EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 2, Number 1 (2011), 5 – 31

OPTIMAL EMBEDDINGS OF GENERALIZED BESOV SPACES

Z. Bashir, G.E. Karadzhov[1]

Communicated by M.L. Goldman

Key words: Besov spaces, optimal embeddings, rearrangement invariant function spaces, real interpolation.

AMS Mathematics Subject Classification: 46E30, 46E35.

Abstract. We prove optimal embeddings of generalized Besov spaces built-up over rearrangement invariant function spaces defined on $\mathbb{R}^n$ with the Lebesgue measure into other rearrangement invariant spaces in the subcritical or critical cases and into generalized Hölder-Zygmund spaces in the supercritical case. The investigation is based on some real interpolation techniques and estimates of the rearrangement of f in terms of the modulus of continuity of f.

1 Introduction

To highlight the key issues around this paper, let us start with some background material.

1.1 Background

Let L_{loc} be the space of all locally integrable functions f on $\mathbb{R}^n$ with the Lebesgue measure. Denote by $\mathbf{M}^+$ the space of all locally integrable functions $g \geq 0$ on $(0, \infty)$ with the Lebesgue measure.

Let ρ_F be a quasi-norm, defined on $\mathbf{M}^+$ with values in $[0, \infty]$, which is monotone in the sense that $g_1 \leq g_2$ implies $\rho_F(g_1) \leq \rho_F(g_2)$. Denote by F the quasi-normed space, consisting of all locally integrable functions in $(0, \infty)$ with the Lebesgue measure such that $\|g\|_F := \rho_F(|g|) < \infty$.

There is an equivalent quasi-norm ρ_p, called a $p-$norm, that satisfies the triangle inequality $\rho_p^p(g_1 + g_2) \leq \rho_p^p(g_1) + \rho_p^p(g_2)$ for some $p \in (0, 1]$ that depends only on the space F (see [22]).

We say that the quasi-norm ρ_F satisfies Minkowski's inequality if for the equivalent quasi-norm ρ_p,

$$\rho_p^p \left(\sum g_j \right) \lesssim \sum \rho_p^p(g_j), \; g_j \in \mathbf{M}^+. \tag{1.1}$$

[1]Research of the second author was partially supported by the Abdus Salam School of Mathematical Sciences, GC University Lahore as well as by a grant from HEC, Pakistan.

Let $h_F(u)$ be the dilation function generated by ρ_F

$$h_F(u) = \sup\left\{\frac{\rho_F(g_u)}{\rho_F(g)} : g \in L_m\right\},\ g_u(t) := g(tu),$$

where

$$L_m := \{g \in \mathbf{M}^+,\ t^m g(t) \text{ is increasing }\}, m > 2.$$

The function $u^m h_F(u)$ is increasing, submultiplicative and

$$h_F(1) = 1,\ h_F(u)h_F\left(\frac{1}{u}\right) \geq 1.$$

We suppose that it is finite. Therefore if α_F and β_F are the Boyd indices of F:

$$\alpha_F := \sup_{0<t<1} \frac{\log h_F(t)}{\log t} \text{ and } \beta_F := \inf_{1<t<\infty} \frac{\log h_F(t)}{\log t},$$

then $-m \leq \alpha_F \leq \beta_F$. We suppose that $\alpha_F = \beta_F$.

Let φ be a quasi-concave function in $\mathbf{M}^+$. This means that φ is non-decreasing and $\varphi(t)/t$ is non-increasing. Let $\varphi(\infty) = \infty$. Define the dilation function h_φ, generated by φ:

$$h_\varphi(u) = \sup_{0<t<\infty} \frac{\varphi(tu)}{\varphi(t)}.$$

Then h_φ is quasi-concave, submultiplicative and

$$h_\varphi(1) = 1,\ 1 \leq h_\varphi(u)h_\varphi\left(\frac{1}{u}\right),\ h_\varphi(u) \leq \max(1, u).$$

Therefore the lower and upper Boyd indices α_φ, β_φ, defined by

$$\alpha_\varphi := \sup_{0<t<1} \frac{\log h_\varphi(t)}{\log t} \text{ and } \beta_\varphi := \inf_{1<t<\infty} \frac{\log h_\varphi(t)}{\log t},$$

satisfy $0 \leq \alpha_\varphi \leq \beta_\varphi \leq 1$. We suppose that $\alpha_\varphi = \beta_\varphi > 0$. Then $\varphi(+0) = 0$.

Using the monotonicity of h_F and h_φ, we see that for any $p > 0$ (cf. [3], p. 147)

$$\int_0^1 h_\varphi^p(u)h_F^p(u)\frac{du}{u} < \infty \text{ if } \alpha_\varphi + \alpha_F > 0; \tag{1.2}$$

$$\int_1^\infty h_\varphi^p(u)h_F^p(u)u^{-pk/n}\frac{du}{u} < \infty \text{ if } \alpha_\varphi + \alpha_F < k/n. \tag{1.3}$$

We shall also consider rearrangement invariant quasi-normed spaces G with a monotone quasi-norm $\|f\|_G = \rho_G(f^*)$, $f \in L_{loc}$, $f^*(\infty) = 0$, f^* being the decreasing rearrangement of f, given by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\},\ t > 0,$$

where μ_f is the distribution function of f, defined by

$$\mu_f(\lambda) = |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|_n\,,$$

$|\cdot|_n$ denoting the Lebesgue $n-$measure. Let

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(u)du.$$

The lower and upper Boyd indices of G are defined similarly to [3]. Let $h_G(u)$ be the dilation function generated by ρ_G

$$h_G(u) = \sup\left\{\frac{\rho_G(g_u^*)}{\rho_G(g^*)} : g \in \mathbf{M}^+\right\},\ g_u(t) := g\left(\frac{t}{u}\right).$$

The function h_G is increasing, submultiplicative,

$$h_G(1) = 1, \quad h_G(u)h_G\left(\frac{1}{u}\right) \geq 1.$$

Therefore, if α_G and β_G are the Boyd indices of G:

$$\alpha_G := \sup_{0<t<1} \frac{\log h_G(t)}{\log t} \text{ and } \beta_G := \inf_{1<t<\infty} \frac{\log h_G(t)}{\log t},$$

then $0 \leq \alpha_G \leq \beta_G$. We shall suppose that $\alpha_G = \beta_G \leq 1$.

Recall that w is slowly varying on $(1,\infty)$ (in the sense of Karamata), if for all $\varepsilon > 0$ the function $t^\varepsilon w(t)$ is equivalent to a non-decreasing function, and the function $t^{-\varepsilon}w(t)$ is equivalent to a non-increasing function. By symmetry, we say that w is slowly varying on $(0,1)$ if the function $t \mapsto w(\frac{1}{t})$ is slowly varying on $(1,\infty)$. Finally, w is slowly varying if it is slowly varying on $(0,1)$ and $(1,\infty)$.

We use the notation $a_1 \lesssim a_2$ or $a_2 \gtrsim a_1$ for nonnegative functions or functionals to mean that the quotient a_1/a_2 is bounded above; also, $a_1 \approx a_2$ means that $a_1 \lesssim a_2$ and $a_1 \gtrsim a_2$. We say that a_1 is equivalent to a_2 if $a_1 \approx a_2$.

1.2 Basic definitions and main results

The classical homogeneous Besov spaces $b^s_{r,q}$, $0 < s < k$, $1 \leq r < \infty$, $0 < q \leq \infty$, are defined by finiteness of the quasi-norms

$$\|f\|_{b^s_{r,q}} = \left(\int_0^\infty [t^{-s}\omega_r^k(t,f)]^q \frac{dt}{t}\right)^{1/q},$$

where $\omega_r^k(t,f) := \sup_{|h|\leq t} \|\Delta_h^k f\|_{L^r}$ is the standard modulus of continuity and L^r is the Lebesgue space on $\mathbb{R}^n$. The following embedding is well known:

$$b^s_{r,q} \hookrightarrow L^{u,q},\ 1/u = 1/r - s/n > 0,$$

where $L^{u,q}$ is the Lorentz space [4]. We can replace the base space L^r in the definition of the Besov spaces by the Lorentz space $L^{r,v}$ and define more general homogeneous Besov spaces $b^s_q L^{r,v}$, $1 \leq v \leq \infty$. Then by interpolation,

$$b^s_q L^{r,v} = (L^{r,v}, w^k L^{r,v})_{s/k,q},$$

where $w^k L^{r,v}$ is the homogeneous Sobolev space. Let $k < n/r$. Then $w^k L^{r,v} \hookrightarrow L^{r_1,v}$, $1/r_1 = 1/r - k/n$, hence

$$b^s_q L^{r,v} \hookrightarrow L^{u,q},\ 1/u = 1/r - s/n > 0,$$

We prove below that $L^{u,q}$ is the optimal rearrangement invariant target space. Observe that it does not depend on $v \in [1,\infty]$, but only on the fundamental function of the base space $L^{r,v}$, which is $t^{1/r}$.

For the inhomogeneous Besov spaces $B^s_q L^{r,v} := b^s_q L^{r,v} \cap L^{r,v}$ with the usual quasi-norm, we clearly have the embedding

$$B^s_q L^{r,v} \hookrightarrow L^{u,q} \cap L^{r,v},\ 1/u = 1/r - s/n > 0$$

and in [15], [16], [13] it is proved that this is the optimal rearrangement invariant target space.

The above discussion suggests to define the generalized homogeneous Besov spaces replacing L^r as a base space by an arbitrary rearrangement invariant Banach function space on $\mathbb{R}^n$ with a fundamental function $\varphi_E \approx \varphi$. Then

$$\Lambda_\varphi \hookrightarrow E \hookrightarrow M_\varphi,$$

where M_φ is the Marcinkievicz space with a norm

$$\|f\|_{M_\varphi} := \sup_{0<t<\infty} f^{**}(t)\varphi(t)$$

and Λ_φ is the Lorentz space with a norm

$$\|f\|_{\Lambda_\varphi} := \int_0^\infty f^*(t)d\varphi(t) = \int_0^\infty f^*(t)\varphi'(t)dt.$$

Here we suppose that φ is concave and $\varphi(+0) = 0$.

Definition 1.1 (Besov spaces)**.** Let E be a rearrangement invariant Banach function space on $\mathbf{R}^n$ as in [23], with a fundamental function $\varphi_E \approx \varphi$. We denote by $b^k(E,F)$ the generalized homogeneous Besov space, consisting of all functions $f \in L_{loc}$, $f^*(\infty) = 0$, such that

$$\|f\|_{b^k(E,F)} := \rho_F\left(\omega^k_E(t^{1/n}, f)\right) < \infty,$$

where $\omega^k_E(t,f) = \sup_{|h|\le t} \|\Delta^k_h f\|_E$ is the modulus of continuity of $f \in L_{loc}$ of order k and Δ^k_h is the difference operator with step h of order k.

