W_p^s(G)} = C \left(\sum_{|\alpha|=s} \|D^\alpha f\|_{L_p(G)} + \|f\|_{L_p(G)} \right), \quad (1.1)$$

$$s \in \mathbb{N}, \quad 1 < p < q < \infty,$$

was proved by Sobolev in 1938 (see [15]) for a domain $G \subset \mathbb{R}^n$ satisfying the cone condition and for

$$s - \frac{n}{p} + \frac{n}{q} \geq 0. \quad (1.2)$$

Relation (3.2) (which determines the maximal possible q in theorem (3.1)) is also a necessary condition for the embedding. Sobolev's result has been generalized to domains of more general form: $J_{\frac{n-1}{n}}$ domains, $I_{p, \frac{1}{p} - \frac{1}{n}}$ domains (V.G. Maz'ya, 1960, 1975, see [12]), John domains (Yu.G. Reshetnyak [14, 6]), and domains satisfying the flexible cone condition (O.V. Besov, 1983, see [4]).

Definition 1 ([2]). For $\sigma \geq 1$, a domain $G \subset \mathbb{R}^n$ is said to satisfy the *flexible σ -cone condition* if for some $T > 0$ and $0 < \kappa_0 \leq 1$ to any point $x \in G$ a piecewise smooth path

$$\gamma = \gamma_x: [0, T] \rightarrow G, \quad \gamma(0) = x, \quad |\gamma'| \leq 1 \quad \text{a.e.},$$

can be assigned such that

$$\text{dist}(\gamma(t), \mathbb{R}^n \setminus G) \geq \kappa_0 t^\sigma \quad \text{for } 0 < t \leq T.$$

In the case $\sigma = 1$ this condition is also known as just the flexible cone condition. The domains that do not satisfy the flexible cone condition will below be called irregular.

Note that in a neighborhood of a boundary point, an irregular domain may look like an exterior peak. For an irregular domain, embedding (3.1) may fail for any relation between the parameters, or may be valid only under more restrictive conditions on n , s , p , and q compared with (3.2), depending on the geometric properties of the domain G . V.G. Maz'ya introduced special classes of domains, I_α (1960) and $J_{p,\alpha}$ (1975), for which he proved the embedding theorem (3.1) with $p = 1$ and $p > 1$, respectively, in the case of $s = 1$ with a maximal possible q . The classes I_α and $J_{p,\alpha}$ are defined in terms of isoperimetric or capacity inequalities.

In [2] it is shown that for a domain satisfying the flexible σ -cone condition embedding (3.1) holds under the following relation between the parameters:

$$s - \frac{\sigma(n-1)+1}{p} + \frac{n}{q} \geq 0. \quad (1.3)$$

This result for $s = 1$ was earlier obtained by T. Kilpeläinen and J. Malý [8]. D.A. Labutin showed [11] that condition (3.3) is a necessary condition for this embedding as well.

In [8, 3, 2] one can also find weight generalizations of inequality (3.1) for a domain satisfying the flexible σ -cone condition.

For particular domains satisfying the σ -cone condition, the embedding theorem (3.1) is valid not only under condition (3.3) but also for a wider range of the parameters, as is shown by D.A. Labutin [10] and V.G. Maz'ya–S.V. Poborchii [13].

B.V. Trushin [17, 18] considered out special subclasses of the class of domains satisfying the flexible σ -cone condition and established Sobolev's embedding theorem under various sharp relations (different for different subclasses) between the parameters.

In the present paper we extend the class of domains considered in [17] while preserving the formulation of the Sobolev embedding theorem as in [17]; thus we generalize the results of B.V. Trushin. The method used in [2, 3, 17, 18] is based, in particular, on weak (1,1) type estimates for the Hardy–Littlewood maximal operator and for its anisotropic analogue. Here we exploit a generalization of the anisotropic Hardy–Littlewood maximal operator constructed with the use of a differential basis composed of rectangular parallelepipeds such that the edges of some of them are not necessarily parallel to the edges of others.

The scheme of the present paper is as follows. For a domain G of a certain type, we construct a family of flexible cones (contained in G) going from every point $x \in G$. Then we construct an integral representation of a function f in terms of its derivatives $D^\alpha f$, $|\alpha| = s$, over the flexible cone, which yields pointwise estimates for the function in terms of integrals containing such derivatives. This reduces deriving estimate (3.1) to obtaining estimates for the corresponding integral operators. Two main integral operators are first estimated in terms of the maximal operator constructed with the use of a differential basis corresponding to the family of flexible cones. The weak (1,1) boundedness of the maximal operator implies the weak (p, q) boundedness of the operators under consideration. By the Marcinkiewicz interpolation theorem, such

boundedness implies the (p, q) boundedness of these operators and, hence, the required estimate of the embedding theorem.

In view of the aforesaid, the implementation of this scheme for a domain G should begin with constructing two objects consistent with each other: a family of flexible cones and a differential basis. These objects are determined by the geometric characteristics of the domain G .

2 Besicovitch type coverings

Everywhere below \mathbb{N} is the set of positive integers; $n \in \mathbb{N}$, $n \geq 2$; \mathbb{R}^n is the n -dimensional Euclidean space; G is a domain of \mathbb{R}^n , $G \neq \mathbb{R}^n$; and χ is the characteristic function of the interval $(0, 1)$. For a Lebesgue measurable set $E \subset \mathbb{R}^n$, we will denote its Lebesgue measure by $|E|$.

For $1 \leq p < \infty$

$$\|f\|_{p,E} = \|f|L_p(E)\|.$$

Furthermore

$$\lambda = (\lambda_1, \dots, \lambda_n) \in [1, \infty)^n, \quad \min_{1 \leq i \leq n} \lambda_i = 1, \quad |\lambda| = \sum_{i=1}^n \lambda_i,$$

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n, \quad |x|_\lambda = \max_{1 \leq i \leq n} |x_i|^{1/\lambda_i}.$$

Note that

$$|x + y|_\lambda \leq |x|_\lambda + |y|_\lambda.$$

We shall extend some properties of Besicovitch type coverings of sets in \mathbb{R}^n (see [7, 16])—the properties well-known for coverings by balls or parallelepipeds with edges parallel to the coordinate axes—to the case of coverings by rotated rectangular parallelepipeds.

We assume that a family of rotation operators $\mathfrak{R}^0(x)$ is defined in \mathbb{R}^n (i.e., linear isometries with $\det \mathfrak{R}^0(x) = 1$); the operators depend on a parameter $x \in \mathbb{R}^n$. By $\mathfrak{R}(x)$ we denote the operator

$$\mathfrak{R}(x)y = x + \mathfrak{R}^0(x)(y - x) \quad \forall y \in \mathbb{R}^n,$$

which will be called a rotation operator around x corresponding to the operator $\mathfrak{R}^0(x)$. The operator $\mathfrak{R}^0(x)$ ($\mathfrak{R}(x)$) may also depend on an additional parameter $d > 0$, in which case it will be denoted by $\mathfrak{R}^0(x, d)$ ($\mathfrak{R}(x, d)$).

By $P(x)$ we denote a rectangular parallelepiped centered at x with edges that are not necessarily parallel to the coordinate axes. For $\theta > 0$, $\theta P(x)$ stands for a homothetic rectangular parallelepiped with the center of similarity at x and with the scaling coefficient θ .

For $d > 0$ we set

$$Q_\lambda(x, d) := x + \prod_{i=1}^n [-d^{\lambda_i}, d^{\lambda_i}].$$

Sets of the form

$$\mathfrak{R}(x, d)Q_\lambda(x, d) \tag{2.1}$$

will be called λ -parallelepipeds centered at x .

Definition 2.1. Let E be a bounded set in \mathbb{R}^n , and let each point $x \in E$ be a center of some rectangular parallelepiped $P(x)$ and $\sup\{\text{diam } P(x) : x \in E\} < \infty$. Let $0 < \theta < 1 < \kappa < \infty$.

