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CORRECT AND SELFADJOINT PROBLEMS FOR QUADRATIC OPERATORS

I.N. Parasidis, P.C. Tsekrekos

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Abstract. In this paper we present a simpler method of proving the correctness and selfadjointtness of the operators of the form B^2 corresponding to some boundary value problems. We also give explict representations for the unique solution of these problems.

1 Introduction

Correct selfadjoint operators arise naturally in many boundary value problems with differential or integro-differential operator. Correct extensions of a densely defined minimal, not necessarily symmetric, operator in a Hilbert and Banach space have been investigated by M.I. Vishik [15], A.A. Dezin [3], B.K. Kokebaev, M. Otelbaev and A.N. Shynibekov [8], T.Sh. Kalmenov [5], R. Oinarov and I.N. Parasidi [11], R. Oinarov and S. Sagintaeva [12] and many others. Selfadjoint extensions of a densely defined minimal symmetric operator A_0 have been studied by a number of authors, in particular by J. von Neumann [10], V.I. Gorbachuk and M.L. Gorbachuk [4], E.A. Coddington, A. Dijksma [2], A.N. Kočubei [6], V.A. Mikhailets [9]. Selfadjoint extensions of a nondensely defined symmetric operator have been investigated by E.A. Coddington [1], A. N. Kočubei [7]. Often the above authors described the extensions as restrictions of certain operators, mainly of the adjoint operator A_0^*. Correct selfadjoint and positive extensions of a nondensely defined minimal symmetric operators A_0 have been considered in [13]. Correct restrictions B of a certain maximal operator A defined in a Banach space, when B is a product of two correct restrictions B_1, B_2 of A, have been investigated by A.N. Shynibekov [14].

Correct operators which we consider in general are not restrictions of maximal operators and so Shynibekov's method cannot be applied. In this paper we use the operator B, defined in [13, Theorem 3.10] by

$$Bx = \widehat{A}x - (\widehat{A}F)C_m\langle \widehat{A}x, F^t\rangle_{H^m} = f, \quad D(B) = D(\widehat{A}),$$

where $\widehat{A}$ is a given correct selfadjoint operator on $\mathbb{H}$, $F \in D(\widehat{A})^m$, C_m – is a $m \times m$ matrix. We investigate the correctness and selfadjointness of the operator B_{SG}

corresponding to the boundary value problem

$$B_{SG}x = \widehat{A}^2 x - S\langle \widehat{A}x, F^t\rangle_{H^m} - G\langle \widehat{A}^2 x, F^t\rangle_{H^m} = f,\ D(B_{SG}) = D(\widehat{A}^2),$$

where the vectors

$$G = (\widehat{A}F)C_m \in D(\widehat{A})^m,\ S = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m} \in \mathbb{H}^m.$$

We show that the operator B_{SG} is quadratic, i.e. $B_{SG} = B^2$. For the corresponding problem $B_{SG}x = f$ we prove a criterion of correctness and selfadjointness in terms of the matrices C_m and solve the equation $B_{SG}x = f$ which is essentially simpler than in the general case of non-quadratic operators.

We note that the selfadjointness of B_{SG} can be proved by more general methods developed in [1], [2], [7]. However, here we do not need the full strength of these methods and prove selfadjointness in a simpler and straightforward way.

The paper is organized as follows. In Section 2 we recall some basic terminology and notation about operators. In Section 3 we prove the main result. Finally, in Section 4, we give three examples of integro-differential equations which show the usefulness of this result.

2 Terminology and notation

By $\langle x, f\rangle_H$ we denote the inner product of elements x, f of a complex Hilbert space $\mathbb{H}$. For operators $A : \mathbb{H} \to \mathbb{H}$ we write $D(A)$ and $R(A)$ for the domain and the range of A respectively.

An operator $\widehat{A}$ is called *correct* if $R(\widehat{A}) = \mathbb{H}$ and the inverse $\widehat{A}^{-1}$ exists and is continuous. Let A be an operator with domain $D(A)$ dense in $\mathbb{H}$.

The *adjoint* operator $A^* : \mathbb{H} \longrightarrow \mathbb{H}$ of A with domain $D(A^*)$ is defined by the equation $\langle Ax, y\rangle_H = \langle x, A^*y\rangle_H$ for every $x \in D(A)$ and every $y \in D(A^*)$. The domain $D(A^*)$ of A^* consists of all $y \in \mathbb{H}$ for which the functional $x \longmapsto \langle Ax, y\rangle_H$ is continuous on $D(A)$. An operator A is called *selfadjoint* if $A = A^*$.

If an operator $B : \mathbb{H} \to \mathbb{H}$ is correct (selfadjoint respectively), then we say that the problem $Bx = f$ is correct (selfadjoint respectively).

Let $F_i \in \mathbb{H}, i = 1, \ldots, m$. Then $F = (F_1, \ldots, F_m)$ and $AF = (AF_1, \ldots, AF_m)$ are vectors of $\mathbb{H}^m$. We write $\mathcal{F} = (\widehat{A}^{-1}F, F) = (\widehat{A}^{-1}F_1, \ldots, \widehat{A}^{-1}F_m, F_1, \ldots, F_m)$ and $\widehat{A}^{-2} = (\widehat{A}^{-1})^2$. We also write F^t and $\langle Ax, F^t\rangle_{H^m}$ for the column vectors $col(F_1, \ldots, F_m)$ and $col(\langle Ax, F_1\rangle_H, \ldots, \langle Ax, F_m\rangle_H)$ respectively. We denote by M^t the transpose matrix of M and by $\langle AF^t, F\rangle_{H^m}$ the $m \times m$ matrix whose i, j-th entry is the inner product $\langle AF_i, F_j\rangle_H$. We also denote by I_m and $[0]_m$ the identity $m \times m$ and the zero $m \times m$ matrix respectively.

