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HARDY-TYPE INEQUALITY FOR $0 < p < 1$ AND HYPODECREASING FUNCTIONS

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Abstract. The notion of a hypodecreasing function is introduced. Some properties of hypodecreasing functions are proved and several examples are given. It is established that the Hardy-type inequality for L_p-spaces with $0 < p < 1$ is satisfied for some spaces of hypodecreasing functions. The obtained result is in a certain sense sharp.

1 Introduction

It is well known that for L_p-spaces with $0 < p < 1$ the Hardy inequality is not satisfied for arbitrary non-negative measurable functions, but it is satisfied for non-negative non-increasing functions. Moreover, in [3], pp. 90-91, the sharp constant in the Hardy-type inequality for non-negative non-increasing functions was found. (See [4] for more details.) Namely the following statement was proved there.

Theorem 1. *Let $0 < p < 1$ and $-\frac{1}{p} < \alpha < 1-\frac{1}{p}$. Then for all functions f non-negative and non-increasing on $(0, \infty)$*

$$\left\| x^{\alpha-1} \int_0^x f dy \right\|_{L_p(0,\infty)} \leq \left(1 - \frac{1}{p} - \alpha\right)^{-\frac{1}{p}} \|x^\alpha f(x)\|_{L_p(0,\infty)} , \tag{1}$$

and the constant $(1 - \frac{1}{p} - \alpha)^{-\frac{1}{p}}$ is sharp.

Remark 1. *If $\alpha \geq 1-\frac{1}{p}$, then there exists a function f non-negative and non-increasing on $(0, \infty)$ such that $\|x^\alpha f(x)\|_{L_p((0,\infty))} < \infty$, but $\|x^{\alpha-1} f(x)\|_{L_p((0,\infty))} = \infty$. (For example, this holds for any function f non-negative non-increasing continuous on $[0, \infty)$ which is not equivalent to zero and is such that $\operatorname{supp} f \subset [0, \infty)$.) If $\alpha \leq -\frac{1}{p}$, then for each function which is non-negative non-increasing on $(0, \infty)$ and is not identically equal to zero $\|x^\alpha f(x)\|_{L_p((0,\infty))} = \infty$.*

Later inequalities of type (1) were proved in [1], [2] for non-negative quasi-decreasing functions, also with sharp constants.

In [7] the Hardy-type inequality for $0 < p < 1$ was proved under weaker assumptions on f but still of monotonicity type. The result was proved for the n-dimensional variant of the Hardy operator, namely for the operator H defined for all functions $f \in L_1^{loc}(\mathbb{R}^n)$ by

$$(Hf)(t) = \frac{1}{v_n t^n} \int_{B_t} f dy, \quad 0 < t < \infty,$$

where B_t is the ball centered at the origin of radius r and v_n is the volume of the unit ball in $\mathbb{R}^n$.

Theorem 2. *Let $0 < p < 1, \alpha < n - \frac{1}{p}$ and $M > 0$.*

Moreover, let f be a function non-negative measurable on $\mathbb{R}^n$ such that $\|f(x)|x|^{\frac{n}{p'}}\|_{L_p(B_r)} < \infty$ and

$$\|f(x)\|_{L_1(B_r)} \leq M \|f(x)|x|^{\frac{n}{p'}}\|_{L_p(B_r)} \tag{2}$$

for all $r > 0$, where $p' = \frac{p}{p-1}$.

Then

$$\|t^\alpha (Hf)(t)\|_{L_p((0,\infty))} \leq N \|f(x)|x|^{\alpha - \frac{n-1}{p}}\|_{L_p(\mathbb{R}^n)}\,, \tag{3}$$

where

$$N = v_n^{-1}((n-\alpha)p - 1)^{-\frac{1}{p}} M. \tag{4}$$

The aim of this paper is finding a still wider space of functions such that for all functions f in this space inequality (3) is satisfied with some $N > 0$, independent of f. It will be shown that for $0 < p < 1$ a slightly stronger version of inequality (3) is itself a certain condition of monotonicity type on a function f.

2 Spaces of hypodecreasing functions

Definition 1. *Let $0 < p < 1, \alpha \in \mathbb{R}$ and $M > 0$. We say that a function f is hypodecreasing with the parameters p, α and M if f is a function non-negative and measurable on $\mathbb{R}^n$ for which*

$$\|f(x)|x|^{\alpha - \frac{n-1}{p}}\|_{L_p(B_r)} < \infty$$

and

$$\|f\|_{L_1(B_r)} \leq M r^{n - \frac{1}{p} - \alpha} \|f(x)|x|^{\alpha - \frac{n-1}{p}}\|_{L_p(B_r)} \tag{5}$$

for all $r > 0$.

We also say that a function f is hypodecreasing with the parameters p and α if for some $M > 0$ it is hypodecreasing with the parameters p, α and M.

In order to simplify the formulation of the main result it is convenient to agree that if the right hand side of inequality (5) is infinite, then this inequality is satisfied, and to denote by $HD_p^\alpha(M)$ the space of all functions f non-negative and measurable on $\mathbb{R}^n$ for which inequality (5) holds for all $0 < r < \infty$. So the space $HD_p^\alpha(M)$ contains all functions f hypodecreasing with the parameters p, α and M and also all functions

f non negative and measurable on $\mathbb{R}^n$ for which $\|f(x)|x|^{\alpha-\frac{n-1}{p}}\|_{L_p(B_r)} = \infty$ for all $0<r<\infty$. We also set

$$HD_p^\alpha = \bigcup_{M>0} HD_p^\alpha(M).$$

Remark 2. *If $\|f(x)|x|^{\alpha-\frac{n-1}{p}}\|_{L_p(\mathbb{R}^n)} < \infty$ and $\alpha > n-\frac{1}{p}$, then inequality (5) holds for all $r>0$ only if f is equivalent to zero on $\mathbb{R}^n$. This follows by passing to the limit as $r\to\infty$.*

Remark 3. *If* [1] *$\alpha \le n-\frac{1}{p}$, then inequality (5) implies that for all $0<\rho\le\infty$ and $r>0$*

$$\|f\chi_{B_\rho}\|_{L_1(B_r)} \le M r^{n-\frac{1}{p}-\alpha}\|f(x)\chi_{B_\rho}(x)|x|^{\alpha-\frac{n-1}{p}}\|_{L_p(B_r)}.$$

For $\rho=\infty$ this is inequality (5). If $\rho<\infty$ and $0<r\le\rho$, it again coincides with inequality (5). If $\rho<r<\infty$, then by inequality (5) with $r=\rho$

$$\|f\chi_{B_\rho}\|_{L_1(B_r)} = \|f\|_{L_1(B_r)} \le M\rho^{n-\frac{1}{p}-\alpha}\|f(x)|x|^{\alpha-\frac{n-1}{p}}\|_{L_p(B_\rho)}$$

$$\le M r^{n-\frac{1}{p}-\alpha}\|f(x)\chi_{B_\rho}(x)|x|^{\alpha-\frac{n-1}{p}}\|_{L_p(B_r)},$$

because $n-\frac{1}{p}-\alpha\ge 0$.