The corresponding generalized inhomogeneous Besov space $B^k(E,F)$ has the quasi-norm

$$\|f\|_{B^k(E,F)} := \rho_F\left(\omega^k_E(t^{1/n}, f)\right) + \|f\|_E.$$

Under the following conditions the generalized Besov spaces contain C_0^∞,

$$\rho_F\left(\chi_{(0,1)}(t)t^{k/n}\right)<\infty,\ \rho_F(\chi_{(a,\infty)})<\infty,\ 0<a<1, \tag{1.4}$$

where $\chi_{(a,b)}$ stands for the characteristic function of the interval (a,b).

Then

$$\|f\|_{B^k(E,F)}\approx\rho_F\left(\chi_{(0,1)}(t)\omega_E^k(t^{1/n},f)\right)+\|f\|_E \tag{1.5}$$

We suppose that the following condition is satisfied

$$0\le\alpha_F\le k/n. \tag{1.6}$$

We can take $F=L_*^q(b(t)t^{-s/n})$, where b is slowly varying and $L_*^q(w)$, or simply L_*^q if $w=1$, is the weighted Lebesgue space with the quasi-norm

$$\|g\|_{L_*^q(w)}=\left(\int_0^\infty[w(t)|g(t)|]^q\frac{dt}{t}\right)^{1/q},\ 0<q\le\infty,\ w>0,\ w\in\mathbf{M}^+.$$

Then $\alpha_F=\beta_F=s/n$ and (1.6) means that $0\le s\le k$. For this reason we call the cases $\alpha_F=0$ or $\alpha_F=k/n$ limiting. Since $b^k(E,F)$ is the $K-$interpolation between E and the homogeneous Sobolev space w^kE, the limiting case $\alpha_F=0$ means that $b^k(E,F)$ is "logarithmically close" to E, while in the limiting case $\alpha_F=k/n$ the space $b^k(E,F)$ is "logarithmically close" to w^kE. If $E=L^r$, $1\le r\le\infty$, then we get the classical Besov spaces $b_{r,q}^s=b^k(L^r,L_*^q(t^{-s/n}))$ and $B_{r,q}^s$ if $0<s<k$. It is well-known that the embedding properties of these spaces depend on the conditions: $s<n/r$ (subcritical case), $s=n/r$ (critical case) and $s>n/r$ (supercritical case). Therefore first we extend these definitions for the generalized Besov spaces.

Definition 1.2. A case is said to be *subcritical, critical, supercritical* provided that $\alpha_F<\alpha_\varphi$, $\alpha_F=\alpha_\varphi$, $\alpha_F>\alpha_\varphi$ respectively.

The main goal of this paper is to prove optimal embeddings of the Besov space $b^k(E,F)$, $\alpha_F<\alpha_\varphi$, into rearrangement invariant quasi-normed spaces G. This is the subcritical case.

In the supercritical case $\alpha_F>\alpha_\varphi$ we prove optimal embeddings of the Besov spaces $B^k(E,F)$ into the generalized Hölder-Zygmund spaces C^kH (cf. [33]) with the quasi-norm $\|f\|_{C^kH}:=\|f\|_{L^\infty}+\rho_H(\omega^k(t^{1/n},f))$, where ρ_H is a monotone quasi-norm and

$$\omega^k(t,f):=\sup_{|h|\le t}\sup_{x\in\mathbf{R}^n}|\Delta_h^kf(x)|.$$

We write $\omega(t,f)$ instead of $\omega^1(t,f)$. We suppose that

$$\rho_H\left(\chi_{(0,1)}(t)\int_0^t\frac{u^{k/n}}{\varphi(u)}\frac{du}{u}\right)<\infty\text{ and }\rho_H\left(\chi_{(a,\infty)}\right)<\infty,\ 0<a<1. \tag{1.7}$$

Then

$$\|f\|_{C^kH}\approx\rho_H\left(\chi_{(0,1)}(t)\omega^k(t^{1/n},f)\right)+\|f\|_{L^\infty} \tag{1.8}$$

Let $h_H(u)$ be the dilation function generated by ρ_H

$$h_H(u) = \sup \left\{ \frac{\rho_H(g_u)}{\rho_H(g)} : g \in L_m \right\},\ g_u(t) := g(tu).$$

The function $u^m h_H(u)$ is increasing, submultiplicative and

$$h_H(1) = 1,\ h_H(u) h_H\left(\frac{1}{u}\right) \geq 1.$$

We suppose that h_H is finite. Therefore if α_H and β_H are the Boyd indices of H:

$$\alpha_H := \sup_{0<t<1} \frac{\log h_H(t)}{\log t} \text{ and } \beta_H := \inf_{1<t<\infty} \frac{\log h_H(t)}{\log t},$$

then $-m \leq \alpha_H \leq \beta_H$. We suppose that $\alpha_H = \beta_H$.

The spaces in the critical case $\alpha_F = \alpha_\varphi$ can be divided into two subclasses: in the first subclass the functions may not be continuous - then the respective space $b^k(E, F)$ is embedded in a rearrangement invariant space of type G, while the functions in the second subclass are continuous and the corresponding space $B^k(E, F)$ is embedded in a Hölder-Zygmund space. The separating space for these two subclasses is given by $F = L^1_*(1/\varphi)$ (cf. Theorem 2.1).

Definition 1.3 (admissible couple - non-supercritical case)**.** We say that a couple ρ_F, ρ_G is admissible for the Besov spaces $b^k(E, F)$ if the following continuous embedding is valid:

$$b^k(E, F) \hookrightarrow G. \tag{1.9}$$

Moreover, ρ_F (F) is called the domain quasi-norm (domain space), and ρ_G (G) is called the target quasi-norm (target space).

For example, by Theorem 2.1 below, the couple $F = L^q_*(w\varphi)$, $G = \Lambda^q_0(v)$, $1 \leq q \leq \infty$, is admissible if v is related to w by the Muckenhoupt condition [30]:

$$\left(\int_0^t [v(s)]^q \frac{ds}{s}\right)^{1/q} \left(\int_t^\infty [w(s)]^{-r} \frac{ds}{s}\right)^{1/r} \lesssim 1,\ 1/q + 1/r = 1.$$

The space $\Lambda^q(w)$, $0 < q \leq \infty$ is the Lorentz space with the quasi-norm $\|g\|_{\Lambda^q(w)} = \|g^*\|_{L^q_*(w)}$, $w(2t) \approx w(t)$ and $\Lambda^q_0(w) = \{f \in \Lambda^q(w);\ f^*(\infty) = 0\}$.

Definition 1.4 (admissible couple - supercritical case)**.** We say that a couple ρ_F, ρ_H is admissible for the Besov spaces $B^k(E, F)$ if the following continuous embedding is valid:

$$B^k(E, F) \hookrightarrow C^k H. \tag{1.10}$$

Moreover, ρ_F (F) is called the domain quasi-norm (domain space), and ρ_H (H) is called target the quasi-norm (target space).

Definition 1.5 (optimal target quasi-norm)**.** Given a domain quasi-norm ρ_F, the optimal target quasi-norm, denoted $\rho_{G(F)}$, is the strongest target quasi-norm, i.e.

$$\rho_G(g^*) \lesssim \rho_{G(F)}(g^*),\ g \in \mathbf{M}^+ \tag{1.11}$$

for any target quasi-norm ρ_G such that the couple ρ_F, ρ_G is admissible.

Definition 1.6 (optimal domain quasi-norm)**.** Given a target quasi-norm ρ_G, the optimal domain quasi-norm, denoted by $\rho_{F(G)}$, is the weakest domain quasi-norm, i.e.

$$\rho_{F(G)}(g) \lesssim \rho_F(g),\ g \in L_m, \tag{1.12}$$

for any domain quasi-norm ρ_F such that the couple ρ_F, ρ_G is admissible.

Definition 1.7 (optimal couple)**.** An admissible couple ρ_F, ρ_G is said to be optimal if $\rho_F = \rho_{F(G)}$ and $\rho_G = \rho_{G(F)}$.

In the supercritical case the definitions of optimal quasi-norms are similar, but we have to replace (1.11) and (1.12) by

$$\rho_H(\chi_{(0,1)}g) \lesssim \rho_{H(F)}(\chi_{(0,1)}g), g \in A;$$

$$\rho_{F(H)}(\chi_{(0,1)}g) \lesssim \rho_F(\chi_{(0,1)}g),\ g \in L_m.$$

Here $A := \{g \in \mathbf{M}^+ : g(t) = \frac{1}{t}\int_0^t h(u)du\}$, where $h \in \mathbf{M}^+$ is increasing, $h(2t) \approx h(t)$ and $h(+0) = 0$. This choice of A is motivated by the fact that the function $h(t) = \omega_E^k(t^{1/n}, f)$ is increasing, $h(+0) = 0$ if f is continuous, and $g \approx h$.

The optimal quasi-norms are uniquely determined up to equivalence, while the optimal target quasi-Banach spaces G are unique.

We give a characterization of all admissible couples, optimal target quasi-norms, optimal domain quasi-norms, and optimal couples.

In the subcritical case $\alpha_F < \alpha_\varphi$ the main result is that the optimal target quasi-norm satisfies $\rho_{G(F)}(g) \approx \rho_F(\varphi g^*)$. Moreover, the couple $\rho_F, \rho_{G(F)}$ is optimal. For example, the couple $F = L_*^q(w)$, $0 < q \leq \infty$, $\alpha_F = \beta_F < \alpha_\varphi$, $G = \Lambda_0^q(w\varphi)$ is optimal (see Theorem 2.5 below). In the supercritical case $\alpha_F > \alpha_\varphi$, we have $\rho_{H(F)}(\chi_{(0,1)}g) \approx \rho_F(\chi_{(0,1)}\varphi g)$ and this couple is optimal (see Theorem 3.4). We also prove that the couple ρ_H, $\rho_{F(H)}$, $\rho_{F(H)}(g) := \rho_H(R_\varphi g)$ is optimal if $\alpha_\varphi \leq \alpha_F < k/n$ (see Theorem 3.5).

In the critical case $\alpha_F = \alpha_\varphi$ we use real interpolation similarly to [7], but in a simpler way [1], and consider domain quasi-norms ρ_F,

$$\rho_F(g) := \rho_T((bg/\varphi)_\mu^{**}),$$

where ρ_T is a monotone quasi-norm on $(0,\infty)$, satisfying $\beta_T < 1$, and h_μ^* means the rearrangement of h with respect to the Haar measure on $(0,\infty)$, $d\mu := \frac{dt}{t}$, $h_\mu^{**}(t) := \frac{1}{t}\int_0^t h_\mu^*(u)du$. In this case the optimal target quasi-norm $\rho_{G(F)}$ is

$$\rho_{G(F)}(g) := \rho_T((cg^*)_\mu^{**}).$$

Here b and c belong to a large class of Muckenhoupt slowly varying weights (see Theorem 2.6). For example, if $\rho_T(g) := \left(\int_0^\infty [g(t)]^q dt\right)^{1/q}$, $1 < q \leq \infty$, then $\beta_T = 1/q < 1$, and

$$\rho_F(g) \approx \left(\int_0^\infty [(bg/\varphi)_\mu^*(u)]^q du\right)^{1/q} = \left(\int_0^\infty [b(t)g(t)/\varphi(t)]^q \frac{dt}{t}\right)^{1/q}.$$

Hence $F = L_*^q(b/\varphi)$ and $G(F) = \Lambda_0^q(c)$ (see Example 2.6). Similar results are valid in the critical case for the Besov space $B^k(E,F)$, when they are embedded in C^kH (see Theorem 3.6).