3 Correct and selfadjoint problems with quadratic operators

Throughout this paper we assume that the components of the vectors $F = (F_1, \ldots, F_m)$ and $\mathcal{F} = (\widehat{A}^{-1}F_1, \ldots, \widehat{A}^{-1}F_m, F_1, \ldots, F_m)$ are linearly independent elements of $D(\widehat{A})$ and $D(\widehat{A}^2)$ respectively.

Next theorem is Theorem 3.10 of [13] with the additional information that if the operator B is selfadjoint, then the matrix C_m is Hermitian. This is a criterion of correctness and selfadjointness of the equation $Bx = f$. Also an explict representation of the unique solution to this problem is given. We shall make use of this theorem in the sequel.

Theorem 1. *Let* $B : \mathbb{H} \to \mathbb{H}$ *and*

$$Bx = \widehat{A}x - (\widehat{A}F)C_m\langle \widehat{A}x, F^t\rangle_{H^m} = f, \qquad D(B) = D(\widehat{A}), \tag{1}$$

where $\widehat{A}$ *is correct and selfadjoint on* $\mathbb{H}$, C_m *is an* $m\times m$ *matrix with rank* $C_m = n \leq m$ *and let* $F_1, \ldots, F_m$ *be linearly independent elements of* $D(\widehat{A})$.

Then

(i) $\dim R(B - \widehat{A}) = n$.

(ii) B *is correct if and only if*

$$\det\left[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C_m\right] \neq 0. \tag{2}$$

(iii) B *is selfadjoint if and only if the matrix* C_m *is Hermitian.*

(iv) If B *is correct, then the unique solution of problem (1) for each* $f \in \mathbb{H}$ *is given by the formula*

$$x = B^{-1}f = \widehat{A}^{-1}f + FC_m\left[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C_m\right]^{-1}\langle f, F^t\rangle_{H^m}. \tag{3}$$

Proof of statement (iii). From (1) we have

$$B^*x = \widehat{A}x - \langle x, \widehat{A}F\rangle_{H^m}\overline{C}_m(\widehat{A}F^t) = \widehat{A}x - (\widehat{A}F)\overline{C}\,^t_m\langle \widehat{A}x, F^t\rangle_{H^m}.$$

Since the operator $\widehat{A}$ is correct and selfadjoint and $F_1, \ldots, F_m$ are linearly independent elements of $D(\widehat{A})$, then, by (1), we have that $B^* = B$ if and only if $C_m = \overline{C}\,^t_m$.

□

Remark 1. *It is useful to notice that the Hermitianess of the matrix* C_m *is equivalent to the selfadjointness of* B *and that (2) implies the correctness of* B *and so it is a solvability condition for the problem* $Bx = f$.

By Theorem 1, since $\widehat{A}^2$ is a correct selfadjoint operator and the components of $\mathcal{F}$ are linearly independent, the next theorem easily follows which is a criterion of correctness and selfadjointness of the equation $B_1x = f$ with $D(B_1) = D(\widehat{A}^2)$ and $\dim R(B_1 - \widehat{A}^2) = n$. A representation of the unique solution is also given.

Theorem 2. *Let* $B_1 : \mathbb{H} \to \mathbb{H}$ *and*

$$B_1x = \widehat{A}^2x - (\widehat{A}^2\mathcal{F})\mathbb{C}_{2m}\langle \widehat{A}^2x, \mathcal{F}^t\rangle_{H^{2m}} = f, \quad D(B_1) = D(\widehat{A}^2), \tag{4}$$

where $\widehat{A}$ *is correct and selfadjoint on* $\mathbb{H}$, $\mathbb{C}_{2m}$ *is a* $(2m)\times(2m)$ *matrix with rank* $\mathbb{C}_{2m} = n \leq 2m$ *and let the components of the vector* $\mathcal{F} = (\widehat{A}^{-1}F_1, \ldots, \widehat{A}^{-1}F_m, F_1, \ldots, F_m)$ *be linearly independent elements of* $D(\widehat{A}^2)$.