Thus for $\alpha\le n-\frac{1}{p}$

$$f\in HD_p^\alpha(M) \Longrightarrow f\chi_{B_r}\in HD_p^\alpha(M)$$

for all $0<r<\infty$.

Remark 4. *If $p\ge 1$ and $\alpha<n-\frac{1}{p}$ for $p>1$ $\alpha\le n-1$ for $p=1$, then by Hölder's inequality*

$$\|f\|_{L_1(B_r)} \le \||x|^{-\alpha+\frac{n-1}{p}}\|_{L_{p'}(B_r)}\|f(x)|x|^{\alpha-\frac{n-1}{p}}\|_{L_p(B_r)}$$

$$= c_1 r^{n-\frac{1}{p}-\alpha}\|f(x)|x|^{\alpha-\frac{n-1}{p}}\|_{L_p(B_r)},$$

where

$$c_1 = \sigma_n^{\frac{1}{p'}}\left[\left(\frac{n-1}{p}-\alpha\right)p'+n\right]^{-\frac{1}{p'}}$$

if $p>1$ and $c_1=1$ if $p=1$. Here $\sigma_n = nv_n$ is the surface area of the unit sphere in $\mathbb{R}^n$.

Next note that

$$\sup \frac{\|f\|_{L_1(B_r)}}{r^{n-\frac{1}{p}-\alpha}\|f(x)|x|^{\alpha-\frac{n-1}{p}}\|_{L_p(B_r)}} = \begin{cases} c_1 & \text{if} \quad 1<p\le\infty, \alpha<n-\frac{1}{p}, \\ 1 & \text{if} \quad p=1, \alpha\le n-1, \\ \infty & \text{if} \quad 0<p<1, \alpha\in\mathbb{R}, \end{cases}$$

[1] We are mostly interested in the case $\alpha\le n-\frac{1}{p}$, because the formulation of the main result in Section 4 contains this assumption.

where the supremum is taken with respect to all $r > 0$ and all non-negative measurable functions f for which

$$0 < \|f(x)|x|^{\alpha - \frac{n-1}{p}}\|_{L_p(B_r)} < \infty$$

for all $r > 0$.

The statement for $1 < p \le \infty$ follows since for $f(x) = |x|^{(-\alpha + \frac{n-1}{p})p'}$ the above inequality turns in an equality. The statement for $p = 1$ follows if one takes $f = e^{\nu|x|}$, $1 \le \nu < \infty$ and passes to limit as $\nu \to \infty$. The statement for $0 < p < 1$ follows if one takes $r = 1, f = \chi_{B_1 \setminus B_\nu}, 0 < \nu < 1$, and passes to the limit as $\nu \to 1^-$. Therefore for $1 < p \le \infty, \alpha < n - \frac{1}{p}$ or $p = 1, \alpha \le n - 1$ inequality (5) with $M \ge c_1$ does not impose any further restrictions on a function f for which $\|f(x)|x|^{\alpha - \frac{n-1}{p}}\|_{L_p(B_r)} < \infty$ for all $r > 0$, whilst for $0 < p < 1$ it imposes further restrictions for any $\alpha \in \mathbb{R}$ and $M > 0$.

Remark 5. *In terms of the introduced definition, in Theorem 2 for $0 < p < 1$ and $\alpha < n - \frac{1}{p}$ Hardy-type inequality (3) is proved under the assumption that $f \in HD_p^{n-\frac{1}{p}}(M)$.*

Remark 6. *The spaces $HD_p^\alpha(M)$ possess monotonicity properties in the indices p and α. Namely, if $0 < p < q < 1, \alpha \in \mathbb{R}, M > 0$, then*

$$HD_p^\alpha(M) \subset HD_q^\alpha(\sigma_n^{\frac{1}{p} - \frac{1}{q}} M), \tag{6}$$

and if $0 < p < 1, \alpha \in \mathbb{R}, \beta < \alpha$, then

$$HD_p^\alpha(M) \subset HD_p^\beta(M). \tag{7}$$

More generally, if $0 < \alpha \le q < 1$ and $\beta < \alpha + \frac{1}{p} - \frac{1}{q}$, then

$$HD_p^\alpha(M) \subset HD_q^\beta(c_2 M), \tag{8}$$

where

$$c_2 = \sigma_n^{\frac{1}{p} - \frac{1}{q}} \left[(\alpha - \beta) \left(\frac{1}{p} - \frac{1}{q} \right)^{-1} + 1 \right]^{\frac{1}{q} - \frac{1}{p}}.$$

Inclusion (8) implies that

$$HD_p^\alpha \subset HD_q^\beta. \tag{9}$$

Indeed, let $f \in HD_p^\alpha(M)$. Then by Hölder's inequality with the exponent $\frac{q}{p} > 1$

$$\|f(x)|x|^{\alpha - \frac{n-1}{p}}\|_{L_p(B_r)} \le \||x|^{\alpha - \beta - (n-1)(\frac{1}{p} - \frac{1}{q})}\|_{L_s(B_r)} \|f(x)|x|^{\beta - \frac{n-1}{q}}\|_{L_q(B_r)}$$

where $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$. Since

$$\||x|^{\alpha - \beta - (n-1)(\frac{1}{p} - \frac{1}{q})}\|_{L_s(B_r)} = c_2 r^{\alpha - \beta + \frac{1}{p} - \frac{1}{q}},$$

we have

$$\begin{aligned} \|f\|_{L_1(B_r)} &\le M r^{n - \frac{1}{p} - \alpha} \|f(x)|x|^{\alpha - \frac{n-1}{p}}\|_{L_p(B_r)} \\ &\le c_2 M r^{n - \frac{1}{q} - \beta} \|f(x)|x|^{\beta - \frac{n-1}{q}}\|_{L_q(B_r)}, \end{aligned}$$

hence inclusion (8) follows.