We say that the covering $\{P(x)\}_{x \in E}$ is θ -separable if $(\theta P(x)) \cap (\theta P(y)) = \emptyset$ whenever

$$x, y \in E, \quad P(x) \cap P(y) \neq \emptyset, \quad y \notin P(x), \quad |P(y)| \leq 2|P(x)|.$$

We say that the covering $\{P(x)\}_{x \in E}$ is κ -engulfing if $P(y) \subset \kappa P(x)$ whenever

$$x, y \in E, \quad P(x) \cap P(y) \neq \emptyset, \quad y \notin P(x), \quad |P(y)| \leq 2|P(x)|.$$

Theorem 2.1. Let $\kappa > 1$ and let $\{P(x)\}_{x \in E}$ be a κ -engulfing covering of a bounded set E in \mathbb{R}^n .

Then from this covering one can extract a finite or countable sequence of rectangular parallelepipeds $\{P_k\} = \{P(x^{(k)})\}$ that satisfy the following conditions for $\theta = \frac{1}{1+2\kappa}$:

- (i) $E \subset \bigcup P_k$;
- (ii) $(\theta P_k) \cap (\theta P_m) = \emptyset \quad \forall k, m, \quad k \neq m$.

Lemma 2.1. Let $\{P(x)\}_{x \in E}$ be a θ -separable covering of a bounded set $E \subset \mathbb{R}^n$, $\theta \in (0, 1)$. Then from this covering one can extract a finite or countable sequence of rectangular parallelepipeds $\{P_k\} = \{P(x^{(k)})\}$ that satisfy conditions (i) and (ii).

Proof. The proof of the lemma follows the standard scheme developed for the case of coverings by cubes (see [16]). Set $a_0 = \sup\{|P(x)| : x \in E\}$ and choose a rectangular parallelepiped $P_1 = P(x^{(1)}) \in \{P(x)\}_{x \in E}$ such that $|P(x)| > \frac{a_0}{2}$. Suppose that P_1, \dots, P_m have already been chosen. If $E \setminus \bigcup_{k=1}^m P_k = \emptyset$, then the selection process is complete. Otherwise set

$$a_m = \sup \left\{ |P(x)| : x \in E \setminus \bigcup_{k=1}^m P_k \right\}$$

and choose a rectangular parallelepiped

$$P_{m+1} = P(x^{(m+1)}) \in \left\{ P(x) : x \in E \setminus \bigcup_{k=1}^m P_k \right\}$$

such that $|P_{m+1}| > \frac{a_m}{2}$.

Let us show that condition (i) holds: $E \subset \bigcup P_k$. If the selection process terminates after finite number of steps, then condition (i) holds trivially. If $\{P_k\}$ is an infinite sequence, then $|P_k| \rightarrow 0$ as $k \rightarrow \infty$. Indeed, $\{P(x)\}_{x \in E}$ is a θ -separable covering, so $\sum |\theta P_k| < \infty$. If there were a point $x \in E \setminus \bigcup_{k=1}^{\infty} P_k$, then $|P_{m+1}| < \frac{1}{2}|P(x)|$ for some $m+1$, which would contradict the choice of $x^{(m+1)}$. This proves (i). Property (ii) follows by the θ -separability of the covering $\{P(x)\}_{x \in E}$ and by the way in which the sequence $\{P_k\}$ was constructed. \square

Proof of Theorem 2.1. It suffices to show that if the covering $\{P(x)\}_{x \in E}$ is κ -engulfing, then it is θ -separable for $\theta = \frac{1}{1+2\kappa}$, and then apply Lemma 1.1. Let $x, y \in E$, and let $P(x)$ and $P(y)$ be two rectangular parallelepipeds such that

$$P(x) \cap P(y) \neq \emptyset, \quad y \notin P(x), \quad |P(y)| \leq 2|P(x)|, \quad \kappa P(x) \supset P(y).$$

Let us demonstrate that $(\theta P(x)) \cap (\theta P(y)) = \emptyset$ for $\theta = \frac{1}{1+2\kappa}$. It suffices to consider the case of $x = 0$ and $P(x) = \prod_{i=1}^n [-a_i, a_i]$. Without loss of generality, we can assume that $y_1 > a_1$. Let

$$\theta \in (0, 1), \quad (\theta P(x)) \cap (\theta P(y)) \neq \emptyset, \quad z \in (\theta P(x)) \cap (\theta P(y)).$$

Since $P(y) \subset \kappa P(x)$, we have $|z_1 - y_1| \leq 2\kappa\theta a_1$. As $z \in \theta P(x)$, it follows that $|z_1 - y_1| > (1 - \theta)a_1$. These two inequalities imply $2\kappa\theta a_1 > (1 - \theta)a_1$, so $\theta > \frac{1}{1+2\kappa}$. Therefore, $(\theta P(x)) \cap (\theta P(y)) = \emptyset$ for $\theta = \frac{1}{1+2\kappa}$. \square

An example of a κ -engulfing covering of a (bounded) set $E \subset \mathbb{R}^n$ is the covering of E by balls or parallelepipeds of the form (2.1) with $\mathfrak{R}(x, d(x)) = \text{Id}$, $d(x) \in (0, d_0)$. A more general example of a κ -engulfing covering is given by a covering by comparable rectangular parallelepipeds (see [7, §1, remark (5)]). Below we shall only deal with coverings by λ -parallelepipeds (2.1) with fixed $\lambda = (\lambda_1, \dots, \lambda_n) \in [1, \infty)^n$, $\min \lambda_i = 1$.

Let $\lambda \in [1, \infty)^n$, $0 < d_0 < \infty$, and G be an open set in \mathbb{R}^n . For every $x \in G$, by $\mathcal{B}_\lambda(x)$ we denote a family of λ -parallelepipeds $\mathfrak{R}(x, d)Q_\lambda(x, d)$ of the form (2.1) with $0 < d < d_0 < \infty$. The union of these families, $\mathcal{B}_\lambda = \bigcup_{x \in G} \mathcal{B}_\lambda(x)$, will be called a differential basis on the set G (see [7]). For $\kappa > 1$ a differential basis \mathcal{B}_λ is said to be κ -engulfing if

$$\kappa \mathfrak{R}(x, d_1)Q_\lambda(y, d_1) \supset \mathfrak{R}(y, d_2)Q_\lambda(y, d_2)$$

whenever

$$\begin{aligned} x, y \in G, \quad \mathfrak{R}(x, d_1)Q_\lambda(x, d_1) \cap \mathfrak{R}(y, d_2)Q_\lambda(y, d_2) \neq \emptyset, \\ y \notin \mathfrak{R}(x, d_1)Q_\lambda(x, d_1), \quad |Q_\lambda(y, d_2)| \leq 2|Q_\lambda(x, d_1)|. \end{aligned}$$

Now we introduce the maximal operator constructed with the use of \mathcal{B}_λ on the set G for $f \in L(\overline{G}, \text{loc})$:

$$\mathfrak{M}_{\lambda, \mathfrak{R}} f(x) := \sup_{0 < d < d_0} \frac{1}{|\mathfrak{R}(x, d)Q_\lambda(x, d) \cap G|} \int_{\mathfrak{R}(x, d)Q_\lambda(x, d) \cap G} |f(y)| dy.$$

Theorem 2.2. *Suppose that a differential basis \mathcal{B}_λ on the set G is κ -engulfing for some $\kappa > 1$. Then the inequality*

$$|\{x \in G : \mathfrak{M}_{\lambda, \mathfrak{R}} f(x) > \tau\}| \leq \frac{C}{\tau} \|f\|_{L_1(G)} \quad (2.2)$$

holds for each $\tau > 0$, with C independent from f and τ .

The proof of this theorem is similar to the proof of the respective part of Theorem 1 in [16, §1].

Let us give, for $n = 2$, an example of a domain $G \in \mathbb{R}^2$ and a related κ -engulfing differential basis of the form $\{\mathfrak{R}(x, d)Q_\lambda(x, d)\}_{x \in G, 0 < d < d_0}$. Let $\lambda = (2, 1)$ and $Q_\lambda(x, d) = x + [-d^2, d^2] \times [-d, d]$. First, we describe a domain $G \subset \mathbb{R}^n$ and construct a family of (truncated) flexible cones on it (later these cones will serve as the supports of an integral representation of functions in terms of their derivatives). Then we construct a differential basis consistent with these flexible cones. We begin with some auxiliary constructions.