Then
(i) $\dim R(B_1 - \widehat{A}^2) = n$.
(ii) B_1 *is correct if and only if*

$$\det L_1 = \det \left[I_{2m} - \overline{\langle \widehat{A}^2 \mathcal{F}^t, \mathcal{F} \rangle}_{H^{2m}} \mathbb{C}_{2m} \right] \neq 0. \tag{5}$$

(iii) B_1 *is selfadjoint if and only if* $\mathbb{C}_{2m}$ *is Hermitian.*
(iv) If B_1 *is correct, then the unique solution of (4) for each* $f \in \mathbb{H}$ *is given by the formula*

$$x = B_1^{-1} f = \widehat{A}^{-2} f + \mathcal{F}\mathbb{C}_{2m} \left[I_{2m} - \overline{\langle \widehat{A}^2 \mathcal{F}^t, \mathcal{F} \rangle}_{H^{2m}} \mathbb{C}_{2m} \right]^{-1} \langle f, \mathcal{F}^t \rangle_{H^{2m}}. \tag{6}$$

Lemma 1. *Let the operators* $B, B_2 : \mathbb{H} \to \mathbb{H}$ *be defined by*

$$Bx = \widehat{A}x - (\widehat{A}F) C_m \langle \widehat{A}x, F^t \rangle_{H^m} = f, \qquad D(B) = D(\widehat{A}), \tag{7}$$

$$\begin{aligned} B_2 x =& \widehat{A}^2 x - \left[(\widehat{A}^2 F) C_m - (\widehat{A}F) C_m \overline{\langle \widehat{A}F^t, \widehat{A}F \rangle}_{H^m} C_m \right] \langle \widehat{A}x, F^t \rangle_{H^m} - \\ & -(\widehat{A}F) C_m \langle \widehat{A}^2 x, F^t \rangle_{H^m} = f, \quad x \in D(B_2) = D(\widehat{A}^2), \end{aligned} \tag{8}$$

where $\widehat{A}$ *is a selfadjoint operator on* $\mathbb{H}$, C_m *is an* $m \times m$ *matrix and the components of the vector* $F = (F_1, \ldots, F_m)$ *belong to* $D(\widehat{A}^2)$. *Then* $B_2 = B^2$.

Proof. We put $y = Bx$. Then, by (8) and the selfadjointness of $\widehat{A}$, for each $x \in D(\widehat{A}^2)$, we have

$$B_2 x = \widehat{A}^2 x - (\widehat{A}^2 F) C_m \langle \widehat{A}x, F^t \rangle_{H^m} + (\widehat{A}F) C_m \overline{\langle \widehat{A}F^t, \widehat{A}F \rangle}_{H^m} C_m \langle \widehat{A}x, F^t \rangle_{H^m} -$$

$$-(\widehat{A}F) C_m \langle \widehat{A}^2 x, F^t \rangle_{H^m} = \widehat{A}\left[\widehat{A}x - (\widehat{A}F) C_m \langle \widehat{A}x, F^t \rangle_{H^m} \right] - (\widehat{A}F) C_m \left[\langle \widehat{A}^2 x, F^t \rangle_{H^m} - \right.$$

$$\left. - \overline{\langle \widehat{A}^2 F^t, F \rangle}_{H^m} C_m \langle \widehat{A}x, F^t \rangle_{H^m} \right] = \widehat{A}y - (\widehat{A}F) C_m \langle \widehat{A}y, F^t \rangle_{H^m} = By = B^2 x,$$

where after the second equality we used that

$$\langle \widehat{A}^2 x, F^t \rangle_{H^m} - \overline{\langle \widehat{A}^2 F^t, F \rangle}_{H^m} C_m \langle \widehat{A}x, F^t \rangle_{H^m} = \langle \widehat{A}^2 x, F^t \rangle_{H^m} -$$

$$- \overline{\left\langle \widehat{A}^2 F^t, F C_m \langle \widehat{A}x, F^t \rangle_{H^m} \right\rangle}_{H^m} = \langle \widehat{A}^2 x, F^t \rangle_{H^m} - \left\langle F C_m \langle \widehat{A}x, F^t \rangle_{H^m}, \widehat{A}^2 F^t \right\rangle_{H^m} =$$

$$\langle \widehat{A}^2 x, F^t \rangle_{H^m} - \left\langle \widehat{A}^2 F C_m \langle \widehat{A}x, F^t \rangle_{H^m}, F^t \right\rangle_{H^m} = \left\langle \widehat{A}[\widehat{A}x - (\widehat{A}F) C_m \langle \widehat{A}x, F^t \rangle_{H^m}], F^t \right\rangle_{H^m}.$$

Now we show that $D(B^2) = D(\widehat{A}^2)$. Since

$$D(B^2) = \{ x \in D(\widehat{A}) : \widehat{A}x - (\widehat{A}F) C_m \langle \widehat{A}x, F^t \rangle_{H^m} \in D(\widehat{A}) \}$$

and $\widehat{A}F_i \in D(\widehat{A})$, $i = 1, \ldots, m$ it follows that $D(B^2) = D(\widehat{A}^2)$ and $B_2 = B^2$. □

We now present the main result of this paper.

Theorem 3. *Let the operators $\widehat{A}, B_{SG} : \mathbb{H} \to \mathbb{H}$, where $\widehat{A}$ is a correct selfadjoint operator and the operator B_{SG} is defined by*

$$B_{SG}\,x = \widehat{A}^2 x - S\langle \widehat{A}x, F^t\rangle_{H^m} - G\langle \widehat{A}^2 x, F^t\rangle_{H^m} = f, \quad D(B_{SG}) = D(\widehat{A}^2), \qquad (9)$$

the components of the vector $\mathcal{F} = (\widehat{A}^{-1}F, F)$ $\big(\widehat{A}\mathcal{F} = (F, \widehat{A}F)$ respectively$\big)$ are linearly independent elements of $D(\widehat{A}^2)$ $\big(D(\widehat{A})$ respectively$\big)$, $S = (s_1, \ldots, s_m)$, $G = (g_1, \ldots, g_m)$, $s_i \in \mathbb{H}$, $g_i \in D(\widehat{A})$, $i = 1, \ldots, m$,

$$S = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m},\; G = (\widehat{A}F)C_m$$

and C_m is an $m \times m$ matrix with rank $C_m = n (n \le m)$.