Lemma. *Let $0<p<\infty, \alpha\in\mathbb{R}, c_3>0$, and let f be a function non-negative and measurable on $\mathbb{R}^n$ such that $\|f(y)|y|^{\alpha-\frac{n-1}{p}}\|_{L_p(B_r)}<\infty$ for all $r>0$ and*

$$f(x)\le\frac{c_3}{|x|^{\alpha+\frac{1}{p}}}\|f(y)|y|^{\alpha-\frac{n-1}{p}}\|_{L_p(B_{|x|})} \tag{10}$$

for almost all $x\in\mathbb{R}^n$.

1. *If $\alpha\le n-\frac{1}{p}$, then inequality* (5) *is satisfied with*

$$M=pc_3^{1-p}. \tag{11}$$

(*Hence $f\in HD_p^{\alpha}(pc_3^{1-p})$.*)

2. *If $\alpha=n-\frac{1}{p}$, then this constant is sharp.*

3. *If $\alpha>n-\frac{1}{p}$, then inequality* (10) *does not imply inequality* (5) *with any $M>0$ independent of f and r.*

Proof. 1. Let $\alpha\le n-\frac{1}{p}$. Note that for all $r>0$ for almost all $x\in B_r$

$$\begin{aligned} f(x)&=(f(x)|x|^{\alpha+\frac{1}{p}})^{1-p}(f^p(x)|x|^{\alpha p-n+1})|x|^{n-\frac{1}{p}-\alpha}\\ &\le r^{n-\frac{1}{p}-\alpha}(f(x)|x|^{\alpha+\frac{1}{p}})^{1-p}(f^p(x)|x|^{\alpha p-n+1})\,, \end{aligned} \tag{12}$$

hence by (10)

$$f(x)\le c_3^{1-p}r^{n-\frac{1}{p}-\alpha}\left(\int_{B_{|x|}}f^p(y)|y|^{\alpha p-n+1}dy\right)^{\frac{1}{p}-1}f^p(x)|x|^{\alpha p-n+1}. \tag{13}$$

Integrating over B_r and taking thrice the spherical coordinates we have

$$\|f\|_{L_1(B_r)}\le c_3^{1-p}r^{n-\frac{1}{p}-\alpha}\sigma_n I\,,$$

where

$$I=\int_0^r\left(\int_{B_\rho}f^p(y)|y|^{\alpha p-n+1}dy\right)^{\frac{1}{p}-1}\left(\int_{S^{n-1}}f^p(\rho(\sigma)d\sigma)\right)\rho^{p\alpha}d\rho\,.$$

Moreover, $I=\sigma_n^{\frac{1}{p}-1}J$, where

$$J=\int_0^r\left(\int_0^\rho\left(\int_{S^{n-1}}f^p\left(t\sigma\right)d\sigma\right)t^{p\alpha}dt\right)^{\frac{1}{p}-1}\left(\int_{S^{n-1}}f^p(\rho(\sigma)d\sigma)\right)\rho^{p\alpha}d\rho.$$

Therefore

$$\begin{aligned} &\|f\|_{L_1(B_r)}\\ &\le pc_3^{1-p}r^{n-\frac{1}{p}-\alpha}\sigma_n^{\frac{1}{p}}\int_0^r\left[\left(\int_0^\rho\left(\int_{S^{n-1}}f^p(t\sigma)d\sigma\right)t^{p\alpha}dt\right)^{\frac{1}{p}}\right]'d\rho \end{aligned}$$

$$= pc_3^{1-p} r^{n-\frac{1}{p}-\alpha} \left(\sigma_n \int_0^r \int_0^r \left(\int_{S^{n-1}} f^p(t\sigma) d\sigma \right) t^{p\alpha} dt \right)^{\frac{1}{p}}$$

$$= pc_3^{1-p} r^{n-\frac{1}{p}-\alpha} \left(\int_{B_r} f^p(x) |x|^{\mu p - n + 1} dx \right)^{\frac{1}{p}}$$

$$= pc_3^{1-p} r^{n-\frac{1}{p}-\alpha} \| f(x) |x|^{\alpha - \frac{n-1}{p}} \|_{L_p(B_r)}.$$

Hence the first statement follows.

2. By the above argument it follows that if in inequalities (12) and (13) there are equalities, then

$$\|f\|_{L_1(B_r)} = pc_3^{1-p} \| f(x) |x|^{\alpha - \frac{n-1}{p}} \|_{L_p(B_r)},$$

which implies that inequality (5) is satisfied with $M = pc_3^{1-p}$ and is not satisfied with any $M < pc_3^{1-p}$. Hence the constant pc_3^{1-p} is sharp if there exists a function $f \in L_1^{loc}(\mathbb{R}^n)$ for which inequalities (12) and (13) turn in equalities and which is not equivalent to zero on $\mathbb{R}^n$.

3. Let $\alpha = n - \frac{1}{p}$, then there is equality in (12). Assume that $f(x) = g(|x|)$ where g is a non-negative differentiable function on $(0, \infty)$ and equality (12) is satisfied, i.e.

$$g(\rho) = c_3^{1-p} \left(\int_{B_\rho} g(|y|)^p |y|^{n(p-1)} dy \right)^{\frac{1}{p}-1} g(\rho)^p \rho^{n(p-1)}, \qquad 0 < \rho < \infty.$$

By taking the spherical coordinates it follows for all $\rho > 0$

$$(g(\rho)\rho^n)^p = c_3^p \sigma_n \int_0^\rho g^p(t) t^{np-1} dt.$$

By differentiating this equality and carrying out simple calculations it follows that

$$g'(\rho)\rho + a g(\rho) = 0$$

where

$$a = n - \frac{c_3^p \sigma_n}{p}.$$

This equation is satisfied by $g(\rho) = \rho^{-a}$.

Thus equations (12) and (13) are satisfied for $f(x) = |x|^{-a}$ for all $x \in \mathbb{R}^n$, $x \neq 0$. Also $f \in L_1^{loc}(\mathbb{R}^n)$ since $a < n$. Hence the second statement of the lemma follows.

4. Let $\alpha > n - \frac{1}{p}, -\alpha - \frac{1}{p} < \mu < -n$ and $f(x) = |x|^\mu$ for all $x \in \mathbb{R}^n, x \neq 0$. Then $\| f(x) |x|^{\alpha - \frac{n-1}{p}} \|_{L_p(B_r)} < \infty$ for all $r > 0$ because $\alpha - \frac{n-1}{p} > -\frac{n}{p}$, but $\|f\|_{L_1(B_r)} = \infty$ because $\mu < -n$, hence inequality (5) does not hold for this function f for any $M > 0$. □

Remark 7. *In* [7] *for the case* $\alpha = n - \frac{1}{p}$ *by a simpler argument it was proved that under the assumptions of the Lemma inequality* (5) *is satisfied with* $M = c_3^{1-p}$.