Let $k \in (0, 1)$ and $R \in (\frac{1}{2}, 1)$. We consider the curve

$$\hat{\gamma}(k, R) = \hat{\gamma}^1(k, R) \cup \hat{\gamma}^2(k, R),$$

where $\hat{\gamma}^1(k, R)$ is an arc of the radius R circle centered at

$$(x_{k,R}, y_{k,R}) = \left(\frac{1-R}{\sqrt{1+k^2}}, \frac{k(1-R)}{\sqrt{1+k^2}} \right)$$

that belongs to the angle $\{(x, y) : x > x_{k,R}, y_{k,R} < y < kx\}$, and $\hat{\gamma}^2$ is the vertical ray going from the lowest point of the arc $\hat{\gamma}^1(k, R)$ downwards. Analytically,

$$\hat{\gamma}^1(k, R) = \{(x, y) : (x - x_{k,R})^2 + (y - y_{k,R})^2 = R^2, y_{k,R} \leq y < kx\},$$

$$\hat{\gamma}^2(k, R) = \{(x, y) : x = x_{k,R} + R, y \leq y_{k,R}\}.$$

Note that the ray $\hat{\gamma}^2(k, R)$ is tangent to the arc $\hat{\gamma}^1(k, R)$ at $(x_{k,R} + R, y_{k,R})$. Set

$$G_k = \bigcup_{\frac{1}{2} < R < 1} \hat{\gamma}(k, R) \subset \left\{ (x, y) : \frac{1}{\sqrt{1+k^2}} < x < 1 \right\}$$

(see Fig. 1); G_k lies in a vertical strip of width less than $\frac{1}{2}k^2$. Let $\mathbb{R}_-^2 := \{(x, y) : y < 0\}$, $e^1 = (1, 0)$. Consider the domain

$$G := \left(\bigcup_{j=1}^{\infty} (G_{k_j} - 2^{-j}e^1) \right) \cup \mathbb{R}_-^2,$$

where $2^{-j-1} \leq k_j^2 < 2^{-j}$.

On G we choose the following family of curves

$$\hat{\gamma}_j(R) := \hat{\gamma}(k_j, R) - 2^{-j}e^1, \quad \text{with } j \in \mathbb{N}, \quad \frac{1}{2} < R < 1,$$

and we construct a differential basis on G using these curves. The basis consists of all rectangles of the form

$$\begin{aligned} \mathfrak{R}((x, y), d)Q_\lambda((x, y), d) &= \mathfrak{R}((x, y), d)((x, y) + [-d^2, d^2] \times [-d, d]), \\ (x, y) &\in G, \quad 0 < d < d_0 < 1, \end{aligned}$$

arranged as follows. The greater side of each rectangle is either vertical or deviates from the vertical by an angle less than $\frac{\pi}{4}$. If the point (x, y) lies on the curve $\hat{\gamma}_j(R)$

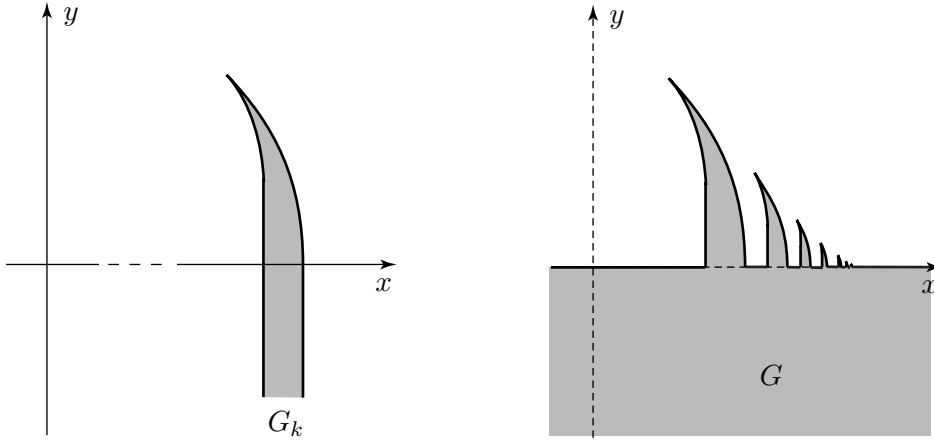


Figure 1: Domain G_k (left) and domain G (right)

and $y \geq 0$, then the midpoint of the lower smaller side of the rectangle also lies on this curve and its ordinate is less than y . Thus, the major semiaxis of the rectangle $\mathfrak{R}((x, y), d)Q_\lambda((x, y), d)$ that lies below its center is a chord of the curve $\hat{\gamma}_j(R)$. If $y < 0$, then $\mathfrak{R}((x, y), d) = \text{Id}$. Geometrically, it is clear (this can also be verified analytically) that the differential basis constructed is κ -engulfing for some $\kappa > 1$.

Let us now build a family of (truncated) flexible cones for representations of functions.

Let $(x_0, y_0) \in G$. We begin with defining the axis of the cone with vertex at (x_0, y_0) . Suppose first that $(x_0, y_0) \in \mathbb{R}_-^2$. Then we take the curve

$$\gamma(x_0, y_0) := \{(x_0, y) : y = y_0 - t, 0 \leq t \leq T\}$$

as the axis of the cone and $\bigcup_{0 < t < T} Q_\lambda((x_0, y_0 - t), \frac{t}{2})$ as the truncated flexible representation cone.

If $(x_0, y_0) \in G \setminus \mathbb{R}_-^2$, then, as the axis of the cone, we take the curve

$$\gamma(x_0, y_0) = \gamma_0(x_0, y_0) \cup \hat{\gamma}_j(x_j, y_0)$$

starting at (x_0, y_0) , where $\gamma_0(x_0, y_0)$ is the horizontal line segment connecting the point (x_0, y_0) to the point (x_j, y_0) of the curve $\hat{\gamma}_j(\frac{3}{4})$, and $\hat{\gamma}_j(x_j, y_0)$ is an arc of finite length of the curve $\hat{\gamma}_j(\frac{3}{4})$ that starts at (x_j, y_0) and lies below this point. We parameterize the curve $\gamma(x_0, y_0)$ by a parameter t as follows: on $\gamma_0(x_0, y_0)$, t is such that t^2 is equal to the distance from the point (x_0, y_0) ($0 \leq t \leq t(x_0, y_0)$), while on $\hat{\gamma}_j(x_j, y_0)$ we set $t = t(x_0, y_0) + u$, u being the arclength of $\hat{\gamma}_j(x_j, y_0)$ calculated from the point (x_j, y_0) in the direction of decreasing ordinate y of the curve $\hat{\gamma}_j(x_j, y_0)$, with $0 \leq u \leq T - t(x_0, y_0)$ (such a special parameterization is needed to satisfy the conditions of the definitions of the next section).

For brevity, the point on $\gamma(x_0, y_0)$ corresponding to the parameter value t will be denoted by $\gamma(t)$, $0 \leq t \leq T$. Let $Q_\lambda^{(t)}$ be the rectangle Q_λ rotated through an angle “sufficiently close” to the angle of deviation of the tangent to γ (on its nonhorizontal

part) from the vertical. Here we currently do not completely specify the rotation operator $\mathfrak{R}_t: Q_\lambda(\gamma(t)) \rightarrow Q_\lambda^{(t)}(\gamma(t))$ (on the horizontal segment the rotation angle is zero). We define the truncated flexible representation cone as

$$\bigcup_{0 < t \leq T} Q_\lambda^{(t)}(\gamma(t), r(x_0, y_0) + \varepsilon t),$$

where $r(x_0, y_0) > 0$ and $\varepsilon > 0$ are so small that

$$\bigcup_{0 < t \leq T} Q_\lambda^{(t)}(\gamma(t), 2(r(x_0, y_0) + \varepsilon t)) \subset G.$$

3 The class of domains and the main theorem

Definition 3.1. Let G_0 and G be open sets in \mathbb{R}^n , $G_0 \subset G$, $\lambda = (\lambda_1, \dots, \lambda_n) \in [1, \infty)^n$, $\min_{1 \leq i \leq n} \lambda_i = 1$, $d_0 \in (0, \infty)$, and $\kappa > 1$.