Then

(i) $\dim R(B_{SG} - \widehat{A}^2) = 2n\,(n \le m)$.

(ii) B_{SG} is a correct operator if and only if

$$\det L = \det \left[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C_m\right] \neq 0. \qquad (10)$$

(iii) B_{SG} is a selfadjoint operator if and only if the matrix C_m is Hermitian.

(iv) If the operator B_{SG} is correct, then the unique solution of problem (9) for each $f \in \mathbb{H}$ is given by the formula

$$\begin{aligned} x = B_2^{-1} =& \widehat{A}^{-2}f + \left[\widehat{A}^{-1}F + FC_mL^{-1}\overline{\langle F^t, F\rangle}_{H^m}\right]C_mL^{-1}\langle f, F^t\rangle_{H^m} + \\ &+ FC_mL^{-1}\langle f, \widehat{A}^{-1}F^t\rangle_{H^m}. \end{aligned} \qquad (11)$$

This theorem is useful in applications and gives a criterion of correctness and selfadjointness of the equation $B_{SG}x = f$. The operator B_{SG} is equal, by Lemma 1, to $B_2 = B^2$ if $S = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m}$. Next we denote by B_2 the operators defined by (4) or by (9). Using (5) and (6) we shall prove (10) and (11) respectively. Formula (11) can also be obtained by using the solution of equation (7).

Proof. Let

$$K = \overline{\langle \widehat{A}F^t, \widehat{A}F\rangle}_{H^m},\; T = \overline{\langle F^t, F\rangle}_{H^m},\; D = \overline{\langle \widehat{A}F^t, F\rangle}_{H^m}.$$

Then the matrix L in (10) is written as $L = I_m - DC_m$. Since $G = (\widehat{A}F)C_m$ and $S = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m}$, equation (9) implies (8) and the equality $B_{SG} = B_2$. Equation (9) can also be written in matrix notation as

$$B_2x = \widehat{A}^2x - (\widehat{A}F, \widehat{A}^2F)\begin{pmatrix} -C_mKC_m & C_m \\ C_m & [0]_m \end{pmatrix}\begin{pmatrix} \langle \widehat{A}x, F^t\rangle_{H^m} \\ \langle \widehat{A}^2x, F^t\rangle_{H^m} \end{pmatrix} = f$$

or

$$B_2x = \widehat{A}^2x - (\widehat{A}^2\mathcal{F})\mathbb{C}_{2m}\langle \widehat{A}^2x, \mathcal{F}^t\rangle_{H^{2m}} = f, \qquad (12)$$

where

$$\mathcal{F} = (\widehat{A}^{-1}F, F), \quad \mathbb{C}_{2m} = \begin{pmatrix} -C_mKC_m & C_m \\ C_m & [0]_m \end{pmatrix}.$$

(i) It is evident, from the last equality, that rank $\mathbb{C}_{2m} = 2n$ if and only if rank $C_m = n$. From (12), since $\widehat{A}$ is correct and the components of $\mathcal{F}$ linearly independent, follows that rank $\mathbb{C}_{2m} = 2n$, $(n \leq m)$.

(iii) It is also evident, from the last equality, that the matrix $\mathbb{C}_{2m}$ is Hermitian if and only if C_m is Hermitian. Hence, by Theorem 2, the operator B_2 is selfadjoint if and only if C_m is Hermitian.

(ii) By Theorem 2 the operator B_2 is correct if and only if (5) holds true, where, we remind, B_1 is denoted by B_2 and L_1 by L_2.

$$L_2 = I_{2m} - \overline{\langle \widehat{A}^2 \mathcal{F}^t, \mathcal{F} \rangle}_{H^{2m}} \mathbb{C}_{2m} =$$

$$= I_{2m} - \overline{\left\langle \mathrm{col}(\widehat{A}F^t, \widehat{A}^2 F^t), (\widehat{A}^{-1}F, F) \right\rangle}_{H^{2m}} \mathbb{C}_{2m} =$$

$$= I_{2m} - \overline{\begin{pmatrix} \langle F^t, F \rangle_{H^m} & \langle \widehat{A}F^t, F \rangle_{H^m} \\ \langle \widehat{A}F^t, F \rangle_{H^m} & \langle \widehat{A}^2 F^t, F \rangle_{H^m} \end{pmatrix}} \mathbb{C}_{2m} =$$

$$= I_{2m} - \begin{pmatrix} T & D \\ D & K \end{pmatrix} \begin{pmatrix} -C_m K C_m & C_m \\ C_m & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} I_m & [0]_m \\ [0] & I_m \end{pmatrix} - \begin{pmatrix} -TC_m K C_m + DC_m & TC_m \\ -DC_m K C_m + KC_m & DC_m \end{pmatrix} =$$

$$= \begin{pmatrix} I_m + TC_m K C_m - DC_m & -TC_m \\ DC_m K C_m - KC_m & I_m - DC_m \end{pmatrix} =$$

$$= \begin{pmatrix} L + TC_m K C_m & -TC_m \\ (DC_m - I_m) K C_m & L \end{pmatrix} = \begin{pmatrix} L + TC_m K C_m & -TC_m \\ -LKC_m & L \end{pmatrix},$$

i.e.