3 Examples of hypodecreasing functions

Example 1. *Let $0 < p < 1, \alpha \in \mathbb{R}$ and $M > 0$. Then the function $|x|^\mu, \mu \in \mathbb{R}$ is hypodecreasing with the parameters p and α if and only if $\mu > \max\{-n, -\alpha - \frac{1}{p}\}$ and it is hypodecreasing with the parameters p and α and M if only if $\mu > \max\{-n, -\alpha - \frac{1}{p}\}$ and $M \geq c_3$, where*

$$c_3 = \frac{\sigma_n^{1-\frac{1}{p}}[(\mu+\alpha)p+1]^{\frac{1}{p}}}{\mu+n}. \tag{14}$$

This follows since,

$$\| \, |x|^\mu \|_{L_1(B_r)} \Longleftrightarrow \mu > -n, \quad \| |x|^{\mu+\alpha-\frac{n-1}{p}} \|_{L_1(B_r)} < \infty \Longleftrightarrow \mu > -\alpha - \frac{1}{p}$$

and the minimal value of $M > 0$ for which inequality (5) is satisfied is equal to

$$\sup_{r>0} \frac{\| |x|^\mu \|_{L_1(B_r)}}{r^{n-\frac{1}{p}-\alpha} \| |x|^{\mu+\alpha-\frac{n-1}{p}} \|_{L_1(B_r)}} = \frac{\sigma_n}{\mu+n} \left(\frac{\sigma_n}{(\mu+\alpha)p+1} \right)^{-\frac{1}{p}} = c_3.$$

Moreover,[2]

$$|x|^\mu \in HD_p^\alpha \Longleftrightarrow \begin{cases} -\infty < \mu < \infty & \text{if } \alpha \leq n - \frac{1}{p}, \\ \mu \leq -\alpha - \frac{1}{p} \quad \text{or} \quad \mu > -n & \text{if } \alpha > n - \frac{1}{p}. \end{cases} \tag{15}$$

It suffices to take into account that in the case $\mu \leq -\alpha - \frac{1}{p}$ one has $\| |x|^{\mu+\alpha-\frac{n-1}{p}} \|_{L_p(B_r)} = \infty$ for all $0 < r < \infty$, hence inequality (5) is trivially satisfied.

Example 2. *Let $0 < p < 1, \alpha \in \mathbb{R}$. Then the function $|x|^\mu \chi_{B_1}(x)$, $\mu \in \mathbb{R}$, is hypodecreasing with the parameters p and α if only if $\alpha \leq n - \frac{1}{p}$ and $\mu > -\alpha - \frac{1}{p}$, and*

$$|x|^\mu \chi_{B_1}(x) \in HD_p^\alpha \quad \Longleftrightarrow \quad \alpha \leq n - \frac{1}{p} \quad \text{and} \quad -\infty < \mu < \infty.$$

This follows because if $\alpha > n - \frac{1}{p}$, then

$$\sup_{r \geq 1} \frac{\| |x|^\mu \chi_{B_1}(x) \|_{L_1(B_r)}}{r^{n-\frac{1}{p}-\alpha} \| |x|^{\mu+\alpha-\frac{n-1}{p}} \chi_{B_1}(x) \|_{L_1(B_r)}} = \infty.$$

Moreover,

$$|x|^\mu \chi_{c_{B_1}}(x) \notin HD_p^\alpha \quad \text{for all} \quad \mu, \alpha \in \mathbb{R},$$

because

$$\sup_{r>0} \frac{\| |x|^\mu \chi_{c_{B_1}}(x) \|_{L_1(B_r)}}{r^{n-\frac{1}{p}-\alpha} \| |x|^{\mu+\alpha-\frac{n-1}{p}} \chi_{c_{B_1}}(x) \|_{L_p(B_r)}}$$

[2] Recall that we are mostly interested in the case $\alpha \leq n - \frac{1}{p}$.

$$\geq \lim_{r\to 1^+} \frac{\| |x|^{\mu} \|_{L_1(B_r \setminus B_1)}}{r^{n-\frac{1}{p}-\alpha} \| |x|^{\mu+\alpha-\frac{n-1}{p}} \|_{L_1(B_r \setminus B_1)}}$$

$$\geq \lim_{r\to 1^+} \frac{\min\{r^{\mu},1\} v_n (r^n-1)}{r^{n-\frac{1}{p}-\alpha} \max\{r^{\mu+\alpha-\frac{n-1}{p}},1\} v_n^{\frac{1}{p}} (r^n-1)^{\frac{1}{p}}} = \infty .$$

In particular,

$$\chi_{B_1} \in HD_p^{\alpha} \Longleftrightarrow \alpha \leq n - \frac{1}{p} \quad \text{and} \quad \chi_{B_1} \notin HD_p^{\alpha} \quad \text{for all} \quad \alpha \in \mathbb{R} .$$

Moreover, if $-\frac{1}{p} < \alpha \leq n - \frac{1}{p}$ *, then*

$$\chi_{B_1} \in HD_p^{\alpha}(M) \Longleftrightarrow M \geq c_4 ,$$

where

$$c_4 = \sigma_n^{1-\frac{1}{p}} n^{-1} (p\alpha+1)^{\frac{1}{p}} .$$

If $\alpha \leq -\frac{1}{p}$*, then*

$$\chi_{B_1} \in HD_p^{\alpha}(M) \quad \text{for all} \quad M > 0 .$$

Example 3. *If* $0 < p < \infty, \alpha > -\frac{1}{p}$ *and* $f(x) = g(|x|), x \in \mathbb{R}^n$*, where g is a non-negative non-increasing function on* $(0,\infty)$ *such that* $\|g(\rho)\rho^{\alpha}\|_{L_p(0,1)} < \infty$*, then the right hand side of inequality* (10) *is finite for all* $x \neq 0$ *and inequality* (10) *is satisfied with*

$$c_3 = \left(\frac{p\alpha+1}{\sigma_n} \right)^{\frac{1}{p}} .$$

Moreover, this constant is sharp.