The problem of the optimal embeddings of Sobolev type spaces is considered in [1], [6], [7], [8], [9], [10], [12], [13], [18], [26], [27] and the same problem for Sobolev or Besov type spaces is treated in [14], [15], [16], [17], [19], [21], [25], [26], [27], [28], [29], [31], [11], [32], [33] by somewhat different methods. In [15], [16], [13] the main object is the generalized Calderon space $\Lambda(E,F)$, where the optimal rearrangement invariant target space is characterized. In [16] the anisotropic Calderon spaces are also investigated. As in [16], Section 2, it can be proved that $B^k(E,F)=\Lambda(E,F_1)$, where $\rho_{F_1}(g)=\rho_F\left(g(t^{-1})\right)$ in the non-limiting case $0<\alpha_F<k/n$. So the results in [15], [16], [13] are valid for the inhomogeneous Besov spaces, at least in the non-limiting case and non-supercritical one. Here in the non-supercritical case we consider only the homogeneous Besov spaces $b^k(E,F)$.

The embedding of $b^k(E,F)$ into rearrangement invariant spaces G is characterized by the continuity of the Hardy operator $Q_\varphi g(t)=\int_t^\infty \frac{g(u)}{\varphi(u)}\frac{du}{u}$ (see Theorem 2.1). In [15], [16], [13], the corresponding Hardy operator H_φ differs by a factor $\frac{t\varphi'(t)}{\varphi(t)}$ and $H_\varphi \lesssim Q_\varphi$. Therefore in the subcritical case $\alpha_F<\alpha_\varphi$, the operator H_φ is bounded in F, thus suggesting that then the optimal rearrangement invariant target space for the inhomogeneous Besov spaces $B^k(E,F)$ is $G(F)\cap E$, where $\rho_{G(F)}(g)=\rho_F(\varphi_E g^*)$. This is confirmed by the Example 9.7 in [16], where $E=L^p$, $F=L^q(bt^{-s/n})$, b - slowly varying, $1/p>s/n>0$, $1\le q\le\infty$. Then the optimal target space is $\Lambda^q(t^{1/p-s/n}b(t))\cap L^p$. In the critical case $s/n=1/p$ the results in [16] are more general then ours.

The embedding of $B^k(E,F)$ into the Hölder-Zygmund space C^kH is characterized by the continuity of the operator $R_\varphi g(t)=\int_0^t \frac{g(u)}{\varphi(u)}\frac{du}{u}$ (see Theorem 3.2).

The plan of the paper is as follows. In Section 2 we consider embeddings in rearrangement invariant spaces and in Section 3 embeddings in Hölder-Zygmund spaces. The main results in a slightly different form are announced in [2].

2 Embeddings in rearrangement invariant spaces

In this section we suppose that $\alpha_F=\beta_F\le\alpha_\varphi$, i.e. here we consider non-supercritical case. Also $\alpha_\varphi=\beta_\varphi>0$. We also suppose that ρ_F satisfies the Minkowski inequality (1.1).

2.1 Pointwise estimates for the rearrangement

Lemma 2.1. *For $k=1$ and $k=2$*

$$\varphi(t)[f^{**}(t)-f^{**}(2t)]\lesssim \omega^k_{M_\varphi}(t^{1/n},f),\ f\in L_{loc}. \tag{2.1}$$

Proof. The case $k=1$ is proved in [25] by another method and for $k\ge 2$ a weaker version is established in [26]. Let $t>0$ and let B_t be the ball in $\mathbb{R}^n$ with center 0, radius h and measure $2t$. Let $u\in\mathbb{R}^n$, $|u|\le h$. Let $\Delta_u f(x):=f(x+u)-f(x)$. Then

$$|f(x)|\le|\Delta_u f(x)|+|f(x+u)|,$$

and, integrating with respect to u over B_t,

$$2t\,|f(x)| \le \int_{B_t} |\Delta_u f(x)|\,du + \int_0^{2t} f^*(s)ds.$$

Now integrate with respect to x over a subset S of $\mathbb{R}^n$ with Lebesgue $n-$measure t and take the supremum over all such sets S. This gives (see [3], p. 53, Proposition 2.3.3)

$$2t[f^{**}(t) - f^{**}(2t)] \le \int_{B_t} (\Delta_u f)^{**}(t)du,$$

whence (2.1) follows for $k = 1$.

In the case $k = 2$ we have $\Delta_u^2 f(x) := f(x+2u) - 2f(x+u) + f(x)$, whence

$$|f(x)| \le \frac{1}{2}\left|\Delta_u^2 f(x-u)\right| + \frac{1}{2}[|f(x+u)| + |f(x-u)|].$$

Integration of this with respect to u over B_t gives

$$2t\,|f(x)| \le \frac{1}{2}\int_{B_t} \left|\Delta_u^2 f(x-u)\right| du + \int_0^{2t} f^*(s)ds.$$

Hence as before we have

$$2t[f^{**}(t) - f^{**}(2t)] \le \int_{B_t} (\Delta_u^2 f)^{**}(t)du \tag{2.2}$$

which implies (2.1) for $k = 2$. □

Lemma 2.2. *Let $k > 2$ and $f \in L_{loc}$, $f^*(\infty) = 0$. If*

$$\int_t^\infty \frac{u^{(k-2)/n}}{\varphi(u)}\frac{du}{u} \lesssim \frac{t^{(k-2)/n}}{\varphi(t)}, \text{ or equivalently, } k < 2 + n\alpha_\varphi, \tag{2.3}$$

then

$$\varphi(t)[f^{**}(t) - f^{**}(2t)] \lesssim \omega^k_{M_\varphi}(t^{1/n}, f). \tag{2.4}$$

Proof. We prove (2.4) by induction for $k > 2$. First we note that $f^*(\infty) = 0$ and

$$f^{**}(t) = \int_t^\infty \delta f^{**}(u)\frac{du}{u} \tag{2.5}$$

and also $\delta f^{**}(t) := f^{**}(t) - f^*(t) \lesssim f^{**}(t) - f^{**}(2t)$. If (2.4) is true for $k-2$, we can write

$$f^{**}(t) \lesssim \int_t^\infty \frac{\omega^{k-2}_{M_\varphi}(u^{1/n}, f)}{u^{(k-2)/n}}\frac{u^{(k-2)/n}}{\varphi(u)}\frac{du}{u}$$

and using the fact that the function $u^{-(k-2)/n}\omega^{k-2}_{M_\varphi}(u^{1/n}, f)$ is equivalent to decreasing, and (2.3), we get

$$\varphi(t)f^{**}(t) \lesssim \omega^{k-2}_{M_\varphi}(t^{1/n}, f).$$

In particular,

$$\varphi(t)(\Delta_u^2 f)^{**}(t) \lesssim \omega_{M_\varphi}^{k-2}(t^{1/n}, \Delta_u^2 f).$$

Applying also (2.2), we get

$$t\varphi(t)[f^{**}(t) - f^{**}(2t)] \lesssim \int_{B_t} \omega_{M_\varphi}^{k-2}(t^{1/n}, \Delta_u^2 f) du. \tag{2.6}$$

By using Lemma 4.11, p. 338 [3], we derive from (2.6) inequality (2.4). □

Lemma 2.3. *Let $\alpha_\varphi = \beta_\varphi$. Then for $f \in L_{loc}$, $f^*(\infty) = 0$,*

$$f^{**}(t) \lesssim \int_t^\infty \frac{\omega_{M_\varphi}^k(u^{1/n}, f)}{\varphi(u)} \frac{du}{u} \lesssim \int_t^\infty \frac{\omega_E^k(u^{1/n}, f)}{\varphi(u)} \frac{du}{u}. \tag{2.7}$$

Proof. If $k \leq 2$ then (2.7) follows from (2.5) and (2.1). Let the integer $m > 2$ satisfy $n\alpha_\varphi < m < 2 + n\alpha_\varphi$. Using Lemma 2.2 and (2.5), we obtain (2.7) for $k = m$. Let now $k > m$. By Marchaud's inequality [3], p. 333, we can write

$$\omega_{M_\varphi}^m(u^{1/n}, f) \lesssim u^{m/n} \int_u^\infty \frac{\omega_{M_\varphi}^k(\sigma^{1/n}, f)}{\sigma^{m/n}} \frac{d\sigma}{\sigma},$$

therefore from (2.7) and Fubini's theorem it follows that

$$f^{**}(t) \lesssim \int_t^\infty \frac{\omega_{M_\varphi}^k(\sigma^{1/n}, f)}{\sigma^{m/n}} \left(\int_0^\sigma \frac{u^{m/n}}{\varphi(u)} \frac{du}{u} \right) \frac{d\sigma}{\sigma}.$$

Since $m > n\beta_\varphi$ we have

$$\int_0^\sigma \frac{u^{m/n}}{\varphi(u)} \frac{du}{u} \lesssim \frac{\sigma^{m/n}}{\varphi(\sigma)}.$$

Therefore (2.7) follows. □

2.2 Admissible couples

Here we give a characterization of all admissible couples ρ_F, ρ_G in the non-supercritical case. We always suppose that $\alpha_\varphi = \beta_\varphi > 0$ and $\alpha_F = \beta_F \leq \alpha_\varphi$, $\alpha_G = \beta_G$.

Theorem 2.1 (non-limiting case)**.** *Let $0 < \alpha_F < k/n$. Then the couple ρ_F, ρ_G is admissible if and only if*

$$\rho_G(Q_\varphi g) \lesssim \rho_F(g),\ g \in M, \tag{2.8}$$

where

$$Q_\varphi g(t) := \int_t^\infty \frac{g(u)}{\varphi(u)} \frac{du}{u},\ t > 0, \tag{2.9}$$

and

$$M := \{g \in L_m \text{ and } Q_\varphi g(t) < \infty.\}$$

Proof. It is clear that (1.9) follows from (2.7) and (2.8).