Suppose that G_0 is equipped with a κ -engulfing differential basis which also satisfies the following monotonicity property:

$$\mathfrak{R}(x, h)Q_\lambda(x, h) \subset \mathfrak{R}(x, d)Q_\lambda(x, d) \quad \text{for } h < d.$$

Suppose that to each $x \in G_0$ there are assigned a piecewise smooth path $\gamma = \gamma_x: [0, t_x] \rightarrow G$, $\gamma(0) = x$, a continuous piecewise smooth function $r = r_\gamma: [0, t_x] \rightarrow (0, \infty)$, a family of rotation operators $\mathfrak{R}_t = \mathfrak{R}_t(\gamma(t))$, and a family of λ -parallelepipeds $\{\mathfrak{R}_t Q_\lambda(\gamma(t), r_\gamma(t))\}_{0 \leq t \leq t_x}$ accompanying γ , with the following properties:

1°. The truncated flexible cone $\bigcup_{0 \leq t \leq t_x} (2\mathfrak{R}_t Q_\lambda(\gamma(t), r_\gamma(t)))$ belongs to G .

2°. $\exists \varepsilon_0 \in (0, 1): r(t_x) \geq \varepsilon_0 \forall x \in G_0$.

3°. $\gamma_x(t) \in \mathfrak{R}(x, t)Q_\lambda(x, t) \forall t \in [0, t_x]$.

4°. The matrix of the transformation $\mathfrak{R}_t = \mathfrak{R}_t(\gamma(t))$ is continuous and piecewise continuously differentiable with respect to t . The t -derivatives of r , γ , and the entries of the matrix of the transformation \mathfrak{R}_t are bounded by a number independent of x and t , and $|\gamma'| \leq 1$.

5°. $\exists \varepsilon_1 > 0: \mathfrak{R}_t Q_t(\gamma(t), r_\gamma(t)) \not\subset \mathfrak{R}(x, d)Q_\lambda(x, d) \Rightarrow r_\gamma(0) + t \geq \varepsilon_1 d$.

6°. $\exists C_0 > 0: \int \frac{1}{r_\gamma(t)} \chi\left(\frac{|\mathfrak{R}_t^{-1}y - \gamma(t)|_\lambda}{r_\gamma(t)}\right) dt \leq C_0 \forall x \in G_0, \forall y \in G$.

Then we shall write $G_0 \in \mathcal{G}(G, \lambda)$.

Definition 3.2. If $G = \bigcup_1^{k_0} G_k$, $G_k \in \mathcal{G}(G, \lambda^k)$, and $\Lambda := \max_{1 \leq k \leq k_0} |\lambda^k|$, then we write $G \in \mathcal{G}(\Lambda)$.

Definition 3.3. Let $\lambda \in [1, \infty)^n$, $\min \lambda_i = 1$, and $\sigma \geq 1$. We say that a set G_0 satisfies the λ -anisotropic flexible σ -cone condition with respect to G , which is written as $G_0 \in \mathcal{G}(G, \lambda, \sigma)$, if $G_0 \in \mathcal{G}(G, \lambda)$ and for every $x \in G_0$ the function r_γ in the definition of the class $\mathcal{G}(G, \lambda)$ satisfies the estimate $r_\gamma(t) \geq c_0 t^\sigma$ for $t \in (0, t_x]$, where $c_0 > 0$ does not depend on $x \in G_0$.

Definition 3.4. If $G = \bigcup_{k=1}^{k_0} G_k$, $G_k \in \mathcal{G}(G, \lambda^k, \sigma)$ for $k = 1, \dots, k_0$, and $\Lambda = \max |\lambda^k|$, then we say that the set G satisfies the flexible (Λ, σ) -cone condition and write $G \in \mathcal{G}(\Lambda, \sigma)$.

For $\delta > 0$ we denote $G_\delta := \{x \in G: \text{dist}(x, \partial G) > \delta\}$.

Theorem 3.1. Let $\sigma \geq 1$, $G \in \mathcal{G}(\Lambda, \sigma)$, $1 < p < q < \infty$, and

$$s - \frac{\sigma(\Lambda - 1) + 1}{p} + \frac{\Lambda}{q} \geq 0.$$

Then for any $f \in W_p^s(G)$ the following estimate holds:

$$\|f|L_q(G)\| \leq C \left(\sum_{|\alpha|=s} \|D^\alpha f|L_p(G)\| + \|f|L_p(G_\delta)\| \right), \quad \delta = \varepsilon_0^\Lambda, \quad (3.1)$$

where the constant C does not depend on f .

If, in addition, $s - \frac{\sigma(\Lambda-1)+1}{p} + \frac{\Lambda}{q} > 0$, then the statement is valid for $1 \leq p < q < \infty$.

The proof of the theorem is based on all subsequent considerations and is given at the end of the paper.

4 Integral representation of functions and pointwise estimates

When deriving the integral representation of a function, we will assume that it is infinitely differentiable on its domain of definition, which is an open set.

Let $x \in G_0$, $\Gamma = (\Gamma_1, \dots, \Gamma_n): [0, t_x] \rightarrow G$ and $\bar{r} = (r_1, \dots, r_n): [0, t_x] \rightarrow (0, \infty)^n$ be continuous piecewise smooth functions such that $|\Gamma'| \leq 1$, $r_i(0) = 0$, and $0 < r_i(t) \leq t$ for $t > 0$ ($i = 1, \dots, n$), and the rotation operator $\mathfrak{R}_t = \mathfrak{R}_t(\Gamma(t))$ around the point $\Gamma(t)$ is a continuous piecewise smooth (with respect to t) transformation. Set $\mathfrak{R}_t^0(y) := \mathfrak{R}_t(\Gamma(t) + y) - \Gamma(t)$. Let $\{e_i\}_1^n$ be the standard basis in \mathbb{R}^n , $\mathfrak{R}_t^0 e_i = \sum_{j=1}^n a_{ij}(t) e_j$, and

$$\mathfrak{R}_t^0 y = \mathfrak{R}_t^0 \left(\sum_{i=1}^n y_i e_i \right) = \sum_{i=1}^n y_i \mathfrak{R}_t^0 e_i = \sum_{i=1}^n \sum_{j=1}^n y_i a_{ij}(t) e_j = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}(t) y_i \right) e_j.$$

Let

$$\omega \in C_0^\infty(\mathbb{R}^1), \quad \text{supp } \omega \in [0, 1], \quad \int \omega(t) dt = 1, \quad \Omega(y) = \prod_{i=1}^n \omega(y_i).$$

Set

$$f_t(x) = \int \prod_{i=1}^n \frac{1}{r_i(t)} \omega\left(\frac{y_i}{r_i(t)}\right) f(\Gamma(t) + \mathfrak{R}_t^0 y) dy = \int \Omega(y) f(\Gamma(t) + \mathfrak{R}_t^0(\bar{r}(t)y)) dy, \quad (4.1)$$

where $\bar{r}(y) := (r_1(t), \dots, r_n(t))$. Note that $f_t(x) \rightarrow f(x)$ as $t \rightarrow 0$, and

$$\frac{\partial}{\partial t} f_t(x) = \int \Omega(y) \sum_{j=1}^n D_j f(\Gamma(t) + \mathfrak{R}_t^0(\bar{r}(t)y)) \left\{ \Gamma'_j(t) + \sum_{i=1}^n [a'_{ij}(t) r_i(t) y_i + a_{ij}(t) r'_i(t) y_i] \right\} dy$$

$$= \int \prod_{i=1}^n \frac{1}{r_i(t)} \omega\left(\frac{y_i}{r_i(t)}\right) \sum_{j=1}^n D_j f(\Gamma(t) + \mathfrak{R}_t^0 y) \left\{ \Gamma'_j(t) + \sum_{i=1}^n \left[a'_{ij}(t) y_i + a_{ij}(t) \frac{r'_i(t)}{r_i(t)} y_i \right] \right\} dy.$$