$$L_2 = I_{2m} - \overline{\langle \widehat{A}^2 \mathcal{F}^t, \mathcal{F} \rangle}_{H^{2m}} \mathbb{C}_{2m} = \begin{pmatrix} L & -TC_m \\ [0]_m & L \end{pmatrix} \begin{pmatrix} I_m & [0]_m \\ -KC_m & I_m \end{pmatrix} \tag{13}$$

and

$$\det L_2 = \det \begin{pmatrix} L & -TC_m \\ [0]_m & L \end{pmatrix} \det \begin{pmatrix} I_m & [0]_m \\ -KC_m & I_m \end{pmatrix} =$$

$$= (\det L)^2 \neq 0 \Leftrightarrow \det L \neq 0. \tag{14}$$

So, by Theorem 2, because of (13) and (14), the operator B_2 is correct if and only if (10) holds true.

(iv) Now, using solution (6), we will find solution (11). From (13) we get

$$L_2^{-1} = \left[I_{2m} - \overline{\langle \widehat{A}^2 \mathcal{F}^t, \mathcal{F} \rangle}_{H^{2m}} \mathbb{C}_{2m} \right]^{-1} =$$

$$= \begin{pmatrix} I_m & [0]_m \\ -KC_m & I_m \end{pmatrix}^{-1} \begin{pmatrix} L & -TC_m \\ [0]_m & L \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} I_m & [0]_m \\ KC_m & I_m \end{pmatrix} \begin{pmatrix} I_m & L^{-1}TC_m \\ [0]_m & I_m \end{pmatrix} \begin{pmatrix} L^{-1} & [0]_m \\ [0]_m & L^{-1} \end{pmatrix} =$$

$$= \begin{pmatrix} I_m & [0]_m \\ KC_m & I_m \end{pmatrix} \begin{pmatrix} L^{-1} & L^{-1}TC_mL^{-1} \\ [0]_m & L^{-1} \end{pmatrix}.$$

Then

$$\mathcal{F}\mathbb{C}_{2m}\big[I_{2m} - \overline{\langle \widehat{A}^2\mathcal{F}^t, \mathcal{F}\rangle}_{H^{2m}}\mathbb{C}_{2m}\big]^{-1}\langle f, \mathcal{F}^t\rangle_{H^{2m}} = (\widehat{A}^{-1}F, F)\cdot$$

$$\cdot\begin{pmatrix} -C_mKC_m & C_m \\ C_m & [0]_m \end{pmatrix} \begin{pmatrix} I_m & [0]_m \\ KC_m & I_m \end{pmatrix} \begin{pmatrix} L^{-1} & L^{-1}TC_mL^{-1} \\ [0]_m & L^{-1} \end{pmatrix} \begin{pmatrix} \langle f, \widehat{A}^{-1}F^t\rangle_{H^m} \\ \langle f, F^t\rangle_{H^m} \end{pmatrix} =$$

$$= \langle \widehat{A}^{-1}F, F) \begin{pmatrix} [0]_m & C_m \\ C_m & [0]_m \end{pmatrix} \begin{pmatrix} L^{-1} & L^{-1}TC_mL^{-1} \\ [0]_m & L^{-1} \end{pmatrix} \begin{pmatrix} \langle f, \widehat{A}^{-1}F^t\rangle_{H^m} \\ \langle f, F^t\rangle_{H^m} \end{pmatrix} =$$

$$= FC_mL^{-1}\langle f, \widehat{A}^{-1}F^t\rangle_{H^m} + \big[(\widehat{A}^{-1}F) + FC_mL^{-1}T\big]C_mL^{-1}\langle f, F^t\rangle_{H^m}.$$

Replacing the above expression in (6), we get (11). □

4 Examples

In the following examples $H^1(0,1)$, $H^2(0,1)$ and $H^4(0,1)$ denote the Sobolev spaces of all complex functions in $L_2(0,1)$ which have generalized derivatives up to the first, second and fourth order respectively, belonging to $L_2(0,1)$.

We recall [12, p. 780] that the operator $\widehat{A} : L_2(0,1) \to L_2(0,1)$ defined by

$$\widehat{A}u = iu' = f, \quad D(\widehat{A}) = \{u(t) \in H^1(0,1) : u(0) + u(1) = 0\} \tag{15}$$

is correct and selfadjoint and the unique solution u of the problem (15) is given by the formula

$$\widehat{A}^{-1}f = \frac{i}{2}\int_0^1 f(x)dt - i\int_0^t f(x)dx \quad \text{for all } f \in H. \tag{16}$$

Then it easily follows [12, p. 781] that the operator $\widehat{A}^2$ defined by

$$\widehat{A}^2u = -u'' = f, \quad D(\widehat{A}^2) = \{u \in H^2(0,1) : u(0) + u(1) = 0,\ u'(0) + u'(1) = 0\} \tag{17}$$

is correct and selfadjoint and for every $f \in L_2(0,1)$ the unique solution u of the problem (17) is given by the formula

$$u = \widehat{A}^{-2}f = -\int_0^t (t-x)f(x)dx + \frac{1}{4}\int_0^1 (2t-2x+1)f(x)dx. \tag{18}$$