Now we prove that (1.9) implies (2.8). To this end we choose the test function of the form

$$f(x) = \int_0^\infty \frac{g(u)}{\varphi(u)} \psi\left(|x|u^{-1/n}\right) \frac{du}{u},$$

where $g \in M$ and $\psi \geq 0$ is a smooth function with compact support such that $\psi(|x|) = 1$ if $|x| \leq c^{-1/n}$ and the constant c is chosen in such a way that if $h(x) := g(c|x|^n)$ then $h^* = g^*$. We have

$$f(x) \geq \int_{c|x|^n}^\infty \frac{g(u)}{\varphi(u)} \frac{du}{u} = (Q_\varphi g)(c|x|^n), \text{ whence } f^*(t) \gtrsim Q_\varphi g(t). \tag{2.10}$$

Let

$$f_{0t}(x) := \int_0^t \frac{g(u)}{\varphi(u)} \psi\left(|x|u^{-1/n}\right) \frac{du}{u}, \; f_{1t}(x) := \int_t^\infty \frac{g(u)}{\varphi(u)} \psi\left(|x|u^{-1/n}\right) \frac{du}{u}.$$

Then

$$\|f_{0t}\|_{\Lambda_\varphi} \lesssim \int_0^t \frac{g(u)}{\varphi(u)} \|\psi(|x|u^{-1/n})\|_{\Lambda_\varphi} \frac{du}{u}, \; a > 1,$$

$$\|(D^k f_{1t}\|_{\Lambda_\varphi} \lesssim \int_t^\infty \frac{g(u)}{\varphi(u)} u^{-k/n} \|\psi(|x|u^{-1/n})\|_{\Lambda_\varphi} \frac{du}{u},$$

$D^k f := \sum_{|\alpha|=k} |D^\alpha f|$. Since $\|\psi(|x|u^{-1/n})\|_{\Lambda_\varphi} \lesssim \varphi(u)$, we get

$$\|f_{0t}\|_E \lesssim \int_0^t g(u) \frac{du}{u}, \; \|D^k f_{1t}\|_E \lesssim \int_t^\infty u^{-k/n} g(u) \frac{du}{u}.$$

Thus

$$\omega_E^k(t^{1/n}, f) \lesssim \int_0^t g(u) \frac{du}{u} + t^{k/n} \int_t^\infty u^{-k/n} g(u) \frac{du}{u}. \tag{2.11}$$

If (1.9) is given then the above and (2.10) imply

$$\begin{aligned} \rho_G(Q_\varphi g) \;&\lesssim\; \rho_F(g) \\ &\times \left(\int_0^1 h_F^p(u) \frac{du}{u} + \int_1^\infty h_F^p(u)(u) u^{-pk/n} \frac{du}{u} \right)^{1/p}. \end{aligned}$$

Here we are using the monotonicity properties of $g \in M$ and the Minkowski inequality for ρ_F. Since $0 < \alpha_F < k/n$, we obtain (2.8) due to (1.2), (1.3).

□

In the limiting cases we suppose that $E = M_\varphi$ and in addition $\alpha_\varphi < 1$. Then

$$\|f\|_{M_\varphi} \approx \sup f^*(t)\varphi(t). \tag{2.12}$$

Theorem 2.2 (limiting cases)**.** *Let $\alpha_F = 0$ or $\alpha_F = k/n \le \alpha_\varphi < 1$. Then the couple ρ_F, ρ_G is admissible if and only if (2.8) is satisfied for all $g \in M_0$, where M_0 consists of all such $g \in \mathbf{M}^+$ that $g(t)$ is increasing and $t^{-k/n}g(t)$ is decreasing as well as $Q_\varphi g(t) < \infty$.*

Proof. It is clear that we need to prove only that (1.9) implies (2.8). To this end we use the same test function as in (2.9) and split f as before: $f = f_{0t} + f_{1t}$. Then using the monotonicity of $g \in M_0$ and

$$\int_t^\infty \frac{1}{\varphi(u)}\frac{du}{u} \lesssim \frac{1}{\varphi(t)} \text{ if } \alpha_\varphi > 0, \tag{2.13}$$

we get the estimates

$$f_{0t}(x) \lesssim \frac{g(t)}{\varphi(c|x|^n)},\ |D^k f_{1t}(x)| \lesssim \frac{t^{-k/n}g(t)}{\varphi(c|x|^n)},$$

whence, using also (2.12),

$$\|f_{0t}\|_{M_\varphi} \lesssim g(t),\ \|D^k f_{1t}\|_{M_\varphi} \lesssim t^{-k/n}g(t).$$

Therefore

$$\omega^k_{M_\varphi}(t^{1/n}, f) \lesssim \omega^k_{M_\varphi}(t^{1/n}, f_{0t}) + \omega^k_{M_\varphi}(t^{1/n}, f_{1t}) \lesssim g(t). \tag{2.14}$$

If (1.9) is given then the above and (2.10) imply

$$\rho_G(Q_\varphi g) \lesssim \rho_G(f^*) \lesssim \rho_F\left(\omega^k_{M_\varphi}(t^{1/n}, f)\right) \lesssim \rho_F(g).$$

□

2.3 Optimal quasi-norms

Here we give a characterization of the optimal domain and optimal target quasi-norms in the non-supercritical case $\alpha_F \le \alpha_\varphi$.

We can define an optimal target quasi-norm by using Theorem 2.1 or Theorem 2.2. We put $N = M$ in the non-limiting case and $N = M_0$ in the limiting cases.

Definition 2.1 (construction of the optimal target quasi-norm)**.** For a given domain quasi-norm ρ_F, satisfying (1.4) and

$$(Q_\varphi h)(a) \lesssim \rho_F(h),\ h \in N,\ 0 < a < 1, \tag{2.15}$$

we set

$$\rho_{G(F)}(g) := \inf\{\rho_F(h) : g^* \le Q_\varphi h,\ h \in N\},\ g \in \mathbf{M}^+,\ g^*(\infty) = 0. \tag{2.16}$$

Theorem 2.3. *Let $\alpha_F = \beta_F \le \alpha_\varphi$. Then the couple $\rho_F, \rho_{G(F)}$ is admissible, the target quasi-norm is optimal and $h_{G(F)}(u) \le h_F(\frac{1}{u})h_\varphi(u)$, therefore $\alpha_{G(F)} = \beta_{G(F)} = \alpha_\varphi - \alpha_F$. Also*

$$\rho_{G(F)}(Q_\varphi(\chi_{(0,1)}(t)t^{k/n})) < \infty,\ \rho_{G(F)}(Q_\varphi(\chi_{(a,\infty)})) < \infty,\ 0 < a < 1. \tag{2.17}$$

Proof. Since ρ_F is a monotone quasi-norm it follows that $\rho_{G(F)}$ is also a monotone quasi-norm. The couple is admissible due to the inequality $\rho_{G(F)}(Q_\varphi h) \le \rho_F(h),\ h \in N$ and Theorem 2.1 or Theorem 2.2. Suppose that the couple ρ_F, ρ_G is admissible. Then by the same theorems, $\rho_G(Q_\varphi h) \lesssim \rho_F(h),\ h \in N$. Therefore if $g^* \le Qh,\ h \in N$, then $\rho_G(g^*) \le \rho_G(Q_\varphi h) \lesssim \rho_F(h)$, whence $\rho_G(g^*) \lesssim \rho_{G(F)}(g^*)$. □

We construct an optimal domain quasi-norm by Theorem 2.1 or Theorem 2.2 as follows.

Definition 2.2 (construction of an optimal domain quasi-norm)**.** For a given target quasi-norm ρ_G, satisfying Minkowski's inequality, we put

$$\rho_{F(G)}(g) := \rho_G(Q_\varphi g),\ g \in N.$$

Theorem 2.4. *Let G be a rearrangement invariant space, satisfying (2.17) and $\alpha_\varphi - k/n \le \alpha_G = \beta_G \le \alpha_\varphi$. Then $\rho_{F(G)}$ is an optimal domain quasi-norm and $h_{F(G)}(u) \le h_\varphi(u) h_G(\frac{1}{u})$, therefore $\alpha_{F(G)} = \beta_{F(G)} = \alpha_\varphi - \alpha_G$. Moreover, in the non-limiting case the couple $\rho_{F(G)}$, ρ_G is optimal if $\beta_G < 1$. Also $F(G)$ satisfies (1.4), (2.15).*

Proof. The couple $\rho_{F(G)}, \rho_G$ is admissible since $\rho_{F(G)}(g) \ge \rho_G(Q_\varphi g)$. Moreover, $\rho_{F(G)}$ is optimal, since for any admissible couple ρ_F, ρ_G we have $\rho_G(Q_\varphi g) \lesssim \rho_F(g),\ g \in N$. Therefore,

$$\rho_{F(G)}(g) = \rho_G(Q_\varphi g) \lesssim \rho_F(g).$$

In the non-limiting case we use $g^{**} = Q_\varphi(\varphi \delta g^{**})$ if $g^*(\infty) = 0$. Since $\varphi\delta g^{**} \in M$, we have

$$\rho_{G(F(G)}(g^{**}) \le \rho_{F(G)}(\varphi\delta g^{**}) = \rho_G(Q_\varphi(\varphi\delta g^{**})) = \rho_G(g^{**}) \lesssim \rho_G(g^*).$$

Here we use $\rho_G(g^{**}) \lesssim \rho_G(g^*)$ if $\beta_G < 1$. Hence the target quasi-norm is also optimal. □

Now we give some examples. In the limiting cases we suppose that $\alpha_\varphi < 1$.

Example 2.1. Consider the space $G = \Lambda_0^1(v)$, satisfying (2.17), $v(2t) \approx v(t)$, $\beta_G = \alpha_G \le \alpha_\varphi$. Using Theorem 2.4, we can construct an optimal domain quasi-norm

$$\rho_F(g) = \rho_G(Q_\varphi g) = \int_0^\infty v(t) \left(\int_t^\infty \frac{g(u)}{\varphi(u)} \frac{du}{u} \right) \frac{dt}{t} = \int_0^\infty w(t) \frac{g(t)}{\varphi(t)} \frac{dt}{t},$$

where $w(t) = \int_0^t v(u) \frac{du}{u}$. Hence $F = L_*^1(w/\varphi)$. If v is slowly varying, then $\alpha_G = \beta_G = 0$ and $\alpha_F = \beta_F = \alpha_\varphi$. In the non-limiting case, $0 < \alpha_\varphi < k/n$, the couple F, G is optimal if $\beta_G < 1$.

Example 2.2. Let $G = C_0$ consist of all bounded functions such that $f^*(\infty) = 0$ and $\rho_G(g) = g^*(0)$. Suppose G satisfies (2.17). Then $\alpha_G = \beta_G = 0$ and $\rho_{F(G)}(g) = \int_0^\infty \frac{g(t)}{\varphi(t)} \frac{dt}{t}$, i.e. $F(G) = L_*^1(1/\varphi)$ and the couple is optimal in the non-limiting case.

Example 2.3. Let $G = \Lambda_0^\infty(v)$ satisfy (2.17) and $v(2t) \approx v(t)$, $\beta_G = \alpha_G \leq 1$. Then

$$\rho_{F(G)}(g) = \sup v(t) \int_t^\infty \frac{g(u)}{\varphi(u)} \frac{du}{u}.$$

If v is slowly varying, then $\alpha_G = \beta_G = 0$ and $\alpha_{F(G)} = \beta_{F(G)} = \alpha_\varphi$. Hence this couple is optimal in the non-limiting case.