Indeed, for all $x \in \mathbb{R}^n, x \neq 0$

$$\begin{aligned} \|f(y)|y|^{\alpha-\frac{n-1}{p}}\|_{L_p(B_{|x|})} &= \|g(|y|)|y|^{\alpha-\frac{n-1}{p}}\|_{L_p(B_{|x|})} \\ &\geq g(|x|) \left(\int_{B_{|x|}} |y|^{p\alpha-n+1} dy \right)^{\frac{1}{p}} \\ &= \left(\frac{\sigma_n}{p\alpha+1} \right)^{\frac{1}{p}} f(x)|x|^{\alpha+\frac{1}{p}}, \end{aligned}$$

and the first statement follows. To verify the second one it suffices to consider $f(x) \equiv 1$.

Example 4. *Let* $0 < p < 1$ *and* $f(x) = g(|x|)$ *where g is non-negative non-increasing on* $(0,\infty)$ *and* $g \not\equiv 0$*. If* $-\frac{1}{p} < \alpha \leq n - \frac{1}{p}$*, then* $f \in HD_p^{\alpha}(c_5)$*, where* $c_5 = p^{\frac{1}{p}} \left(\frac{\alpha+\frac{1}{p}}{\sigma_n} \right)^{\frac{1}{p}-1}$. *If* $\alpha = n - \frac{1}{p}$*, then* $f \in HD_p^{n-\frac{1}{p}}(p^{\frac{1}{p}} v_n^{\frac{1}{p}-1})$ *and there exists a function f satisfying the above conditions such that* $f \notin HD_p^{\alpha}(M)$ *for any* $0 < M < p^{\frac{1}{p}} v_n^{\frac{1}{p}-1}$.

The first statement immediately follows by the Lemma and Example 1*. One should also note that* $1 \in HD_p^{\alpha}(M)$ *if and only if* $M \geq p^{\frac{1}{p}} v_n^{\frac{1}{p}-1}$ *(by Example* 1*).*

Example 5. *Let $\beta \geq 0$. Consider the space of quasi-decreasing function QD^β which consists of all functions f such that $f(x) = g(|x|), x \in \mathbb{R}^n$, where g is a non-negative function on $(0,\infty)$ and such that the function $\rho^{-\beta} g(\rho)$ is non-increasing on $(0,\infty)$. Similarly to Example 3, if $0 < p \leq \infty, \alpha > -\beta - \frac{1}{p}$ and $\|\rho^\alpha g(\rho)\|_{L_p(0,1)} < \infty$, then the right-hand side of inequality (10) is finite for all $x \neq 0$ and inequality (10) is satisfied with*

$$c_3 = \left(\frac{(\alpha+\beta)p + 1}{\sigma_n}\right)^{\frac{1}{p}}.$$

Moreover, this constant is sharp.

Indeed, for all $x \in \mathbb{R}^n, x \neq 0$

$$\begin{aligned} \|f(y)|y|^{\alpha - \frac{n-1}{p}}\|_{L_p(B_{|x|})} &= \|g(|y|)|y|^{-\beta}|y|^{\alpha+\beta-\frac{n-1}{p}}\|_{L_p(B_{|x|})} \\ &\geq g(|x|)|x|^{-\beta}\left(\int_{B_{|x|}} |y|^{p(\alpha+\beta)-n+1} dy\right)^{\frac{1}{p}} \\ &= \left(\tfrac{\sigma_n}{p(\alpha+\beta)+1}\right)^{\frac{1}{p}} f(x)|x|^{\alpha+\frac{1}{p}} \end{aligned}$$

and equality is attained if $f(x) = |x|^\beta$.

Similarly to Example 4, if $0 < p < 1, \alpha \leq n - \frac{1}{p}$, then

$$QD^\beta \subset HD_p^\alpha(c_6), \quad \text{where} \quad c_6 = p^{\frac{1}{p}}\left(\frac{\alpha+\beta+\frac{1}{p}}{\sigma_n}\right)^{\frac{1}{p}-1}, \tag{16}$$

hence

$$QD^\beta \subset HD_p^\alpha. \tag{17}$$

In particular for any $\beta \geq 0$

$$QD^\beta \subset HD_p^{n-\frac{1}{p}}(c_7), \quad \text{where} \quad c_7 = p^{\frac{1}{p}}\left(\frac{n+\beta}{\sigma_n}\right)^{\frac{1}{p}-1}. \tag{18}$$

Also

$$QD^\beta \not\subset HD_p^{n-\frac{1}{p}}(M) \quad \text{for any} \quad 0 < M < c_7. \tag{19}$$

This follows since $|x|^\beta \in QD^\beta$ but by Example 1 $|x|^\beta \notin HD_p^{n-\frac{1}{p}}(M)$ for any $0 < M < c_7$.

Example 6. *Let $0 < p < 1$ and $\alpha \leq n - \frac{1}{p}$. Example 1 shows that a hypodecreasing function with the parameters p and α can be a radially increasing function. Moreover, any power function $|x|^\mu, \mu > 0$, is hypodecreasing with the parameters p and α.*

However, there are restrictions on the rapidness of growth of a hypodecreasing function. For example, if g is a positive continuous function on $(0,\infty)$ for which both limits

$$\lim_{r\to 0^+} \frac{g(r)r^n}{\int_0^r g(\rho)\rho^{n-1}d\rho}, \qquad \lim_{r\to+\infty} \frac{g(r)r^n}{\int_0^r g(\rho)\rho^{n-1}d\rho} \tag{20}$$

are finite, then the function $g(|x|)$ is hypodecreasing with the parameters p and $n-\frac{1}{p}$ (and hence, by Remark 6, also with the parameters p and $\alpha < n-\frac{1}{p}$), but if at least one of these limits is infinite, then it is not hypodecreasing with the parameters p and $n-\frac{1}{p}$.

Indeed, for $\alpha = n-\frac{1}{p}$

$$S^p \equiv \left(\sup_{r>0} \frac{\|g(|x|)\|_{L_1(B_r)}}{\left\|g(|x|)|x|^{\frac{n}{p'}}\right\|_{L_p(B_r)}} \right)^p = \sigma_n^{p-1} \sup_{r>0} \frac{\left(\int_0^r g(\rho)\rho^{n-1}d\rho\right)^p}{\int_0^r g(\rho)^p \rho^{np-1}d\rho}.$$

If both limits (20) are finite, then by the L'Hospital rule

$$\lim_{r\to 0+} \frac{\left(\int_0^r g(\rho)\rho^{n-1}d\rho\right)^p}{\int_0^r g(\rho)^p\rho^{np-1}d\rho} = p \lim_{r\to 0+} \left(\frac{g(r)r^n}{\int_0^r g(\rho)\rho^{n-1}d\rho} \right)^{1-p} < \infty$$

and similarly the limit as $r \to +\infty$ is finite, hence $S < \infty$. If at least one of limits (20) is infinite, then for the same reason $S = \infty$, and the statement follows.