Hence

$$\left| \frac{\partial}{\partial t} f_t(x) \right| \leq C \int \prod_{i=1}^n \frac{1}{r_i(t)} \left| \omega\left(\frac{y_i}{r_i(t)}\right) \right| \sum_{j=1}^n |D_j f(\Gamma(t) + \mathfrak{R}_t^0 y)| dy. \quad (4.2)$$

By the Newton–Leibniz formula,

$$|f(x)| \leq C \int_0^{t_x} \prod_{i=1}^n r_i(t)^{-1} \int_{0 \leq y_i \leq r_i(t), i=1, \dots, n} \sum_{j=1}^n |D_j f(\Gamma(t) + \mathfrak{R}_t^0 y)| dy dt + |f_{t_x}(x)|. \quad (4.3)$$

Below we restrict ourselves to the case of $\bar{r}(t) = (r(t)^{\lambda_1}, \dots, r(t)^{\lambda_n})$ with fixed $\lambda = (\lambda_1, \dots, \lambda_n) \in [1, \infty)^n$, $\min \lambda_i = 1$.

Lemma 4.1. *Let a domain $G \subset \mathbb{R}^n$, $R > 0$, $\lambda \in [0, \infty)^n$, $\min \lambda_i = 1$, $x \in G$ and $\Gamma: [0, t_x] \rightarrow G$ be a piecewise smooth path with $\Gamma(0) = x$. Moreover, let $r: [0, t_x] \rightarrow [0, \infty)$ be a continuous piecewise smooth function such that $r(0) = 0$ and $r(t) > 0$ for $t > 0$. Suppose that $\mathfrak{R}_t Q_\lambda(\Gamma(t), r(t)) \subset G$, $|r'(t)| \leq C_1$, $|\Gamma'(t)| \leq 1$ for a.e. $t \in [0, t_x]$, the coefficients a_{ij} of the matrix of the transformation \mathfrak{R}_t^0 are continuous piecewise smooth functions of t , and $|a'_{ij}| \leq C_2$.*

Then

$$\begin{aligned} |f(x)| &\leq C \int_0^{t_x} t^{s-1} r(t)^{-|\lambda|} \int_{|y| < r(t)} \sum_{|\alpha|=s} |D^\alpha f(\Gamma(t) + \mathfrak{R}_t^0 y)| dy dt + \\ &\quad + C \int_{|y|_\lambda < r(t_x)} |f(\Gamma(t_x) + \mathfrak{R}_{t_x}^0 y)| dy, \end{aligned} \quad (4.4)$$

where $C = C(C_1, C_2)$ is independent of f and $x \in G$.

Proof. First, let us prove that under the hypothesis of the lemma

$$|f(x)| \leq C \int_0^{t_x} t^{s-1} r(t)^{-|\lambda|} \int_{|y| < r(t)} \sum_{|\alpha|=s} |D^\alpha f(\Gamma(t) + \mathfrak{R}_t^0 y)| dy dt + C \sum_{|\beta| \leq s-1} |(D_\beta f)_{t_x}(x)|, \quad (4.5)$$

where C is independent of f and $x \in G$.

Note that for $s = 1$ estimate (4.5) coincides with (4.3). Suppose that for some $s \geq 2$ estimate (4.5) is valid with s replaced by $s - 1$. Let us prove that in this case it is valid in the form (4.5). Fix t and y , and hence \mathfrak{R}_t^0 , in the integrand ($y \in Q_\lambda^{(t)}(\Gamma(t), r(t))$). For $y \neq \Gamma(t)$ we construct a path $\Gamma_t: [0, t_x - t + r(t)] \rightarrow G$ and a vector function $\bar{\rho} = (\rho^{\lambda_1}, \dots, \rho^{\lambda_n})$, where $\rho: [0, t_x - t + r(t)] \rightarrow [0, \infty)$.

Set $u_* = |y - \Gamma(t)|_\lambda$, $u^* := t_x - t + r(t)$, and

$$\Gamma_t(u) = \begin{cases} y + \mathfrak{R}_t^0 \left[\left(\frac{u}{u_*} \right)^\lambda (\mathfrak{R}_t^0)^{-1} (y - \Gamma(t)) \right], & 0 \leq u \leq u_*, \\ \Gamma(t), & u_* \leq u \leq r(t), \\ \Gamma(t + u - r(t)), & r(t) \leq u \leq u^*, \end{cases} \quad (4.6)$$

$$\rho(u) = \begin{cases} u & \text{for } 0 < u < r(t), \\ r(t + u - r(t)) & \text{for } r(t) \leq u \leq u^*. \end{cases} \quad (4.7)$$

Estimate (4.3) for $y \in Q_\lambda^{(t)}(\Gamma(t), r(t))$, the path Γ_t , the function $D^\beta f$ instead of f with $|\beta| = s - 1$, and the vector function ρ yields

$$\begin{aligned} |D^\beta f(y)| &\leq C \int_0^{u_*} u^{-|\lambda|} \int_{|z|_\lambda < u} \sum_{|\alpha|=s} |D^\alpha f(\Gamma_t(u) + \mathfrak{R}_t^0 z)| dz du \\ &+ C \int_{u_*}^{r(t)} u^{-|\lambda|} \int_{|z|_\lambda < u} \sum_{|\alpha|=s} |D^\alpha f(\Gamma(t) + \mathfrak{R}_t^0 z)| dz du \\ &+ C \int_{r(t)}^{u^*} (r(t - r(t) + u))^{-|\lambda|} \int_{|z|_\lambda \leq r(t - r(t) + u)} \sum_{|\alpha|=s} |D^\alpha f(\Gamma(t - r(t) + u) + \mathfrak{R}_{t-r(t)+u}^0 z)| dz du \\ &+ |(D^\beta f)_{t_x}(x)|. \end{aligned} \quad (4.8)$$

The first of the integrals over z in (4.8) is

$$I_1 = \int_{|(\mathfrak{R}_t^0)^{-1}(z - \Gamma_t(u))|_\lambda < u, |(\mathfrak{R}_t^0)^{-1}(z - \Gamma(t))|_\lambda < r(t)} \sum_{|\alpha|=s} |D^\alpha f(z)| dz.$$

We have

$$(\mathfrak{R}_t^0)^{-1}(z - \Gamma_t(u)) = (\mathfrak{R}_t^0)^{-1} \left(z - y - \left(\frac{u}{u_*} \right)^\lambda (y - \Gamma(t)) \right),$$

hence inequality $|(\mathfrak{R}_t^0)^{-1}(z - \Gamma_t(u))|_\lambda < u$ implies the inequality

$$|(\mathfrak{R}_t^0)^{-1}(z - y)|_\lambda \leq \left| \left(\frac{u}{u_*} \right)^\lambda (\mathfrak{R}_t^0)^{-1}(y - \Gamma(t)) \right|_\lambda + u \leq \frac{u}{u_*} |(\mathfrak{R}_t^0)^{-1}y - \Gamma(t)|_\lambda + u \leq 2u,$$

which gives

$$I_1 \leq \int_{|(\mathfrak{R}_t^0)^{-1}(z - y)|_\lambda \leq 2u, |(\mathfrak{R}_t^0)^{-1}(z - \Gamma(t))|_\lambda < r(t)} \sum_{|\alpha|=s} |D^\alpha f(z)| dz.$$

The second of the integrals over z in (4.8) is

$$I_2 = \int_{|(\mathfrak{R}_t^0)^{-1}(z - \Gamma(t))|_\lambda < u} \sum_{|\alpha|=s} |D^\alpha f(z)| dz, \quad u \in [u_*, r(t)].$$