Example 2.4. Let G be as in the previous example and $0 < \alpha_\varphi < k/n$. Since

$$\rho_{F(G)}(g) \leq \sup \frac{w(t)}{\varphi(t)} g(t), \ \frac{1}{v(t)} = \int_t^\infty \frac{1}{w(u)} \frac{du}{u},$$

it follows that the couple $F_1 = L_*^\infty(w/\varphi), G = \Lambda_0^\infty(v)$ is admissible. Let w be slowly varying and let F_1 satisfy (2.15). In order to prove that ρ_G is optimal, take any $g \in \mathbf{M}^+$, and define h from $\frac{w(t)}{\varphi(t)} h(t) = \sup_{0<u\leq t} v(u) g^*(u)$. Then $h \in M$ and $\rho_{F_1}(h) \lesssim \rho_G(g^*)$. On the other hand

$$Q_\varphi h(t) = \int_t^\infty \sup_{0<x\leq u} v(x) g^*(x) \frac{1}{w(u)} \frac{du}{u} \geq \sup_{0<u\leq t} v(u) g^*(u) \frac{1}{v(t)} \geq g^*(t).$$

Hence $\rho_{G(F)}(g^*) \leq \rho_{F_1}(h) \lesssim \rho_G(g^*)$, therefore ρ_G is optimal.

2.4 Subcritical case

Here we suppose that $\alpha_F = \beta_F < \alpha_\varphi$, F satisfies (1.4), (2.15) and as before, $\alpha_\varphi = \beta_\varphi > 0$. Also, in the limiting cases $\alpha_F = 0$ or $\alpha_F = k/n$, we suppose that $\alpha_\varphi < 1$.

Theorem 2.5. *The optimal target quasi-norm $\rho_{G(F)}$ is given by*

$$\rho_{G(F)}(g) \approx \rho(g), \text{ where } \rho(g) := \rho_F(\varphi g^{**}), \ g \in \mathbf{M}^+, \ g^*(\infty) = 0.$$

Moreover, the couple $\rho_F, \rho_{G(F)}$ is optimal and $\alpha_{G(F)} = \beta_{G(F)} = \alpha_\varphi - \alpha_F < 1$.

Proof. First we prove that the beta index β of ρ satisfies $\beta < 1$. Indeed,

$$\rho(g_u^*) \leq h_F(\frac{1}{u}) h_\varphi(u) \rho_F(\varphi g^{**}),$$

hence

$$\rho(g_u^*) \lesssim h_F(\frac{1}{u}) h_\varphi(u) \rho(g^*).$$

Therefore $\beta = \alpha_\varphi - \alpha_F$, in particular $\beta < 1$. As a consequence, $\rho(g) \approx \rho_F(\varphi g^*)$.

Since

$$\rho_F(\varphi Q_\varphi g) \lesssim \rho_F(g) \left(\int_1^\infty h_\varphi^p(u) h_F^p(u) \frac{du}{u} \right)^{1/p} \lesssim \rho_F(g) \text{ if } \alpha_F < \alpha_\varphi, \ g \in N,$$

it follows that the couple ρ_F, ρ is admissible. Therefore, $\rho(g) \lesssim \rho_{G(F)}(g)$.

On the other hand, $g \lesssim Q_\varphi(\varphi g)$, $g \in N$, hence $g^* \lesssim Q_\varphi(\varphi g)$ and since $g \lesssim g^{**}$ for $g \in N$, we have

$$\rho_{G(F)}(g^*) \lesssim \rho_F(\varphi g^{**}) \lesssim \rho(g^*).$$

The couple $\rho_F, \rho_{G(F)}$ is optimal, since

$$\rho_{F(G(F))}(g) = \rho_{G(F)}(Q_\varphi g) \approx \rho_F(\varphi Q_\varphi g) \gtrsim \rho_F(g),\ g \in L_m.$$

□

Example 2.5. Let $F = L_*^q(w)$ with $0 < q \le \infty$, $\alpha_F = \beta_F < \alpha_\varphi$ satisfy (1.4), (2.15), $G = \Lambda_0^q(\varphi w)$, $w(2t) \approx w(t)$. Then this couple is optimal. In particular, if $w = b$ is slowly varying, then $\alpha_F = \beta_F = 0 < \alpha_\varphi$, i.e. this is a subcritical and limiting case. Thus if $\alpha_\varphi < 1$, then

$$\left(\int_0^\infty [b(t)\varphi(t)f^*(t)]^q \frac{dt}{t}\right)^{1/q} \lesssim \left(\int_0^\infty [b(t)\omega_{M_\varphi}^k(t^{1/n}, f)]^q)\frac{dt}{t}\right)^{1/q}.$$

Analogous result is valid if $w(t) = t^{-k/n}b(t)$, $k/n < \alpha_\varphi < 1$. Then $\alpha_F = \beta_F = k/n < \alpha_\varphi$, i.e. this is the other limiting case.

2.5 Critical case

Here we are going to use real interpolation for quasi-normed spaces, similarly to [1], [8], [7]. Let (A_0, A_1) be a couple of two quasi-Banach spaces (see [4], [5]) and let

$$K(t,f) = K(t,f;A_0,A_1) = \inf_{f=f_0+f_1}\{\|f_0\|_{A_0} + t\,\|f_1\|_{A_1}\},\ f \in A_0 + A_1,$$

be the $K-$functional of Peetre (see [4]). Then, the $K-$interpolation space $A_\Phi = (A_0, A_1)_\Phi$ has a quasi-norm

$$\|f\|_{A_\Phi} = \|K(t,f)\|_\Phi\,,$$

where Φ is a quasi-normed function space with a monotone quasi-norm on $(0,\infty)$ with the Lebesgue measure and such that $\min\{1,t\} \in \Phi$. Then (see [5])

$$A_0 \cap A_1 \hookrightarrow A_\Phi \hookrightarrow A_0 + A_1.$$

If $\Phi = L_*^q(t^{-\theta})$, $0 < \theta < 1$, $0 < q \le \infty$, we write $(A_0, A_1)_{\theta,q}$ instead of $(A_0, A_1)_\Phi$. (see [4])

Now we construct the required couples of Muckenhoupt weights. Let the function b satisfy the following properties:

$$\text{it increases and slowly varies on } (0,\infty) \text{ with } b(t^2) \approx b(t) \tag{2.18}$$

and for some $\varepsilon > 0$,

$$(1+\ln t)^{-1-\varepsilon} b(t) \text{ is increasing for } t > 1. \tag{2.19}$$

Let

$$c(t) = \frac{b(t)}{1 + |\ln t|)}. \tag{2.20}$$

Then

$$\int_t^\infty \frac{1}{b(u)} \frac{du}{u} \lesssim \frac{1}{c(t)}, \ t > 0. \tag{2.21}$$

Indeed, if $0 < t < 1$ we can write:

$$\int_t^\infty \frac{1}{b(u)} \frac{du}{u} = \int_t^1 \frac{1}{b(u)} \frac{du}{u} + \int_1^\infty \frac{(1 + \ln u)^{-1-\varepsilon}}{b(u)(1 + \ln u)^{-1-\varepsilon}} \frac{du}{u}.$$

Using monotonicity properties (2.18), (2.19) and the fact that $c(t) \lesssim 1$ for $0 < t < 1$, we get (2.21). The case $t > 1$ is analogous, but simpler.

Theorem 2.6. *Let ρ_T be a monotone quasi-norm on $\mathbf{M}^+$ with $\beta_T < 1$, satisfying Minkowski's inequality. Here the index β_T is defined in the same way as for G. Let b, c be given by (2.18) - (2.20). Let ρ_F be defined by*

$$\rho_F(g) := \rho_S(bg/\varphi),$$

$$S := (L^1_*, L^\infty_*)_{T(\frac{1}{t})}, \tag{2.22}$$

and $T(\frac{1}{t})$ has the quasi-norm $\|g\|_{T(\frac{1}{t})} := \rho_T(g(t)/t)$. If $0 < \alpha_\varphi < k/n$, then the optimal target quasi-norm is given by

$$\rho_{G(F)}(g) := \rho_S(g^* c), \ g^*(\infty) = 0.$$

Proof. Let L^∞_v be the weighted Lebesgue space on $(0, \infty)$ with the Lebesgue measure and the norm

$$\|g\|_{L^\infty_v} := \sup |g(t)v(t)|.$$

Then the operator Q_φ, defined by (2.9) is bounded in the following couple of spaces:

$$Q_\varphi : L^1_*(b/\varphi) \mapsto L^\infty_b \ \text{ and } \ Q_\varphi : L^\infty_*(b/\varphi) \mapsto L^\infty_c,$$

where b, c are given by (2.18), (2.20).

Define S by (2.22). It is well known that ([4])

$$\rho_S(g) = \rho_T(g^{**}_\mu) \approx \rho_T(g^*_\mu), \tag{2.23}$$

where $g^{**}_\mu(t) = \frac{1}{t} \int_0^t g^*_\mu(s) ds$. The equivalence in (2.23) is true because $\beta_T < 1$.

By interpolation,

$$Q : F_1 \mapsto G_1,$$

where

$$F_1 := (L^1_*(b/\varphi), L^\infty_*(b/\varphi))_{T(\frac{1}{t})}, \ G_1 := (L^\infty_b, L^\infty_c)_{T(\frac{1}{t})}.$$

Denote the quasi-norm in F_1 by ρ_F. We have

$$\rho_F(g) = \rho_S(bg/\varphi) = \rho_T((bg/\varphi)^{**}_\mu) \approx \rho_T((bg/\varphi)^*_\mu).$$

Hence ρ_F is a monotone quasi-norm and $\alpha_F = \beta_F = \alpha_\varphi$; this is because b is slowly varying and $\alpha_S = \beta_S = 0$. Also F satisfies (1.4), (2.15).

Now we characterize the space G_1. Since (see [4])

$$K(t, g; L_b^\infty, L_c^\infty) = tK\left(\frac{1}{t}, g; L_c^\infty, L_b^\infty\right) = t \sup_s |g(s)| \min(c(s), b(s)/t),$$

we get the formula

$$\rho_{G_1}(g) = \rho_H(h_g),\ h_g(u) := \sup_s |g(s)| \min(c(s), b(s)/u). \tag{2.24}$$

Also, since $L_b^\infty \hookrightarrow L_c^\infty$ it follows $h_g(u) \approx \sup |g(s)|c(s)$ if $0 < u < 1$. Let

$$H_g(t) := h_g(1 + |\ln t|),\ 0 < t < \infty.$$

Then $(H_g)_\mu^*(t) \leq h_g(t/2)$, hence by (2.23) and (2.24)

$$\rho_S(H_g) \lesssim \rho_{G_1}(g).$$

Note that $H_g \gtrsim gc$, hence, if we define the quasi-norm $\rho_G(g) := \rho_S(g^* c)$, we get the relation

$$\rho_G(Q_\varphi g) \lesssim \rho_{G_1}(Q_\varphi g) \lesssim \rho_F(g),\ g \in M.$$

Theorem 2.1 shows that the couple ρ_F, ρ_G is admissible. Also $\alpha_G = \beta_G = 0$.