Example 1. The operator $B_1 : L_2(0,1) \to L_2(0,1)$ which corresponds to the problem

$$B_1u = -u'' + 12ic_1(2t-1)\int_0^1 u(x)(x^2-x)dx - \frac{2}{5}c_1^2(t^2-t)\int_0^1 u'(x)(4x^3- \tag{19}$$

$$-6x^2+1)dx - ic_1(t^2-t)\int_0^1 u''(x)(4x^3-6x^2+1)dx = f(t), \quad D(B_1) = D(\widehat{A}^2)$$

is correct and selfadjoint if and only if c_1 is a real nonzero constant. The unique solution of (19), for each $f \in L_2(0,1)$, is given by the formula

$$u(t) = \widehat{A}^{-2}f(t) + \frac{c_1}{48}\Big[i(4t^4 - 8t^3 + 4t) + \frac{17c_1}{105}(4t^3 - 6t^2 + 1)\Big]\int_0^1 (4x^3 - 6x^2 + 1)f(x)dx - \frac{ic_1}{12}(4t^3 - 6t^2 + 1)\int_0^1 (4x^4 - 8x^3 + 4x + 1)f(x)dx. \qquad (20)$$

Proof. We refer to Theorem 3. If we compare equation (19) with equation (9) it is natural to take

$$\widehat{A}^2 u = -u'' \text{ with } D(\widehat{A}^2) = D(B_1),\ m = 1,\ F = 4t^3 - 6t^2 + 1.$$

Then we can take $\widehat{A}$ defined by (15). It follows that $F \in D(\widehat{A}^2)$. By simple calculations we find

$$\widehat{A}F = 12i(t^2 - t),\ \widehat{A}^2F = -12(2t - 1),$$

$$\langle \widehat{A}F^t, F\rangle_H = 0, \quad \langle F^t, F\rangle_H = 17/35, \quad \langle \widehat{A}u, F\rangle_H = \int_0^1 iu'(x)(4x^3 - 6x^2 + 1)dx,$$

$$\langle \widehat{A}^2 u, F\rangle_H = -\int_0^1 u''(x)(4x^3 - 6x^2 + 1)dx.$$

Equation (19) can now be rewritten as follows:

$$B_1 u = -u'' - \big[c_1(2t - 1) - \frac{2}{5}ic_1^2(t^2 - t)\big]\int_0^1 iu'(x)(4x^3 - 6x^2 + 1)dx - \big[-ic_1(t^2 - t)\big]\int_0^1 [-u''(x)](4x^3 - 6x^2 + 1)dx = f(t). \qquad (21)$$

By comparing again (21) with (9) we take $S = c_1(2t - 1) - \frac{2}{5}ic_1^2(t^2 - t)$ and $G = -ic_1(t^2 - t)$. It is evident that $G \in D(\widehat{A})$ and $F, \widehat{A}F$ are linearly independent elements of $D(\widehat{A})$. Next

$$\widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m} = (2t - 1)c_1 - \frac{2}{5}ic_1^2(t^2 - t) = S.$$

From $G = (\widehat{A}F)C_m$ it follows $-ic_1(t^2 - t) = 12i(t^2 - t)C_m$. This equation implies that $C_m = -c_1/12$. Therefore by Theorem 3 the operator B_{SG} is correct and selfadjoint if and only if c_1 is a real number and

$$\det L = \det[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C_m] = 1 \neq 0.$$

Hence $L^{-1} = 1$. So B_1 is correct and selfadjoint if and only if c_1 is a real nonzero constant. If we substitute in (16) $f = F = 4t^3 - 6t^2 + 1$, then we get that $\widehat{A}^{-1}F = -i(t^4 - 2t^3 + t)$ and

$$\langle f, \widehat{A}^{-1}F\rangle_H = -i\int_0^1 (x^4 - 2x^3 + x)f(x)dx.$$

From this and (11), (18) we obtain formula (20) for the solution of problem (19). □

Example 2. The operator $B_1 : L_2(0,1) \to L_2(0,1)$ which corresponds to the problem

$$B_1 u = u^{(4)} + \Big[12t - 6 - \frac{52c_1}{7}(2t^3 - 3t^2 + 1)\Big] c_1 \int_0^1 (2x^5 - 5x^4 + 10x^2 - 7)u'' dx +$$

$$+10c_1(2t^3 - 3t^2 + 1) \int_0^1 (x^4 - 2x^3 + 2x)u''' dx = f(t) \qquad (22)$$

with

$$D(B_1) = \{u \in H^4(0,1) : u(1) = u'(0) = u''(1) = u'''(0) = 0\},$$

is correct and selfadjoint if and only if c_1 is a real constant such that $c_1 \neq -126/367$. The unique solution of (22), for each $f \in L_2(0,1)$, is given by the formula

$$\begin{aligned} u(t) &= \frac{1}{6}\int_0^t (t-x)^3 f(x)dx + \frac{1}{6}\int_0^1 (1-x)(2+2x-x^2-3t^2) f(x)dx \\ &+ \Big[\frac{2t^7 - 7t^6 + 35t^4 - 147t^2 + 117}{42} + \frac{16301c_1(2t^5 - 5t^4 + 10t^2 - 7)}{110(126 + 367c_1)}\Big] \\ &\times \frac{63c_1}{10(126+367c_1)} \int_0^1 (2x^5 - 5x^4 + 10x^2 - 7) f(x)dx \qquad (23) \\ &+ \frac{63c_1(2t^5 - 5t^4 + 10t^2 - 7)}{420(126 + 367c_1)} \int_0^1 (2x^7 - 7x^6 + 35x^4 - 147x^2 + 117) f(x)dx. \end{aligned}$$