We have $(\mathfrak{R}_t^0)^{-1}(z - \Gamma(t)) = (\mathfrak{R}_t^0)^{-1}(z - y) + (\mathfrak{R}_t^0)^{-1}(y - \Gamma(t))$, which implies that

$$|(\mathfrak{R}_t^0)^{-1}(z - y)|_\lambda \leq u + |(\mathfrak{R}_t^0)^{-1}(y - \Gamma(t))|_\lambda = u + u_* \leq 2u$$

for $|(\mathfrak{R}_t^0)^{-1}(z - \Gamma(t))|_\lambda < u$. Therefore,

$$I_2 \leq \int_{|(\mathfrak{R}_t^0)^{-1}(z-y)|_\lambda \leq 2u, |(\mathfrak{R}_t^0)^{-1}(z-\Gamma(t))|_\lambda < r(t)} \sum_{|\alpha|=s} |D^\alpha f(z)| dz.$$

Consequently, it follows by (4.8) that

$$\begin{aligned} |D^\beta f(\mathfrak{R}_t^0 y)| &\leq C \int_0^{r(t)} u^{-|\lambda|} \int_{|z-y|_\lambda \leq 2u, |z-\Gamma(t)|_\lambda < r(t)} \sum_{|\alpha|=s} |D^\alpha f(\mathfrak{R}_t^0 z)| dz du \\ &+ C \int_t^{t_x} r(u)^{-|\lambda|} \int_{|z|_\lambda < r(u)} \sum_{|\alpha|=s} |D^\alpha f(\Gamma(u) + \mathfrak{R}_u^0 z)| dz du + |(D_\beta f)_{t_x}(x)|. \end{aligned}$$

For $|y - \Gamma(t)|_\lambda < r(t)$ this gives

$$\begin{aligned} |D^\beta f(\mathfrak{R}_t^0 y)| &\leq C_1 \int_{|z-y|_\lambda \leq 2r(t), |z-\Gamma(t)|_\lambda < r(t)} |z-y|_\lambda^{1-|\lambda|} \sum_{|\alpha|=s} |D^\alpha f(\mathfrak{R}_t^0 z)| dz \\ &+ C \int_t^{t_x} r(u)^{-|\lambda|} \int_{|z|_\lambda < r(u)} \sum_{|\alpha|=s} |D^\alpha f(\Gamma(u) + \mathfrak{R}_u^0 z)| dz du + |(D^\beta f)_{t_x}(x)|. \end{aligned} \tag{4.9}$$

Integrating over $y \in \{y: |y - \Gamma(t)|_\lambda < r(t)\} \subset \{y: |y - z|_\lambda \leq 2r(t)\}$ and taking into account that

$$\int_{|y-\Gamma(t)|_\lambda < r(t)} |z-y|_\lambda^{1-|\lambda|} dy \leq \int_{|w|_\lambda \leq 2r(t)} |w|_\lambda^{1-|\lambda|} dw \leq C_2 r(t),$$

we obtain

$$\begin{aligned} \int_{|y-\Gamma(t)|_\lambda < r(t)} |D^\beta f(\mathfrak{R}_t^0 y)| dy &\leq C_3 r(t) \int_{|z-\Gamma(t)|_\lambda < r(t)} \sum_{|\alpha|=s} |D^\alpha f(\mathfrak{R}_t^0 z)| dz \\ &+ Cr(t)^{|\lambda|} \int_t^{t_x} r(u)^{-|\lambda|} \int_{|z|_\lambda < r(u)} \sum_{|\alpha|=s} |D^\alpha f(\Gamma(u) + \mathfrak{R}_u^0 z)| dz du + r(t)^{|\lambda|} |(D^\beta f)_{t_x}(x)|. \end{aligned}$$

Substituting this estimate in inequality (4.5) with s replaced by $s - 1$, we arrive at the

estimate

$$\begin{aligned}
|f(x)| &\leq C_4 \int_0^{t_x} t^{s-2} r(t)^{1-|\lambda|} \int_{|z|_\lambda < r(t)} \sum_{|\alpha|=s} |D^\alpha f(\Gamma(t) + \mathfrak{R}_t^0 z)| dz dt \\
&+ C_4 r(t)^{|\lambda|} \int_0^{t_x} t^{s-2} \int_t^{t_x} r(u)^{-|\lambda|} \int_{|z|_\lambda < r(u)} \sum_{|\alpha|=s} |D^\alpha f(\Gamma(u) + \mathfrak{R}_u^0 z)| dz du dt \\
&+ C \sum_{|\beta| \leq s-1} |(D^\beta f)_{t_x}(x)|. \tag{4.10}
\end{aligned}$$

Changing the order of integration in the second term and evaluating the integral over t , in view of the boundedness of $r(t)$ we obtain (4.5).

Let us estimate the $|D^\beta|$ terms, $|\beta| \leq s-1$, in the right-hand side of (4.5). Using the definition of f_t and integrating by parts for $0 < |\beta| \leq s-1$, we find

$$|(D^\beta f)_{t_x}(x)| \leq c_\beta \int_{|y|_\lambda < r(t_x)} |f(\Gamma(t_x) + \mathfrak{R}_t^0 y)| dy. \tag{4.11}$$

Now (4.10) and (4.11) imply (4.4). \square

Lemma 4.2. *Let G_0 and G be open sets in \mathbb{R}^n , $G_0 \in G$, $\lambda = (\lambda_1, \dots, \lambda_n) \in [1, \infty)$, and $\min \lambda_i = 1$.*

Suppose that to each $x \in G_0$ there are assigned a piecewise smooth path $\gamma = \gamma_x: [0, t_x] \rightarrow G$, $\gamma(0) = x$, a continuous piecewise smooth function $r = r_\gamma: [0, t_x] \rightarrow (0, \infty)$, and a family of λ -parallelepipeds $\{\mathfrak{R}_t Q_\lambda(\gamma(t), r(t))\}_{0 \leq t \leq t_x}$ accompanying γ with properties 1°, 2°, and 4° of Definition 3.1.

Then, for $f \in C^\infty(G)$ and $x \in G_0$, the following estimate holds:

$$|f(x)| \leq CA_1 \left(\sum_{|\alpha|=s} |D^\alpha f| \right) (x) + CA_2 \left(\sum_{|\alpha|=s} |D^\alpha f| \right) (x) + CA_3 f(x), \tag{4.12}$$

where

$$A_1 g(x) = \int_0^{r_\gamma(0)} t^{s-1-|\lambda|} \int_{|y|_\lambda < t} g(x + \mathfrak{R}_0^0 y) dy dt, \tag{4.13}$$

$$A_2 g(x) = \int_0^{t_x} (t + r_\gamma(0))^{s-1} r_\gamma(t)^{-|\lambda|} \int_{|y|_\lambda < r_\gamma(t_x)} |g(\gamma(t) + \mathfrak{R}_t^0 y)| dy dt, \tag{4.14}$$

$$A_3 f(x) = \int_{|y| < r_\gamma(t_x)} |f(\gamma(t) + \mathfrak{R}_{t_x}^0 y)| dy, \tag{4.15}$$

with a constant C independent of f , x , γ , and r_γ .

Proof. Given a path γ and a function $r = r_\gamma$, we construct the path Γ :

$$\Gamma(t) = \begin{cases} \gamma(0) & \text{for } 0 \leq t \leq r(0), \\ \gamma(t - r(0)) & \text{for } r(0) \leq t \leq t_x + r(0). \end{cases}$$

With the path Γ , we associate the piecewise smooth function r_Γ :

$$r_\Gamma(t) = \begin{cases} t & \text{for } 0 \leq t \leq r(0), \\ r(t - r(0)) & \text{for } r(0) \leq t \leq t_x + r(0) \end{cases}$$

and the rotation operator

$$\mathfrak{R}_t(\Gamma(t)) = \begin{cases} \mathfrak{R}_0(\gamma(0)) & \text{for } 0 < t \leq r(0), \\ \mathfrak{R}_{t-r(0)}(\gamma(t - r(0))) & \text{for } r(0) \leq t \leq t_x + r(0). \end{cases}$$

Replacing Γ , r_Γ , and $\mathfrak{R}_t(\Gamma(t))$ in (4.4) with their expressions in terms of γ , r , and $\mathfrak{R}_t(\gamma(t))$, we obtain the required assertion. \square

5 Some estimates for integral operators

Let G_0 and G be open sets in \mathbb{R}^n such that $G_0 \subset G$. Consider the operator

$$Kf(x) = \int_G k(x, y) f(y) dy, \quad x \in G_0, \quad (5.1)$$

where $k: G_0 \times G \rightarrow \mathbb{R}$ is a measurable nonnegative function.