Now we want to prove that ρ_G is an optimal target quasi-norm. It is sufficies to see that

$$\rho_G(g^{**}) \approx \rho_{G(F)}(g^{**}),\ g \in \mathbf{M}^+,\ g^*(\infty) = 0,$$

where $\rho_{G(F)}$ is defined by (2.16). And since the quasi-norm $\rho_{G(F)}$ is optimal, we need only to prove that $\rho_{G(F)}(g^{**}) \lesssim \rho_G(g^{**})$. To this end first for any such g we construct $h \in M$ such that $g^* \lesssim Q_\varphi h$ and $\rho_F(h) \lesssim \rho_G(g^{**})$. Let $bh/\varphi = g_1$, where $g_1(t) = g^{**}(t^2/e^2)c(t^2)$ for $0 < t < 1$ and $g_1(t) = g^{**}(\sqrt{t/e})c(\sqrt{t})$ if $t > 1$. Then $h \in M$ and $\rho_F(h) \approx \rho_S(g^{**}c) = \rho_G(g^{**})$. On the other hand,

$$Q_\varphi h(t) \geq \int_t^{\sqrt{te}} g^{**}(s^2/e) \frac{c(s^2)}{b(s)} \frac{ds}{s} \geq g^{**}(t)A(t) \gtrsim g^{**}(t),$$

since

$$A(t) = \int_t^{\sqrt{te}} \frac{c(s^2)}{b(s)} \frac{ds}{s} \approx \int_t^{\sqrt{te}} \frac{1}{1 + |\ln s|} \frac{ds}{s} \gtrsim 1.$$

Similarly, for $t > 1$ we obtain

$$Q_\varphi h(t) \geq \int_t^{et^2} g^{**}(\sqrt{s/e}) \frac{1}{1 + \ln s} \frac{ds}{s} \gtrsim g^{**}(t).$$

Thus $Q_\varphi h \gtrsim g^{**}$ and $\rho_F(h) \approx \rho_G(g^{**})$. Then by the definition of $\rho_{G(F)}$ we get $\rho_{G(F)}(g^{**}) \lesssim \rho_G(g^{**})$. □

Example 2.6. Let $G = \Lambda_0^q(c)$, $1 < q < \infty$, $F = L_*^q(b/\varphi)$, where b and c are slowly varying on $(0, \infty)$, $b(t^2) \approx b(t)$, $b(t) \lesssim (1 + |\ln t|)c(t)$ and

$$\left(\int_0^t c^q(s)\frac{ds}{s}\right)^{1/q}\left(\int_t^\infty [b(s)]^{-r}\frac{ds}{s}\right)^{1/r} \lesssim 1,\ 1/q + 1/r = 1.$$

Then the couple F, G is admissible by [30] and using the same argument as above, we see that G is an optimal target space if $0 < \alpha_\varphi < k/n$.

3 Embeddings in Hölder-Zygmund spaces

In this section we consider the non-subcritical case, i.e. $\alpha_F = \beta_F \geq \alpha_\varphi$. Also $\alpha_\varphi = \beta_\varphi > 0$ and in the limiting case $\alpha_F = k/n$ we suppose in addition that $\alpha_\varphi < 1$ and $\alpha_\varphi \leq k/n$.

3.1 Equivalent quasi-norms in Hölder-Zygmund spaces

We suppose that $0 \leq \alpha_H = \beta_H \leq k/n$ and that ρ_H satisfies Minkowski's inequality for some equivalent $p-$norm, denoted again by ρ_H for simplicity. Let $\chi_{(1,\infty)} \in H$, where χ stands for the characteristic function of the corresponding interval.

Theorem 3.1. *Let $k \geq 2$ and $0 \leq j \leq k-1$.*

- *If $j/n < \alpha_H < (j+1)/n$ for $1 \leq j \leq k-2$, $k \geq 3$, or $\alpha_H < 1/n$ for $j = 0$, or $\alpha_H > (k-1)/n$ for $j = k-1$, then*

$$\|f\|_{C^kH} \approx \sum_{l=0}^{j} \|D^l f\|_{L^\infty} + \rho_H(t^{j/n}\omega(t^{1/n}, D^j f)). \tag{3.1}$$

- *If $\alpha_H = (j+1)/n$, $0 \leq j \leq k-2$, then*

$$\|f\|_{C^kH} \approx \sum_{l=0}^{j} \|D^l f\|_{L^\infty} + \rho_H(t^{j/n}\omega^2(t^{1/n}, D^j f)). \tag{3.2}$$

Proof. Since $\omega^k(t^{1/n}, f) \lesssim t^{j/n}\omega(t^{1/n}, D^j f)$, the left-hand side in (3.1) is bounded by the right one. For the converse, consider first the case $j/n < \alpha_H < (j+1)/n$, $1 \leq j \leq k-2$, $k \geq 3$. By Marchaud's inequality,

$$t^{j/n}\omega(t^{1/n}, D^j f) \lesssim t^{(j+1)/n}\int_t^\infty u^{-1/n}\omega^k(u^{1/n}, D^j f)\frac{du}{u}.$$

Using also the estimate (cf. [3], p. 342)

$$\omega^k(t^{1/n}, D^j f) \lesssim \int_0^t u^{-j/n}\omega^k(u^{1/n}, f)\frac{du}{u},$$

and Fubini's theorem, we get $t^{j/n}\omega(t^{1/n}, D^j f) \lesssim A(t)$, where

$$\begin{aligned} A(t) &= t^{(j+1)/n}\int_t^\infty u^{-(j+1)/n}\omega^k(u^{1/n}, f)\frac{du}{u} \\ &\quad + t^{j/n}\int_0^t u^{-j/n}\omega^k(u^{1/n}, f)\frac{du}{u}. \end{aligned}$$

Applying Minkowski's inequality, we obtain

$$\rho_H(t^{j/n}\omega(t^{1/n}, D^j f)) \lesssim \rho_H(\omega^k(t^{1/n}, f)),$$

since

$$\int_0^1 h_H^p(u)u^{-pj/n}\frac{du}{u} + \int_1^\infty h_H^p(u)u^{-p(j+1)/n}\frac{du}{u} < \infty$$

due to $j/n < \alpha_H < (j+1)/n$ (cf. (1.2), (1.3)).

On the other hand (see [3], p. 341),

$$\|D^j f\|_{L^\infty} \lesssim \int_0^\infty u^{-j/n}\omega^k(u^{1/n}, f)\frac{du}{u},$$

whence

$$\|D^j f\|_{L^\infty} \lesssim \int_0^1 u^{-j/n}\omega^k(u^{1/n}, f)\frac{du}{u} + \|f\|_{L^\infty}. \tag{3.3}$$

Since $\rho_H(g) \geq g(t)\rho_H(\chi_{(t,\infty)})$ for increasing g and

$$\rho_H(\chi_{(1,\infty)}) \leq h_H(u)\rho_H(\chi_{(u,\infty)}),$$

we have

$$g(t) \lesssim h_H(t)\rho_H(g), \ g \in L_m. \tag{3.4}$$

Therefore

$$\int_0^1 u^{-j/n}\omega^k(u^{1/n}, f)\frac{du}{u} \lesssim \int_0^1 u^{-j/n}h_H(u)\frac{du}{u}\ \rho_H(\omega^k(t^{1/n}, f)).$$

Hence (3.3) can be rewritten as

$$\|D^j f\|_{L^\infty} \lesssim \rho_H(\omega^k(u^{1/n}, f)) + \|f\|_{L^\infty}. \tag{3.5}$$

Finally, using the estimate $\|D^l f\|_{L^\infty} \lesssim \|f\|_{L^\infty} + \|D^j f\|_{L^\infty}$, $1 \leq l \leq j-1$, we get (3.1). The proof of (3.2) is similar.

Let now $j = 0$ and $\alpha_H < 1/n$. Then as above, but using only Marshaud inequality, we get (3.1).

It remains to consider the case $j = k-1$, $\alpha_H > (k-1)/n$. Let w_∞^k be the homogeneous Sobolev space with a norm $\|f\|_{w_\infty^k} = \|D^k f\|_{L^\infty}$. Since $(L^\infty, w_\infty^k)_{(k-1)/k,1} \hookrightarrow w_\infty^{k-1}$ (cf. [4]), we have

$$\begin{aligned} \omega(t^{1/n}, D^{k-1}f) &\lesssim K(t^{1/n}, f; w_\infty^{k-1}, w_\infty^k) \\ &\lesssim K(t^{1/n}, f; (L^\infty, w_\infty^k)_{(k-1)/k,1}, w_\infty^k) \end{aligned}$$

and by the Holmstedt reiteration formulae for the $K-$functional (see [4]), we obtain

$$\omega(t^{1/n}, D^{k-1}f) \lesssim \int_0^t u^{-(k-1)/n}\omega^k(u^{1/n}, f)\frac{du}{u}.$$

Hence applying Minkowski's inequality as above, we get

$$\rho_H(t^{(k-1)/n}\omega(t^{1/n}, D^{k-1}f)) \lesssim \rho_H(\omega^k(u^{1/n}, f)).$$

Using also (3.5) for $j = k-1$, we finish the proof. □

As an example, let $\rho_H(g) = \sup t^{-\gamma/n}b(t)g(t)$, where $0 \le \gamma \le k$, b is slowly varying. Then $\alpha_H = \beta_H = \gamma/n$ and C^kH is the usual Hölder-Zygmund space C^γ if $0 < \gamma < k$ and $b = 1$ (cf. [33]).

3.2 Admissible couples

Here we give a characterization of all admissible couples ρ_F, ρ_H. We always suppose that $\alpha_\varphi = \beta_\varphi > 0$ and $\alpha_F = \beta_F \ge \alpha_\varphi$, $\alpha_H = \beta_H$. Also let H satisfy (1.7), and let F satisfy (1.4). Moreover, let

$$\int_0^a \frac{g(u)}{\varphi(u)}\frac{du}{u} \lesssim \rho_F(g),\ g \in M_1,\ 1 < a < \infty, \tag{3.6}$$

and

$$\rho_H(\chi_{(0,1)}\frac{g}{\varphi}) \lesssim \rho_F(\chi_{(0,1)}g),\ g \in M_1. \tag{3.7}$$

Theorem 3.2 (non-limiting case)**.** *Let $0 < \alpha_F < k/n$. Then the couple ρ_F, ρ_H is admissible if and only if*

$$\rho_H(\chi_{(0,1)}R_\varphi g) \lesssim \rho_F(\chi_{(0,1)}g),\ g \in M_1, \tag{3.8}$$

where

$$R_\varphi g(t) := \int_0^t \frac{g(u)}{\varphi(u)}\frac{du}{u},\ t > 0,$$

and

$$M_1 := \{g \in L_m\ g(2t) \approx g(t),\ \ \textit{and}\ \ R_\varphi g(t) < \infty.\}$$

Proof. We shall use (1.8). Next we prove that

$$\omega^k(t^{1/n}, f) \lesssim \int_0^t \frac{\omega^k_{M_\varphi}(u^{1/n}, f)}{\varphi(u)}\frac{du}{u} \text{ if } \alpha_\varphi > 0. \tag{3.9}$$

From (2.7) it follows

$$|f(x)| \lesssim \int_0^\infty \frac{\omega^k_{M_\varphi}(t^{1/n}, f)}{\varphi(t)}\frac{dt}{t}.$$

For $|h| \le t^{1/n}$ we get (using also (2.13))

$$|\Delta_h^k f(x)| \lesssim \int_0^t \frac{\omega_{M_\varphi}^k(u^{1/n}, f)}{\varphi(u)} \frac{du}{u} + \frac{\omega_{M_\varphi}^k(t^{1/n}, f)}{\varphi(t)}.$$

Since

$$\int_0^t \frac{\omega_{M_\varphi}^k(u^{1/n}, f)}{\varphi(u)} \frac{du}{u} \gtrsim \frac{\omega_{M_\varphi}^k(t^{1/n}, f)}{\varphi(t)},$$

we obtain (3.9).