Proof. Now we refer to Theorem 3. If we compare equation (22) with equation (9) it is natural to take

$$\widehat{A}^2 u = u^{(4)} \quad \text{with} \;\; D(\widehat{A}^2) = D(B_1),\; m = 1,\; F = 2t^5 - 5t^4 + 10t^2 - 7.$$

Then we can take $\widehat{A}$ defined by

$$\widehat{A}u = u''(t) \;\; \text{with} \;\; D(\widehat{A}) = \{u \in H^2(0,1) : u(1) = u'(0) = 0\}.$$

It is easy to verify that $\widehat{A}$ is a correct and selfadjoint operator and that

$$\widehat{A}^{-1} f = \int_0^t (t-x) f(x)dx - \int_0^1 (1-x) f(x)dx, \qquad (24)$$

$$\widehat{A}^{-2} f = \frac{1}{6}\int_0^t (t-x)^3 f(x)dx + \frac{1}{6}\int_0^1 (1-x)(2+2x-x^2-3t^2) f(x)dx, \qquad (25)$$

give the solutions of equations $\widehat{A}u = f$ and $\widehat{A}^2 u = f$ respectively. By simple calculations we find

$$\widehat{A}F = 20(2t^3 - 3t^2 + 1) \in D(\widehat{A}),\; \langle F^t, F\rangle_H = 16301/693,$$

$$\langle \widehat{A}F^t, F\rangle_H = -3670/63, \quad \langle \widehat{A}u, F\rangle_H = \int_0^1 u''(x)(2x^5 - 5x^4 + 10x^2 - 7)dx.$$

With integration by parts we find

$$\langle \widehat{A}^2 u, F\rangle_H = \int_0^1 u^{(4)}(x)(2x^5 - 5x^4 + 10x^2 - 7)dx = -10\int_0^1 u'''(x)(x^4 - 2x^3 + 2x)dx.$$

Equation (22) can now be rewritten as follows

$$\begin{aligned} B_1 u =& u^{(4)} - \Big[12t - 6 - \frac{52c_1}{7}(2t^3 - 3t^2 + 1)\Big]c_1 \int_0^1 (2x^5 - 5x^4 + 10x^2 - 7)u'' dx \\ & -c_1(2t^3 - 3t^2 + 1)\int_0^1 (2x^5 - 5x^4 + 10x^2 - 7)u^{(4)} dx = f(t). \end{aligned} \tag{26}$$

By comparing again (26) with (9) we take $S = \big[12t - 6 - \frac{52c_1}{7}(2t^3 - 3t^2 + 1)\big]c_1$ and $G = c_1(2t^3 - 3t^2 + 1)$. It is evident that $G \in D(\widehat{A})$ and $F, \widehat{A}F$ are linearly independent elements of $D(\widehat{A})$. Then

$$\widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m} = (12t - 6)c_1 - (2t^3 - 3t^2 + 1)c_1 52c_1/7 = S.$$

From $G = (\widehat{A}F)C_m$ it follows $c_1(2t^3 - 3t^2 + 1) = 20(2t^3 - 3t^2 + 1)C_m$. This equation implies that $C_m = c_1/20$. Then by Theorem 3 the operator B_{SG} is correct and selfadjoint if and only if c_1 is a real number and

$$\det L = \det[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C_m] = (126 + 367c_1)/126 \neq 0,$$

i.e. if and only if $c_1 \neq -126/367$. Then $L^{-1} = 126/(126 + 367c_1)$. If we substitute in (24) $f = F = 2t^5 - 5t^4 + 10t^2 - 7$, we get

$$\widehat{A}^{-1}F = (2t^7 - 7t^6 + 35t^4 - 147t^2 + 117)/42$$

and

$$\langle f, \widehat{A}^{-1}F\rangle_H = \frac{1}{42}\int_0^1 (2x^7 - 7x^6 + 35x^4 - 147x^2 + 117)f(x)dx.$$

We also have

$$\langle f, F\rangle_H = \int_0^1 (2x^5 - 5x^4 + 10x^2 - 7)f(x)dx.$$

From this and (11), (25) we obtain formula (23) for the solution of problem (22). □

Let $\Omega = \{x \in \mathbb{R}^2 : |x| = r < 1\}$, $\partial\Omega = \gamma$ and $H^2(\Omega)$ – the Sobolev space of all functions in $L_2(\Omega)$ which have their partial generalized derivatives up to the second order belonging $L_2(0,1)$. The problem

$$\triangle u = f, \quad u_{|\gamma} = 0, \quad u \in H^2(\Omega), \quad f \in L_2(\Omega), \tag{27}$$

is the well known Dirichlet problem. It is known that the corresponding operator $\widehat{A}$ is correct and self-adjoint and

$$u = \widehat{A}^{-1}f = \int_\Omega G(x,y)f(y)dy, \quad \forall f \in L_2(\Omega), \tag{28}$$

where $G(x,y)$ – is Green's function. Using (28) it is easy to verify that

$$\widehat{A}^{-2}f = \int_\Omega G(x,y)\left(\int_\Omega G(y,z)f(z)dz\right)dy. \tag{29}$$