By applying the L'Hospital rule once more it follows that if g is a positive continuously differentiable function on $(0,\infty)$ for which both limits

$$\lim_{r\to 0+} \frac{rg'(r)}{g(r)}, \qquad \lim_{r\to +\infty} \frac{rg'(r)}{g(r)}$$

are finite, then the function $g(|x|)$ is hypodecreasing with the parameters p and $\alpha \le n-\frac{1}{p}$, and if one of these limits is infinite, then g is not hypodecreasing with the parameters p and $n-\frac{1}{p}$.

In particular, if ψ is a positive continuously differentiable function on $(0,\infty)$ such that $\psi'(r) > 0$ for all $r > 0$ and $\lim_{r\to+\infty} \psi(r) = \infty$, then

$$|x|^{\psi(|x|)} \notin HD_p^{n-\frac{1}{p}}$$

for any $0 < p < 1$ because of too rapid growth at infinity. For example, for any $\varepsilon > 0$ $e^{|x|^\varepsilon} \notin HD_p^{n-\frac{1}{p}}$.

(Roughly speaking, any radially increasing function which grows at infinity quicker than any power function is not hypodecreasing with the parameters p and $n-\frac{1}{p}$.)

Example 7. *Let $\sigma \in \mathbb{N}$, $h \in \mathbb{R}^n$, $0 < q \le \infty$, $\Delta_h^\sigma \varphi$ be the difference of order σ with step h of a function $\varphi \in L_q(\mathbb{R}^n)$ and*

$$f(h) = \|\Delta_h^\sigma \varphi\|_{L_q(\mathbb{R}^n)}.$$

Then for all $0 < p \le \infty$ and $\alpha \in \mathbb{R}$ the function f satisfies inequality (10) with some $c_3 > 0$ depending only on p, q, σ, α and h (see, for example [5], [7] and [6]). Hence by the lemma $f \in HD_p^\alpha$ for any $0 < p < 1$ and $\alpha \le n-\frac{1}{p}$.

4 The main result

In this section, for $0 < p < 1$, we give sufficient conditions close to necessary ones in terms of spaces of hypodecreasing functions ensuring that the following stronger version of the Hardy-type equality is satisfied for all functions f non-negative and measurable on $L_p(\mathbb{R}^n)$

$$\|t^\alpha (H(f\chi_{B_r}))(t)\|_{L_p(0,\infty)} \le N \|f(x)\chi_{B_r}(x)|x|^{\alpha - \frac{n-1}{p}}\|_{L_p(\mathbb{R}^n)} \tag{21}$$

for all $0 < r \le \infty$, where $N > 0$ is independent of f and r. If $r = \infty$ this is inequality (5).

Remark 8. *If $\alpha \ge n - \frac{1}{p}$, then for any continuous function with compact support the right hand side of inequality (5) is finite for all $r > 0$ whilst the left hand side is infinite. Hence, for any space $Z(\mathbb{R}^n)$ of functions defined on $\mathbb{R}^n$, containing at least one continuous function with compact support, inequality (5) cannot be satisfied for all functions $f \in Z(\mathbb{R}^n)$. Since all spaces $Z(\mathbb{R}^n)$ under consideration contain some continuous functions with compact supports, it is natural to assume that $\alpha < n - \frac{1}{p}$.*

Definition 2. *Given $0 < p < 1$, $\alpha < n - \frac{1}{p}$ and $N > 0$, we denote by $H_p^\alpha(N)$ the space of all functions f non-negative and measurable on $\mathbb{R}^n$ for which inequality (21) is satisfied [3] for all $0 < r \le \infty$. We also set*

$$H_p^\alpha = \bigcup_{N>0} H_p^\alpha(N).$$

Example 8. *Let $0 < p < 1, \alpha < n - \frac{1}{p}$. Then*

$$|x|^\mu \in H_p^\alpha \quad \text{for all} \quad \mu \in \mathbb{R}\,.$$

The case $\mu \le -\alpha - \frac{1}{p}$ is trivial since in this case $\||x|^{\mu+\alpha-\frac{n-1}{p}}\|_{L_p(B_r)} = \infty$ for all $0 < r \le \infty$. If $\mu > -\alpha - \frac{1}{p}$ and $N > 0$, then

$$|x|^\mu \in H_p^\alpha(N) \iff \quad N \ge c_{11},$$

where

$$c_{11} = n\sigma_n^{-\frac{1}{p}} \Big(n - \frac{1}{p} - \alpha\Big)^{-1} (\mu + n)^{\frac{1}{p}-1}\,.$$

Indeed, in this case for all $0 < r < \infty$

$$\||x|^{\mu+\alpha-\frac{n-1}{p}}\|_{L_p(B_r)} = \sigma_n^{\frac{1}{p}} [(\mu+\alpha)p + 1]^{-\frac{1}{p}} r^{\mu+\alpha+\frac{1}{p}}\,,$$

for all $0 < r, t < \infty$ and $\mu > -n$

$$\big(H(|x|^\mu \chi_{B_r}(x))\big)(t) = \frac{\sigma_n}{v_n t^n} \int_0^{\min\{t,r\}} \rho^{\mu+n-1} d\rho$$

---

[3] As in the case of inequality (5), the convention is that if the right hand side of inequality (21) is infinite, then this inequality is satisfied. This convention implies that any function f non-negative and measurable on $\mathbb{R}^n$ for which $\|f(x)|x|^{\alpha-\frac{n-1}{p}}\|_{L_p(B_r)} = \infty$ for all $r > 0$ belongs to the space $H_p^\alpha(N)$ for all $N > 0$.

$$= \frac{n}{\mu+n} t^{-n} \min\{t, r\}^{\mu+n}$$

and

$$\|t^{\alpha}\left(H(|x|^{\mu}\chi_{B_r}(x))\right)(t)\|_{L_p(0,\infty)}$$

$$= \frac{n}{\mu+n}\left(\int_0^r t^{(\alpha+\mu)p}dt + r^{\mu+n}\int_r^{+\infty} t^{(\alpha-n)p}dt\right)^{\frac{1}{p}}$$

$$= np^{\frac{1}{p}}((\mu+\alpha)p+1)^{-\frac{1}{p}}((n-\alpha)p-1)^{-\frac{1}{p}}(\mu+n)^{\frac{1}{p}-1} < \infty$$

if and only if $\alpha < n - \frac{1}{p}$ *(hence* $\mu > -n$*). Therefore the minimal value of* N *for which inequality* (21) *is satisfied for the function* $|x|^{\mu}$ *is equal to*

$$\sup_{0<r<\infty} \frac{\|t^{\alpha}\left(H(|x|^{\mu}\chi_{B_r}(x))\right)(t)\|_{L_p(0,\infty)}}{\||x|^{\mu}\chi_{B_r}(x)|x|^{\alpha-\frac{n-1}{p}}\|_{L_p(0,\infty)}} = c_{11}.$$