Now we prove that (3.8) implies (1.10). From (3.8) and (3.9) it follows

$$\rho_H \left(\chi_{(0,1)}(t)\omega^k(t^{1/n}, f)\right) \lesssim \rho_F \left(\chi_{(0,1)}(t)\omega_{M_\varphi}^k(t^{1/n}, f)\right). \tag{3.10}$$

Using (2.7) and (2.13), we can write

$$\sup |f(x)| \lesssim \int_0^1 \frac{\omega_{M_\varphi}^k(t^{1/n}, f)}{\varphi(t)} \frac{dt}{t} + \|f\|_{M_\varphi}.$$

Hence (3.6) gives $\sup |f(x)| \lesssim \|f\|_{B^k(M_\varphi, F)} \lesssim \|f\|_{B^k(E,F)}$, which together with (3.10) imply (1.10).

Moreover, if $f \in B^k(E, F)$ then f is continuous: $\omega(t^{1/n}, f) \to 0$ as $t \to 0$. Indeed, by Marchaud's inequality and (3.9),

$$\omega(t^{1/n}, f) \lesssim t^{1/n} \left(\int_0^t \frac{\omega_{M_\varphi}^k(u^{1/n}, f)}{\varphi(u)} \frac{du}{u} + \int_t^\infty \frac{\omega_{M_\varphi}^k(u^{1/n}, f)}{\varphi(u)} u^{-1/n} \frac{du}{u} \right).$$

Let $0 < t < 1$. Clearly,

$$\int_1^\infty \frac{\omega_{M_\varphi}^k(u^{1/n}, f)}{\varphi(u)} u^{-1/n} \frac{du}{u} < \infty.$$

Let

$$h(t) := t^{1/n} \int_t^1 \frac{\omega_{M_\varphi}^k(u^{1/n}, f)}{\varphi(u)} u^{-1/n} \frac{du}{u}.$$

Since $\int_0^1 h(t) \frac{dt}{t} < \infty$ it follows $h(t) = 0(1)$ as $t \to 0$. Therefore

$$\omega(t^{1/n}, f) \lesssim t^{1/n} + o(1),\ t \to 0.$$

Now we prove that (1.10) implies (3.8). To this end we choose the test function

$$f(x) = \int_0^1 \frac{g(u)}{\varphi(u)} \psi\left(|x| u^{-1/n}\right) \frac{du}{u},$$

where $g \in M_1$ and $\psi \ge 0$ is in C_0^∞ such that $\psi(|x|) = 1$ for $|x| \le 1/2$ and $\psi(|x|) = 0$ if $|x| \ge 1$.

Then

$$\|f\|_{\Lambda_\varphi} \lesssim \int_0^1 \frac{g(u)}{\varphi(u)} \|\psi(|x| u^{-1/n})\|_{\Lambda_\varphi} \frac{du}{u},$$

hence

$$\|f\|_E \lesssim \|f\|_{\Lambda_\varphi} \lesssim \int_0^1 g(u)\frac{du}{u}. \tag{3.11}$$

Therefore, using also (2.17), we get

$$\|f\|_E \lesssim \rho_F(g).$$

Let $|h| = t^{1/n}$, $0 < t < 1$. We estimate $|\Delta_h^k \psi(|x|u^{-1/n})|$ from below for $x = 0$ and $u < t$. Namely, we have

$$\frac{g(ct)}{\varphi(ct)} + \omega^k(t^{1/n}, f) \gtrsim R_\varphi g(t),\ 0 < t < 1/c,\ c = (2k)^n \tag{3.12}$$

and

$$\int_0^1 \frac{g(t)}{\varphi(t)}\frac{dt}{t} + \omega^k(t^{1/n}, f) \gtrsim R_\varphi g(t),\ 1/c < t < 1. \tag{3.13}$$

Further, we use (3.7) and the same arguments as in the proof of Theorem 2.1 and conclude that (1.10) implies (3.8) due to (3.12), (2.11) and (3.11). □

Theorem 3.3 (limiting case)**.** *Let $E = M_\varphi$, $\alpha_F = k/n \geq \alpha_\varphi$ and let (1.4), (3.6) be satisfied, $0 < \alpha_\varphi < 1$. Then the couple ρ_F, ρ_H is admissible if and only if (3.8) is satisfied for all $g \in M_2$, where M_2 is the set of all $g \in \mathbf{M}^+$ with $g(t)$ increasing and $t^{-k/n}g(t)$ decreasing as well as $R_\varphi g(t) < \infty$.*

Proof. The arguments are the same as in the proof of Theorem 3.2, using also (2.14). □

3.3 Optimal quasi-norms

Here we give a characterization of the optimal domain and optimal target quasi-norms when $\alpha_F \geq \alpha_\varphi$, hence $\alpha_\varphi \leq k/n$. In the limiting case we also require $\alpha_\varphi < 1$.

We can define an optimal domain quasi-norm by using Theorem 3.2 or Theorem 3.3. Let $S = M_1$ in the non-limiting case and $S = M_2$ in the limiting cases.

Definition 3.1 (construction of the optimal target quasi-norm)**.** For a given domain quasi-norm ρ_F we set

$$\rho_{H(F)}(g) := \inf\{\rho_F(h) : g \leq R_\varphi h,\ h \in S\},\ g \in A.$$

Theorem 3.4. *Let $\alpha_F = \beta_F \geq \alpha_\varphi$ and let ρ_F satisfy (1.4), (2.17).*

Then $\rho_{H(F)}$ satisfies (1.7), the couple $\rho_F, \rho_{H(F)}$ is admissible, satisfies (3.7), the target quasi-norm is optimal and $h_{H(F)}(u) \leq h_F(u)h_\varphi(1/u)$, therefore $\alpha_{H(F)} = \beta_{H(F)} = \alpha_F - \alpha_\varphi$.

Moreover, if $\alpha_F > \alpha_\varphi$, then the couple is optimal and

$$\rho_{H(F)}(\chi_{(0,1)}g) \approx \rho_F(\chi_{(0,1)}\varphi g).$$

Proof. The proof follows by arguments similar to those in the proof of Theorem 2.3. To prove optimality of the couple when $\alpha_F > \alpha_\varphi$, let $g \le R_\varphi h$. Then $\rho_F(\varphi g) \le \rho_F(\varphi R_\varphi h) \lesssim \rho_F(h)$, whence $\rho_F(\varphi g) \lesssim \rho_{H(F)}(g)$. On the other hand, $g \lesssim R_\varphi(\varphi g)$, whence $\rho_{H(F)}(g) \lesssim \rho_F(\varphi g)$. Finally, since

$$\rho_{F(H(F))}(g) = \rho_{H(F)}(R_\varphi g) \gtrsim \rho_{H(F)}(g/\varphi) \gtrsim \rho_F(g),\ g \in L_m,$$

it follows that the domain quasi-norm is also optimal. □

Definition 3.2 (construction of an optimal domain quasi-norm)**.** For a given target quasi-norm ρ_H, satisfying Minkowski's inequality, (1.7) and $\alpha_H \le k/n - \alpha_\varphi$, we put

$$\rho_{F(H)}(g) := \rho_H(R_\varphi g),\ g \in S.$$

Theorem 3.5. *Let $\alpha_H = \beta_H \le k/n - \alpha_\varphi$, $\alpha_\varphi < k/n$, and let (1.7) be satisfied for H. Then $\rho_{F(H)}$ satisfies (1.4), (3.6), (3.7), it is an optimal domain quasi-norm and $h_{F(H)}(u) \le h_\varphi(u)h_H(u)$, therefore $\alpha_{F(H)} = \beta_{F(H)} = \alpha_H + \alpha_\varphi$. Moreover, this couple is optimal in the non-limiting case.*

Proof. The proof is similar to that of Theorem 2.4. We only need to prove (1.4) and optimalitty of ρ_H. We have

$$\rho_{F(H)}\left(\chi_{(a,\infty)}\right) = \rho_H\left(\int_0^t \frac{\chi_{(a,\infty)}(u)}{\varphi(u)}\frac{du}{u}\right) \le$$
$$\rho_H(\chi_{(a,\infty)})\int_a^\infty \frac{1}{\varphi(u)}\frac{du}{u} \lesssim \frac{1}{\varphi(a)}\rho_H(\chi_{(a,\infty)}).$$

The other condition in (1.4) follows from (1.7). To check optimality of ρ_H, let $g \in A$, then by definition, $g(t) = \frac{1}{t}\int_0^t h(u)du$, h is increasing and $h(+0) = 0$. Hence g is increasing, equivalent to h and $g(+0) = 0$. If $h_1(t) := tg'(t)$, then $g = R_\varphi(\varphi h_1)$. Moreover, $th_1(t)$ is increasing, since $th_1(t) = h(t) - g(t) = \int_0^t udh(u)$. Therefore $\varphi h_1 \in M_1$ and $\rho_{H(F(H))}(g) \le \rho_{F(H)}(\varphi h_1) = \rho_H(g)$.

□

Now we give examples. In the limiting case $\alpha_F = k/n$, we suppose that $0 < \alpha_\varphi < 1$ and $\alpha_\varphi \le k/n$.

Example 3.1. The couple $F = L_*^q(w)$, $H = L_*^q(\varphi w)$, $\alpha_F > \alpha_\varphi$, satisfying (1.4), (3.6), (3.7) is optimal. In particular, we can take $w(t) = t^{-s/n}b(t)$, b slowly varying, $s/n > \alpha_\varphi$.

Example 3.2. Consider the space $H = L_*^1(v)$, satisfying (1.7) and $\beta_H = \alpha_H \le k/n - \alpha_\varphi$, $\alpha_\varphi < k/n$. Using Theorem 3.5, we can construct an optimal domain quasi-norm

$$\rho_F(g) = \rho_H(R_\varphi g) = \int_0^\infty v(t)\left(\int_0^t \frac{g(u)}{\varphi(u)}\frac{du}{u}\right)\frac{dt}{t} = \int_0^\infty w(t)\frac{g(t)}{\varphi(t)}\frac{dt}{t},$$

where $w(t) = \int_t^\infty v(u)\frac{du}{u}$. Hence $F = L_*^1(w/\varphi)$. If v is slowly varying, then $\alpha_H = \beta_H = 0$ and $\alpha_F = \beta_F = \alpha_\varphi$, i.e. this is a critical case. Moreover, this couple is optimal.

Example 3.3. Let $F = L^1(1/\varphi)$ satisfy (1.4), (3.6), (3.7) with $H = L^\infty$ and $\beta_H = \alpha_H \le k/n - \alpha_\varphi$, $\alpha_\varphi < k/n$. Then this couple is optimal.