Introduce

$$k(x, y, d) := \left(1 - \chi \left(\frac{|(\mathfrak{R}^0(x, d))^{-1}(y - x)|_\lambda}{d} \right) \right) k(x, y) \quad \text{for } x \in G_0, \ y \in G, \ d > 0,$$

$$\| \| k \| \|_{p,q} := \sup_{x \in G_0, 0 < d < \infty} \| k(x, \cdot, d) \|_{L_{p'}(G)} \| |\mathfrak{R}Q_\lambda(x, d)|^\frac{1}{q}.$$

Lemma 5.1. *Let $1 \leq p < q < \infty$ and K be integral operator (5.1) with the kernel k . Then*

$$|Kf(x)| \leq 4 \left(\frac{1}{p} - \frac{1}{q} \right)^{-\frac{p}{q}} \| \| k \| \|_{p,q} \| f \|_{L_p(G)}^{1-\frac{p}{q}} \mathfrak{M}_{\lambda, \mathfrak{R}}(|f|^p)(x)^\frac{1}{q} \quad (5.2)$$

for $x \in G_0$.

This lemma extends the nonweighted results of V.M. Kokilashvili, M.A. Gabidzashvili [9, 5] and B.V. Trushin [17] to the case of Besicovitch type coverings and the corresponding maximal operator used in the present paper.

Proof. We can assume that $\|f\|_{L_p(G)} > 0$ and that the right-hand side of (5.2) is finite. To simplify the notation, set $\mathfrak{R}Q_\lambda(x, d) := \mathfrak{R}(x, d)Q_\lambda(x, d)$. Consider the sequence $\{d_i\}_0^\infty : d_0 = d, |Q_\lambda(x, d_i)| = 2^{-i}|Q_\lambda(x, d)|$. We represent $Kf(x)$ as

$$Kf(x) = \int_{G \setminus \mathfrak{R}Q_\lambda(x, d)} k(x, y)f(y) dy + \sum_{i=0}^{\infty} \int_{\mathfrak{R}Q_\lambda(x, d_i) \setminus \mathfrak{R}Q_\lambda(x, d_{i+1})} k(x, y)f(y) dy.$$

Applying Hölder's inequality with the exponents p and p' to each term on the right-hand side, we obtain

$$\begin{aligned} |Kf(x)| &\leq \| \|k\|_{p,q} |Q_\lambda(x, d)|^{-\frac{1}{q}} \|f\|_{L_p(G \setminus \mathfrak{R}Q_\lambda(x, d))} \| \\ &\quad + \sum_{i=0}^{\infty} \| \|k\|_{p,q} |Q_\lambda(x, d_i)|^{-\frac{1}{q}} \|f\|_{L_p(\mathfrak{R}Q_\lambda(x, d_i) \setminus \mathfrak{R}Q_\lambda(x, d_{i+1}))} \| \\ &\leq \| \|k\|_{p,q} |Q_\lambda(x, d)|^{-\frac{1}{q}} \\ &\quad \times \left[\|f\|_{L_p(G \setminus \mathfrak{R}Q_\lambda(x, d))} + 2^{\frac{1}{p}} (2^{\frac{1}{p} - \frac{1}{q}} - 1)^{-1} (\mathfrak{M}_{\lambda, \mathfrak{R}}(|f|^p)(x))^{\frac{1}{p}} |Q_\lambda(x, d)|^{\frac{1}{p}} \right]. \end{aligned}$$

In view of the monotonicity and continuity with respect to d , it is clear that for some d the two terms in the square brackets are equal to each other. Denote their common value by κ . Raising the first term to the power $\frac{1}{p} - \frac{1}{q}$ and the second term to the power $\frac{1}{q}$ and multiplying them, we arrive at

$$2\kappa = 2^{1+\frac{1}{q}} \|f\|_{L_p(G \setminus \mathfrak{R}Q_\lambda(x, d))}^{1-\frac{p}{q}} (2^{\frac{1}{p} - \frac{1}{q}})^{\frac{p}{q}} (\mathfrak{M}_{\lambda, \mathfrak{R}}(|f|^p)(x))^{\frac{1}{q}} |Q_\lambda(x, d)|^{\frac{1}{q}},$$

which implies (5.2). \square

Lemma 5.2. *Let $1 \leq p < q < \infty$ and K be the operator (5.1) with kernel k . If $\| \|k\|_{p,q} < \infty$, then the operator K is of weak (p, q) type.*

The proof follows by estimates (5.1) and (2.2).

6 Estimates for the operators A_1 , A_2 , and A_3 and the proof of Theorem 3.1

Suppose that $G \in \mathcal{G}(\Lambda, \sigma)$, hence $G = \bigcup_{k=1}^{k_0} G_k$ with $G_k \in \mathcal{G}(G, \lambda^k, \sigma)$ for $k = 1, \dots, k_0$. Take an arbitrary set G_k ($1 \leq k \leq k_0$) and denote it by G_0 and λ^k by λ . Consider the operators $A_i : L_p(G) \rightarrow L_q(G_0)$ in (4.13)–(4.15) ($i = 1, 2, 3$).

First we estimate A_3 . Since

$$|x - (\gamma(t_x) + \mathfrak{R}_{t_x} y)| \leq |x - \gamma(t_x)| + |\mathfrak{R}_{t_x} y| \leq R_0 + |\mathfrak{R}_{t_x y}|_\lambda \leq R_0 + C_0$$

for $\gamma = \gamma_x$, we have

$$|A_3 f(x)| \leq \bar{A}_3 f(x) := \int_{G_\delta} \chi \left(\frac{y-x}{R_0 + C_0} \right) |f(y)| dy.$$

Applying the Young inequality, we obtain

$$\|A_3 f|_{L_q(G_0)}\| \leq C \|f|_{L_p(G_\delta)}\| \quad (6.1)$$

for $1 \leq p < q < \infty$.

Let us estimate $A_i f$, $i = 1, 2$. We write $A_i f$ in the form

$$A_i f(x) = \int_G k_i(x, y) f(y) dy, \quad x \in G_0, \quad i = 1, 2.$$

We estimate $\|k_1\|_{p,q}$ assuming that $s - \frac{|\lambda|}{p} + \frac{|\lambda|}{q} \geq 0$. Recall that $\mathfrak{R}_0 = \mathfrak{R}(x, r_\gamma(0))$. We have

$$\begin{aligned} |k_1(x, y, d)| &= \chi(d|\mathfrak{R}^{-1}(x, d)(y-x)|_\lambda^{-1}) \int_0^{r_\gamma(0)} t^{s-1-|\lambda|} \chi\left(\frac{|\mathfrak{R}_0^{-1}(y-x)|_\lambda}{t}\right) dt \\ &\leq C_1 \chi(d|\mathfrak{R}^{-1}(x, d)(y-x)|_\lambda^{-1}) \chi(r_\gamma(0)^{-1}|\mathfrak{R}^{-1}(x, r_\gamma(0))(y-x)|_\lambda) \int_0^{r_\gamma(0)} t^{\frac{|\lambda|}{p} - \frac{|\lambda|}{q} - 1 - |\lambda|} dt. \end{aligned}$$