Example 9. *Let* $0 < p < 1, \alpha < n - \frac{1}{p}$. *Consider the space* $\widetilde{H}_p^{\alpha}$ *of all functions* f *measurable on* $\mathbb{R}^n$ *for which* $\|f(x)|x|^{\alpha-\frac{n-1}{p}}\|_{L_p(B_r)} < \infty$ *for all* $0 < r < \infty$ *and for some* $N > 0$ *inequality* (21) *is satisfied for all* $0 < r < \infty$. *Example* 8 *shows that the space* $\widetilde{H}_p^{\alpha}$ *contains some radially increasing functions. Moreover, any power function* $|x|^{\mu}, \mu > 0$, *belongs to* $\widetilde{H}_p^{\alpha}$.

However, there are restrictions on the rapidness of growth of a function belonging to $\widetilde{H}_p^{\alpha}$. *For example, if* g *is a positive continuous function on* $(0,\infty)$ *for which both limits* (20) *are finite, then* $g(|x|) \in \widetilde{H}_p^{\alpha}$, *but if at least one of limits* (20) *is infinite, then* $g(|x|) \notin \widetilde{H}_p^{\alpha}$.

Indeed,

$$T^p \equiv \sup_{0<r<\infty} \frac{\|t^{\alpha}(H(g(|x|)\chi_{B_r}(x))(t)\|^p_{L_p(0,\infty)}}{\|g(|x|)\chi_{B_r}(x)|x|^{\alpha-\frac{n-1}{p}}\|^p_{L_p(0,\infty)}} = \frac{n^p}{\sigma_n}\sup_{0<r<\infty} T(r)^p\,,$$

where

$$T(r)^p = \frac{I_1 + I_2}{\displaystyle\int_0^r g(\varrho)^p \varrho^{\alpha p} d\varrho}$$

and

$$I_1 = \int_0^r t^{(\alpha-n)p}\Big(\int_0^t g(\varrho)\varrho^{n-1}d\varrho\Big)^p dt\,,$$

$$I_2 = \frac{r^{(\alpha-n)p+1}}{(n-\alpha)p-1}\Big(\int_0^r g(\varrho)\varrho^{n-1}d\varrho\Big)^p.$$

If both limits (20) *are finite, then by the L'Hospital rule*

$$\lim_{r\to 0^+} T(r)^p = \big(n - \frac{1}{p} - \alpha\big)^{-1} \lim_{r\to 0^+}\left(\frac{g(r)r^n}{\displaystyle\int_0^r g(\rho)\rho^{n-1}d\rho}\right)^{1-p} < \infty\,,$$

and similarly the limit as $r\to+\infty$ is finite, hence $T<\infty$. If at least one of limits (20) is infinite, then for the same reason $T=\infty$, and the statement follows.

Similarly to Example 6 it follows that if ψ is a positive continuously differentiable function on $(0,\infty)$ such that $\psi'(r)>0$ for all $r>0$ and $\lim\limits_{r\to+\infty}\psi(r)=\infty$, then

$$|x|^{\psi(|x|)}\notin \widetilde{H}_p^\alpha$$

for any $0<p<1$ and $\alpha<n-\frac{1}{p}$.

Theorem 3. *Let $0<p<1$ and $\alpha<n-\frac{1}{p}$.*

1. *For all $\alpha<\beta\le n-\frac{1}{p}$*

$$HD_p^\beta\subset H_p^\alpha\subset HD_p^\alpha. \tag{22}$$

2. *For all $\alpha<\beta\le n-\frac{1}{p}$ and $M>0$*

$$HD_p^\beta(M)\subset H_p^\alpha(c_{12}M) \tag{23}$$

where $c_{12}=v_n^{-1}((\beta-\alpha)p)^{-\frac{1}{p}}$ and this constant is sharp.

3. *For all $N>0$*

$$H_p^\alpha(N)\subset HD_p^\alpha(c_{13}N) \tag{24}$$

where $c_{13}=v_n((n-\alpha)p-1)^{\frac{1}{p}}$ and this constant is sharp.

Remark 9. *In terms of inequalities inclusions (22) mean that if a function f non-negative and measurable on $\mathbb{R}^n$ satisfies for some $N>0$ inequality (21) for all $0<r\le\infty$, then it also satisfies inequality (5) with some $M=M(N,n,\alpha,p)>0$, depending only on N,n,α and p, for all $0<r<\infty$.*

Moreover, if a function f non-negative and measurable on $\mathbb{R}^n$ satisfies for some $M>0$ a slightly stronger inequality than inequality (5), namely inequality (5) where α is replaced by any $\beta\in(\alpha,n-\frac{1}{p}]$, for all $0<r<\infty$, then it also satisfies inequality (21) with some $N=N(M,n,\alpha,\beta,p)>0$, depending only on N,n,α,β and p, for all $0<r\le\infty$. In particular, it satisfies inequality (3).

Hence, all functions f which satisfy inequality (21), a stronger version of inequality (3), and are such that $\|f(x)|x|^{\alpha-\frac{n-1}{p}}\|_{L_p(B_r)}$ for all $r>0$, are hypodecreasing with the parameters p and α.

Statements 2 and 3 are more precise versions of the above. In the first part $M(N,n,\alpha,p)=c_{13}N$, and c_{13} cannot be replaced by a smaller quantity. In the second part $N(M,n,\alpha,\beta,p)=c_{12}M$, and c_{12} cannot be replaced by a smaller quantity.