Example 3.4. Let $H = L^\infty_*(v)$ satisfy (1.7) and $\beta_H = \alpha_H \le k/n - \alpha_\varphi$, $\alpha_\varphi < k/n$. Then

$$\rho_{F(H)}(g) = \sup v(t) \int_0^t \frac{g(u)}{\varphi(u)} \frac{du}{u}.$$

If v is slowly varying, then $\alpha_H = \beta_H = 0$, $\alpha_{F(H)} = \beta_{F(H)} = \alpha_\varphi$ and the couple is optimal.

3.4 Critical case

Here we use the same technique as in Section 2.5. First we construct the required couples of Muckenhoupt weights. Let a slowly varying function $b(t)$ satisfy the following properties:

$$b(t) \text{ is non-increasing, } b(t^2) \approx b(t),\ b(t) = 0 \text{ if } t \ge 1 \tag{3.14}$$

and for some $\varepsilon > 0$,

$$(1 - \ln t)^{-1-\varepsilon} b(t) \text{ is non-increasing if } 0 < t < 1. \tag{3.15}$$

Let

$$c(t) = \frac{b(t)}{1 + |\ln t|)}. \tag{3.16}$$

Then

$$\int_0^t \frac{1}{b(u)} \frac{du}{u} \lesssim \frac{1}{c(t)},\ 0 < t < 1.$$

Indeed, we can write:

$$\int_0^t \frac{1}{b(u)} \frac{du}{u} = \int_0^t \frac{(1 - \ln u)^{-1-\varepsilon}}{b(u)(1 - \ln u)^{-1-\varepsilon}} \frac{du}{u} \lesssim \frac{1}{c(t)}.$$

by using monotonicity property (3.15).

Theorem 3.6. *Let ρ_T be a monotone quasi-norm on $\mathbf{M}^+$ with $\beta_T < 1$, satisfying Minkowski's inequality. Let b, c be given by (3.14) - (3.16). Let ρ_F be defined by*

$$\rho_F(g) := \rho_S(bg/\varphi),$$

$$S := (L^1_*, L^\infty_*)_{T(\frac{1}{t})}.$$

Let $0 < \alpha_\varphi < k/n$. Then the optimal target quasi-norm is given by

$$\rho_{H(F)}(g) := \rho_S(gc).$$

Proof. The operator R_φ, defined by (3.2) is bounded in the following couple of spaces:

$$R : L^1_*(b/\varphi) \mapsto L^\infty_b \text{ and } R_\varphi : L^\infty_*(b/\varphi) \mapsto L^\infty_c,$$

where b, c are given by (3.14), (3.16).

By interpolation,

$$R : F_1 \mapsto H_1,$$

where

$$F_1 := (L^1_*(b/\varphi), L^\infty_*(b/\varphi))_{T(\frac{1}{t})}, \; H_1 := (L^\infty_b, L^\infty_c)_{T(\frac{1}{t})}.$$

Denote the quasi-norm in F_1 by ρ_F. We have

$$\rho_F(g) = \rho_S(bg/\varphi) = \rho_T((bg/\varphi)^{**}_\mu) \approx \rho_T((bg/\varphi)^*_\mu).$$

Hence ρ_F is a monotone quasi-norm and $\alpha_F = \beta_F = \alpha_\varphi$, since $\alpha_S = \beta_S = 0$ and b is slowly varying. Also, (1.4), (3.6) are satisfied.

Analogously to the proof of Theorem 2.6, we characterize the space H_1 and define the quasi-norm $\rho_H(g) := \rho_S(gc)$, hence

$$\rho_H(R_\varphi g) \lesssim \rho_{H_1}(R_\varphi g) \lesssim \rho_F(g), \; g \in M_1.$$

Theorem 3.2 shows that the couple ρ_F, ρ_H is admissible. Finally, arguments similar to those in the proof of Theorem 2.6 show that ρ_H is an optimal target quasi-norm. We only note that if $b(t)h(t)/\varphi(t) = g(\sqrt{te})c(\sqrt{t})$ for $0 < t < 1$ and $h(t) = \varphi(t)g(2t)$ for $t \geq 1$, $g \in A$, then $h \in M_1$ and $R_\varphi h \gtrsim g(t)$. $\square$

Example 3.5. Let $0 < \alpha_\varphi < k/n$. Let $H = L^q_*(c)$, $1 < q < \infty$, $F = L^q_*(b/\varphi)$, where b and c are slowly varying on $(0,1)$, $b(t^2) \approx b(t)$, $b(t) \lesssim (1 + |\ln t|)c(t)$, $c(t) = 0$ for $t \geq 1$ and

$$\left(\int_t^1 c^q(s)\frac{ds}{s}\right)^{1/q} \left(\int_0^t [b(s)]^{-r}\frac{ds}{s}\right)^{1/r} \lesssim 1, \; 1/q + 1/r = 1, \; 0 < t < 1.$$

Then the couple F, H is admissible by [30] and using the same argument as above, we see that H is an optimal target space.

Acknowledgments

We are grateful to the referee for the references [15], [16] and for the useful remarks.

References

[1] I. Ahmed, G.E. Karadzhov, *Optimal embeddings of generalized homogeneous Sobolev spaces.* C.r. de l'Acad. bulgare des Sci., 61 (2008), 967 – 972.

[2] Z. Bashir, G.E. Karadzhov, , *Optimal embeddings of generalized Besov spaces.* C.r. de l'Acad. bulgare des Sci., 63 (2010), 799 – 806.

[3] C. Bennett, R. Sharpley, *Interpolation of operators.* Academic Press, New York, 1988.

[4] J. Berg, J. Löfström, *Interpolation spaces. An introduction.* Springer-Verlag, New York, 1976.

[5] Ju.A. Brudnyi, N.Ja. Krugliak, *Interpolation spaces and interpolation functors.* North-Holland, Amsterdam, 1991.

[6] A. Cianchi, *Symmetrization and second order Sobolev inequalities.* Annali di Matem., 183 (2004), 45 – 77.

[7] M. Cwikel, E. Pustilnik, *Sobolev type embeddings in the limiting case.* J. Fourier Anal. Appl., 4 (1998), 433 – 446.

[8] M. Cwikel, E. Pustilnik, *Weak type interpolation near "endpoint" spaces.* J. Funct. Anal., 171 (2000), 235 – 277.

[9] D.E. Edmunds, R. Kerman, L. Pick, *Optimal Sobolev embeddings involving rearrangement invariant quasinorms.* J. Funct. Anal., 170 (2000), 307 – 355.

[10] D.E. Edmunds, H. Triebel, *Sharp Sobolev embeddings and related Hardy inequalities: the critical case.* Math. Nachr., 207 (1999), 79 – 92.

[11] A. Gogatishvili, J.S. Neves, B. Opic, *Optimal embeddings of Bessel-potential-type spaces into generalized Hölder spaces involving $k-$modulus of smoothness*, preprint, 2008.

[12] A. Gogatishvili, V.I. Ovchinnikov, *Interpolation orbits and optimal Sobolev's embeddings.* J. Funct. Anal., 253 (2007), 1–17.

[13] A. Gogatishvili, V.I. Ovchinnikov, *The optimal embedding for the Calderon type spaces and the $J-$method spaces*, preprint, 2008.

[14] M.L. Goldman, *On embedding for different metrics of Calderon type spaces.* Trudy Mat. Inst. Steklova, Akad. Nauk SSSR, 181 (1988), 70 – 94 (in Russian).

[15] M.L. Goldman, R.A. Kerman, *On optimal embedding of Calderon spaces and generalized Besov spaces.* Proc. Steklov Inst. Math., 243 (2003), 154 – 184.

[16] M.L. Goldman, *Rearrangement invariant envelopes of generalized Besov, Sobolev, and Calderon spaces.* Contemporary Math., 424 (2007), 53 – 81.

[17] P. Gurka, B. Opic, *Sharp embeddings of Besov spaces with logarithmic smoothness.* Rev. Mat. Complut., 18 (2005), 81 – 110.

[18] K. Hansson, *Imbedding theorems of Sobolev type in potential theory.* Math. Scand., 45 (1979), 77 – 102.

[19] D.D. Haroske, S.D. Moura, *Continuity envelopes and sharp embeddings in spaces of generalized smoothness.* J. Funct. Anal., 254 (2008), 1487 – 1521.

[20] R. Kerman, L. Pick, *Optimal Sobolev imbeddings.* Forum Math., 18 (2006), 535 – 579.

[21] V.I. Kolyada, *Rearrangements of functions and embedding theorems.*Russian Math. Surveys, 44 (1989), 73 – 117.

[22] G. Köthe, *Topologische lineare Räume.* Springer, Berlin, 1966.

[23] S.G. Krein, U.I. Petunin, E.M. Semenov, *Interpolation of linear operators.* Nauka, Moscow, 1978 (in Russian). English transl. Amer. Math. Soc., Providence, 1982.

[24] J. Maly, L. Pick, *An elementary proof of sharp Sobolev embeddings.* Proc. Amer. Math. Soc., 130 (2002), 555 – 563.

[25] J. Martin, M. Milman, *Symmetrization inequalities and Sobolev embeddings.* Proc. Amer. Math. Soc., 134 (2006), 2235 – 2247.

[26] J. Martin, M. Milman, *Higher orger symmetrization inequalities and applications.* J. Math. Anal. and Appl., 330 (2007), 91 – 113.

[27] J. Martin, M. Milman, *Self-improving Sobolev-Poincare inequalities and symmetrization.* Potential Anal., 29 (2008), 391 – 408.

[28] J. Martin, M. Milman, E. Pustylnik, *Self improving Sobolev inequalities, truncation and symmetrization.* J. Funct. Anal., 252 (2007), 677 – 695.

[29] M. Milman, E. Pustylnik, *On sharp higher order Sobolev embeddings.* Comm. Contemp. Math., 6 (2004), 495 – 511.

[30] B. Muckenhoupt, *Hardy's inequality with weights.* Studia Math., XLIV (1972), 31 – 38.

[31] Y.V. Netrusov, *Embedding theorems for Besov spaces in symmetric spaces.* Notes Sci. Sem. LOMI, 159 (1987), 69 – 102.

[32] S.M. Nikolskii, *Approximation of functions and embedding theorems.* Nauka, Moscow, 1977 (in Russian). English transl. Springer-Verlag, New York – Heidelberg, 1975.

[33] H. Triebel, *Theory of function spaces III.* Birkhäuser, Basel, 2006.

Zia Bashir
Abdus Salam School of Mathematical Sciences
GC University Lahore
68-B, New MuslimTown
54660 Lahore, Pakistan
E-mail: ziabashir@gmail.com

Georgi Karadzhov
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
8 Acad. G. Bonchev St,
1113 Sofia, Bulgaria
&
Abdus Salam School of Mathematical Sciences
GC University Lahore
68-B, New MuslimTown
54660 Lahore, Pakistan
E-mail: geremika@yahoo.com

Received: 02.10.2010