Hence

$$\begin{aligned} \|k_1(x, \cdot, d)|_{L_{p'}(G)}\|^{p'} &\leq C_1 \int_{d < |\mathfrak{R}^{-1}(x, r_\gamma(0))(y-x)|_\lambda < r_\gamma(0)} |\mathfrak{R}^{-1}(x, r_\gamma(0))(y-x)|_\lambda^{\left(\frac{|\lambda|}{p} - \frac{|\lambda|}{q} - |\lambda|\right)p'} dy \\ &= C_1 \int_{d < |y|_\lambda < r_\gamma(0)} |y|_\lambda^{\left(\frac{|\lambda|}{p} - \frac{|\lambda|}{q} - |\lambda|\right)p'} dy \leq C_2 d^{-\frac{|\lambda|}{q}p'}, \end{aligned}$$

so

$$\|k_1\|_{p,q} \leq C_3. \quad (6.2)$$

For the kernel k_2 under the conditions $1 \leq p < \infty$ and $s - \frac{\sigma(|\lambda|-1)+1}{p} + \frac{|\lambda|}{q} \geq 0$ we have

$$|k_2(x, y, d)| = \chi(d|\mathfrak{R}^{-1}(x, d)(y-x)|_\lambda^{-1}) \int_0^{t_x} (t + r_\gamma(0))^{s-1} r_\gamma(t)^{-|\lambda|} \chi\left(\frac{|\mathfrak{R}_t^{-1}y - \gamma(t)|_\lambda}{r_\gamma(t)}\right) dt.$$

Applying (if $p > 1$) Hölder's inequality, we obtain

$$\begin{aligned} |k_2(x, y, d)|^{p'} &\leq \chi(d|\mathfrak{R}^{-1}(x, d)(y-x)|_\lambda^{-1}) \\ &\quad \times \int_0^{t_x} \chi(r_\gamma(t)^{-1}|\mathfrak{R}_t y - \gamma(t)|_\lambda) (t + r_\gamma(0))^{(s-1)p'} r_\gamma(t)^{\left(\frac{1}{p} - |\lambda|\right)p'} dt \\ &\quad \times \left(\int_0^{t_x} \frac{1}{r_\gamma(t)} \chi(r_\gamma(t)^{-1}|\mathfrak{R}_t^{-1}y - \gamma(t)|_\lambda) dt \right)^{\frac{p'}{p}}. \end{aligned}$$

In view of property 6° in Section 3, we find

$$\begin{aligned} & \|k_2(x, \cdot, d)|_{L_{p'}(G)}\|^{p'} \\ & \leq C_1^{p'} \int_{G \setminus \mathfrak{R}Q_\lambda(x, d)} \int_0^{t_x} \chi(r_\gamma(t)^{-1}|\mathfrak{R}_t^{-1}(y - \gamma(t))|_\lambda) (t + r_\gamma(0))^{(s-1)p'} r_\gamma(t)^{\left(\frac{1}{p} - |\lambda|\right)p'} dt dy. \end{aligned}$$

Property 5° in Section 3 implies that if

$$y \notin \mathfrak{R}(x, d)Q_\lambda(x, d), \quad y \in \mathfrak{R}_t Q_\lambda(\gamma(t), r_\gamma(t)),$$

then $\varepsilon_1 d \leq r_\gamma(0) + t$. Therefore,

$$\begin{aligned} \|k_2(x, \cdot, d)|_{L_{p'}(G)}\|^{p'} & \leq C_1 \int_0^{t_x} \chi\left(\frac{d}{c_2 \max\{r_\gamma(0), t\}}\right) (t + r_\gamma(0))^{(s-1)p'} r_\gamma(t)^{\left(\frac{1}{p} - |\lambda|\right)p' + |\lambda|} dt \\ & = C_1 \int_0^{y_x} \chi\left(\frac{d}{c_2 \max\{r_\gamma(0), t\}}\right) (t + r_\gamma(0))^{(s-1)p'} r_\gamma(t)^{(1-|\lambda|)\frac{p'}{p}} dt. \end{aligned}$$

Then, assuming that $t_\gamma(t) \geq ct^\sigma$, $\sigma \geq 1$, we obtain

$$\| \|k_2\| \|_{p, q}^{p'} \leq C_2 \sup_{x \in G_0, 0 < d \leq d_0} (I_1^{p'}(x, d) + I_2^{p'}(x, d)),$$

where (for $\tau_x = \min\{t_x, r_\gamma(0)\}$)

$$\begin{aligned} I_1(x, d)^{p'} & = d^{\frac{|\lambda|}{q}p'} \int_0^{\tau_x} \chi\left(\frac{d}{c_3 r_\gamma(0)}\right) r(0)^{\left(s-1 + \frac{1-|\lambda|}{p}\right)p'} dt \leq C_4 r_\gamma(0)^{\frac{|\lambda|}{q}p' + \left(s-1 + \frac{1-|\lambda|}{p}\right)p' + 1} \\ & \leq C_5 r_\gamma(0)^{\left[s - |\lambda|\left(\frac{1}{p} - \frac{1}{q}\right)\right]p'} \leq C_6, \end{aligned}$$

$$I_2(x, d)^{p'} = d^{\frac{|\lambda|}{q}p'} \int_{\tau_x}^{t_x} \chi\left(\frac{d}{c_4 t}\right) t^{\left[s - \frac{\sigma(|\lambda|-1)+1}{p}\right]p' - 1} dt \leq C_7 d^{\frac{|\lambda|}{q}p'} \int_{\tau_x}^{t_x} \chi\left(\frac{d}{c_4 t}\right) t^{-\frac{|\lambda|}{q}p' - 1} dt \leq C_8.$$

Combining the results yields

$$\| \|k_2\| \|_{p, q} < \infty \quad \text{for} \quad s - \frac{\sigma(|\lambda| - 1) + 1}{p} + \frac{|\lambda|}{q} \geq 0. \quad (6.3)$$

Lemma 6.1. 1°. Let $s \in \mathbb{N}$, $\lambda \in [1, \infty)^n$, $\min \lambda_i = 1$, $1 \leq p < q < \infty$ and $s - \frac{|\lambda|}{p} + \frac{|\lambda|}{q} \geq 0$.

Then the operator A_1 is of weak (p, q) type.

If, in addition, $p > 1$ or $s - \frac{|\lambda|}{p} + \frac{|\lambda|}{q} > 0$, then the operator A_1 is of strong (p, q) type.

2°. Let $s \in \mathbb{N}$, $\lambda \in [1, \infty)^n$, $\min \lambda_i = 1$, $1 \leq p < q < \infty$, $\sigma \geq 1$, $s - \frac{\sigma(|\lambda|-1)+1}{p} + \frac{|\lambda|}{q} \geq 0$, and $G_0 \in \mathcal{G}(G, \lambda, \sigma)$. Suppose the operator A_2 is constructed with the use of γ and r_γ satisfying requirements 1°–6° in Definition 3.1.

Then the operator $A_2: L_p(G) \rightarrow L_q(G_0)$ is of weak (p, q) type.

If, in addition, $p > 1$ or $s - \frac{\sigma(|\lambda|-1)+1}{p} + \frac{|\lambda|}{q} > 0$, then the operator A_2 is of strong (p, q) type.

Proof. In view of Lemma 5.2, it follows by estimates (6.2) and (6.3) that each of the operators A_1 and A_2 is of weak (p, q) type. Applying the Marcinkiewicz interpolation theorem, we obtain the required strong (p, q) -type assertions. \square

Proof of Theorem 3.1. Let a domain $G \in \mathcal{G}(\Lambda, \sigma)$. Then $G = \bigcup_{k=1}^{k_0} G_k$ with $G_k \in \mathcal{G}(G, \lambda^k, \sigma)$. Let $f \in C^\infty(G)$. For each $k = 1, \dots, k_0$ and every $x \in G_k$ estimate (4.11) holds. By virtue of (6.1) and Lemma 6.1, the operators $A_i: L_p(G) \cap C^\infty(G) \rightarrow L_q(G_k)$ ($i = 1, 2, 3$) are bounded. Therefore, for $1 \leq k \leq k_0$ and $f \in W_p^s(G) \cap C^\infty(G)$ we have the estimate

$$\|f|_{L_q(G_k)}\| \leq C \left(\sum_{|\alpha|=s} \|D^\alpha f|_{L_p(G)}\| + \|f|_{L_p(G_\delta)}\| \right),$$

which implies estimate (3.1) for $f \in C^\infty(G)$. Since $C^\infty(G)$ is dense in $W_p^s(G)$, estimate (3.1) is valid for arbitrary functions f for which the right-hand side of (3.1) is finite. \square

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