Example 3. The operator $B_1 : L_2(\Omega) \to L_2(\Omega)$ which corresponds to the problem

$$B_1u = \triangle^2 u - 12c_1[1-4c_1\pi(|x|^2-1)]\int_\Omega (|y|^4-4|y|^2+3)\triangle u dy - 48c_1(|x|^2 \tag{30}$$
$$-1)\int_\Omega (|y|^2-1)\triangle u dy = f(x), \quad D(B_1) = \{u \in H^4(\Omega) : u_{|\gamma} = 0,\ \triangle u_{|\gamma} = 0\},$$

is correct and selfadjoint if and only if c_1 is a real constant and $c_1 \neq -\frac{4}{11\pi}$. The unique solution of (30), for every $f \in L_2(\Omega)$, is given by the formula

$$\begin{aligned} u(x) = \widehat{A}^{-2}f + \frac{3c_1}{4(4+11c_1\pi)}\Big[\int_\Omega G(x,y)(|y|^4-4|y|^2+3)dy + \\ \frac{19c_1\pi(|x|^4-4|x|^2+3)}{10(4+11c_1\pi)}\Big]\int_\Omega (|y|^4-4|y|^2+3)f(y)dy + \\ \frac{3c_1(|x|^4-4|x|^2+3)}{4(4+11c_1\pi)}\int_\Omega f(z)\left(\int_\Omega \overline{G(z,y)}(|y|^4-4|y|^2+3)dy\right)dz. \end{aligned} \tag{31}$$

Proof. We again refer to Theorem 3 . If we compare equation (30) with equation (9) it is natural to take

$$\widehat{A}^2u = \triangle^2 u \ \text{ with } \ D(\widehat{A}^2) = D(B_1),\ m=1,\ F = |x|^4-4|x|^2+3.$$

Then $\widehat{A}$ is defined by (27) and $F \in D(\widehat{A})$. By simple calculations we find $\widehat{A}F = \triangle F = 16(r^2-1)$, $\widehat{A}^2F = \triangle^2 F = 64$,

$$\langle F^t, F\rangle_H = \int_\Omega (|y|^4-4|y|^2+3)^2 dy = 2\pi\int_0^1 (r^4-4r^2+3)^2 r dr = 38\pi/15,$$

$$\langle \widehat{A}F^t, F\rangle_H = \int_\Omega 16(|y|^2-1)(|y|^4-4|y|^2+3)dy =$$

$$= 32\pi\int_0^1 (r^2-1)(r^4-4r^2+3)rdr = -44\pi/3,$$

$$\langle \widehat{A}F^t, \widehat{A}F\rangle_H = 256\pi/3, \quad \langle \widehat{A}u, F\rangle_H = \int_\Omega (|y|^4-4|y|^2+3)\triangle u dy,$$

and from the selfadjointness of $\widehat{A}$ we have

$$\langle \widehat{A}^2u, F\rangle_H = \langle \widehat{A}u, \widehat{A}F\rangle_H = 16\int_\Omega (|y|^2-1)\triangle u dy.$$

The equation (30) can now be rewritten as follows

$$B_1u = \triangle^2 u - S\int_\Omega (|y|^4-4|y|^2+3)\triangle u dy - 16G\int_\Omega (|y|^2-1)\triangle u dy = f(x),$$

where $S = 12c_1[1 - 4c_1\pi(|x|^2 - 1)]$, $G = 3c_1(|x|^2 - 1)$. We find $\widehat{A}G = 12c_1$, $\overline{\langle F^t, \widehat{A}G\rangle}_H = 16c_1\pi$ and then $\widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m} = S$. The equation $G = (\widehat{A}F)C_m$ implies that $C_m = 3c_1/16$. From the above it follows that

$$\det L = \det[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C_m] = (4 + 11c_1\pi)/4 \neq 0$$

if and only if $c_1 \neq -4/(11\pi)$. So $L^{-1} = 4/(4+11c_1\pi)$. It is evident that the components of $\widehat{A}F, F$ are linearly independent. Then, by Theorem 3, the operator B_{SG} is correct and selfadjoint if and only if $c_1 \neq -4/(11\pi)$ and c_1 is a real constant. If we substitute $f = F = |x|^4 - 4|x|^3 + 3$ in (28), we get

$$\widehat{A}^{-1}F = \int_\Omega G(x,y)(|y|^4 - 4|y|^2 + 3)dy$$

and

$$\langle f, \widehat{A}^{-1}F\rangle_H = \int_\Omega f(z)\left(\int_\Omega \overline{G(z,y)}(|y|^4 - 4|y|^2 + 3)dy\right)dz.$$

From the above and (11), (29) follows formula (31) for the solution of problem (30). □

A comment from the first author. The second author passed away from a heart attack in the autumn of 2009, at the age of 64. I would like to express my deepest sorry for his sudden death.

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Ioannis Parasidis
Technological Educational Institution of Larissa
41110 Larissa, Greece
E-mail: paras@teilar.gr

Panayotis Tsekrekos
Department of Mathematics
National Technical University
15780 Athens, Greece
E-mail: ptsekre@math.ntua.gr

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