Corollary. *Let $0<p<1,\alpha<\beta\le n-\frac{1}{p}$ and $M>0$. Then for each function f hypodecreasing with the parameters p,β and M inequality (3) is satisfied with $N=c_{12}M$, and the quantity c_{12} cannot be replaced by a smaller one.*

Remark 10. *For $\beta=n-\frac{1}{p}$ the first statement of the Corollary coincides with Theorem 2.*

Proof. 1. Let $\alpha < \beta \le n - \frac{1}{p}$, $M > 0$ and $f \in HD_p^\beta(M)$. Then by inequality (5)

$$\|t^\alpha (Hf)(t)\|_{L_p(0,\infty)}$$

$$= v_n^{-1} \left(\int_0^\infty t^{(\alpha-n)p} \left(\int_{B_t} f(x)dx \right)^p dt \right)^{\frac{1}{p}}$$

$$\le v_n^{-1} M \left(\int_0^\infty t^{(\alpha-\beta)p-1} \left(\int_{B_t} f^p(x)|x|^{\beta p - n+1} dx \right) dt \right)^{\frac{1}{p}}$$

$$= v_n^{-1} M \left(\int_{\mathbb{R}^n} f^p(x)|x|^{\beta p-n+1} \int_{|x|}^\infty t^{(\alpha-\beta)p-1} dt \right)^{\frac{1}{p}}$$

$$= v_n^{-1} ((\beta-\alpha)p)^{-\frac{1}{p}} M \left(\int_{\mathbb{R}^n} f^p(x)|x|^{\alpha p-n+1} dx \right)^{\frac{1}{p}}$$

$$= v_n^{-1} ((\beta-\alpha)p)^{\frac{1}{p}} M \| f(x)|x|^{\alpha - \frac{n-1}{p}} \|_{L_p(\mathbb{R}^n)},$$

and inequality (20) for all $r = \infty$ follows with $N = v_n^{-1}((\beta-\alpha)p)^{-\frac{1}{p}} M$.

By Remark 3 $f\chi_{B_r} \in HD_p^\beta(M)$ for all $0 < r < \infty$, therefore in the above argument the function f can be replaced by $f\chi_{B_r}$ which implies inequality (21) for all $0 < r < \infty$.

2. Let c_{12}^* be the minimal value of $c > 0$ for which the inclusion $HD_p^\beta(M) \subset HD_p^\alpha(cM)$ holds. Consider the function $f(x) = |x|^\mu$ with $\mu > -\alpha - \frac{1}{p}$, hence $\mu > -\beta - \frac{1}{p}$ and $\mu > -n$. By Example 1 $|x|^\mu \in HD_p^\alpha(c_3)$. On the other hand by Example 8 $|x|^\mu \in H_p^\alpha(N)$ if and only if $N \ge c_{11}$, hence $c_{12}^* c_3 \ge c_{11}$. Consequently

$$c_{12}^* \ge \sup_{\mu > -\alpha - \frac{1}{p}} \frac{c_{11}}{c_3}$$

$$= \sup_{\mu > -\alpha - \frac{1}{p}} v_n^{-1} \left(n - \frac{1}{p} - \alpha \right)^{-1} \left(\frac{\mu+n}{(\mu+\beta)p+1} \right)^{\frac{1}{p}} = c_{12}.$$

By Step 1 $c_{12}^* \le c_{12}$, so $c_{12}^* = c_{12}$.

3. Let $f \in H_p^\alpha(N)$, i.e. inequality (21) is satisfied for all $0 < r \le \infty$. Note that

$$\begin{aligned} \|t^\alpha (H(f\chi_{B_r})(t)\|_{L_p(0,\infty)} &\ge v_n^{-1} \|t^{\alpha-n} \int_{B_t} f\chi_{B_r} dy \|_{L_p(r,\infty)} \\ &= v_n^{-1} \|f\|_{L_1(B_r)} \|t^{\alpha-n}\|_{L_p(r,\infty)} \\ &= v_n^{-1} ((n-\alpha)p-1)^{-\frac{1}{p}} r^{\alpha-n+\frac{1}{p}} \|f\|_{L_1(B_r)}. \end{aligned}$$

Hence by (21) for all $r > 0$

$$\|f\|_{L_1(B_r)} \le v_n ((n-\alpha)p-1)^{\frac{1}{p}} r^{n-\frac{1}{p}-\alpha} N \| f(x)|x|^{\alpha-\frac{n-1}{p}} \|_{L_p(B_r)}$$

which means that $f \in HD_p^\alpha(c_{13}N)$.

4. Let c_{13}^* be the minimal value of $c > 0$ for which the inclusion $H_p^\alpha(N) \subset HD_p^\alpha(cN)$ holds. As in Step 2 consider the function $f(x) = |x|^\mu$ with $\mu > -\alpha - \frac{1}{p}$. By Example 8 $|x|^\mu \in H_p^\alpha(c_{11})$. On the other hand by Example 1 $|x|^\mu \in HD_p^\alpha(M)$ if and only if $M \geq c_3 \mid_{\beta=\alpha}$, hence $c_{13}^* c_{11} \geq c_3 \mid_{\beta=\alpha}$. Consequently

$$c_{13}^* \geq \sup_{\mu > -\alpha - \frac{1}{p}} \frac{c_3 \mid_{\beta=\alpha}}{c_{11}}$$

$$= \sup_{\mu > -\alpha - \frac{1}{p}} v_n \left(n - \frac{1}{p} - \alpha\right) \left(\frac{(\mu+\beta)p+1}{\mu+n}\right)^{\frac{1}{p}} = c_{13}.$$

By Step 3 $c_{13}^* \leq c_{13}$, so $c_{13}^* = c_{13}$.

5. Finally, inclusions (22) follow by inclusions (23) and (24). □

Remark 11. *By Example 7 Theorem 3 is applicable to the function* $f(h) = \|\Delta_h^\sigma \varphi\|_{L_q(\mathbb{R}^n)}$, *where* $0 < q \leq \infty$. *This fact was used in* [6] *for establishing the equivalence of certain quasi-norms involving differences and similar quasi-norms involving moduli of continuity. At a certain stage of the proof in* [6] *it was required to prove that*

$$\left\| t^{-l-\frac{1}{\theta}} \left(\frac{1}{v_n t^n} \int_{B_t} \|\Delta_h^\sigma \varphi\|_{L_q(\mathbb{R}^n)} dh \right) \right\|_{L_\theta(0,\infty)}$$

$$\leq c_{14} \left\| \, |h|^{-l-\frac{n}{\theta}} \|\Delta_h^\sigma \varphi\|_{L_q(\mathbb{R}^n)} \right\|_{L_\theta(\mathbb{R}^n)},$$

where $0 < l < \sigma, 0 < \theta \leq \infty$ *and* $c_{14} > 0$ *is independent of* φ.

For $\theta \geq 1$ *this inequality follows by applying the standard Hardy inequality. For* $0 < \theta < 1$ *it was deduced by using Theorem* 2, *a particular case of the Corollary.*

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