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ON ASYMPTOTIC DECAY OF THE EIGENFUNCTIONS OF ELLIPTIC OPERATORS

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Abstract. It is known that eigenfunctions of many elliptic operators such as Schrödinger operators decrease exponentially. It this paper we suggest a different idea of the proof of this fact. This idea is based on a special transformation Ψ_ε.

1 Introduction

We consider the operator

$$\big(Au\big)(x) = -\operatorname{div}\big(p(x)\operatorname{grad} u\big) + q(x)u(x), \qquad x \in \mathbb{R}^n.$$

We assume that the function $p\colon \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and satisfies the estimates $0 < p_0 \le p(x) \le P_0 < \infty$, $x \in \mathbb{R}^n$, and $\|\operatorname{grad} p(x)\| \le P_0$, $x \in \mathbb{R}^n$, for some p_0 and P_0; the function $q\colon \mathbb{R}^n \to \mathbb{R}$ belongs to $L_\infty(\mathbb{R}^n)$ if $n > 3$ and q belongs to $L_{2\infty}(\mathbb{R}^3)$ if $n = 1, 2, 3$ (the space $L_{2\infty}$ is defined in Section 2).

The main result is Theorem 2. It states that the eigenfunctions of A associated with isolated eigenvalues of finite multiplicity decrease exponentially. Similar results were proved by many authors (see, e. g., [2, 1, 5]). It this paper we present a different idea of the proof. It is based on a simple special transformation Ψ_ε, described is Section 4. Sections 2 and 3 contain some general auxiliary statements.

2 Spaces and operators

In this section we describe the operator (1), which is the main object of our investigation. We also introduce notation and define some function spaces.

For $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ we set

$$|x| = |x_1| + |x_2| + \dots + |x_n|, \qquad \|x\| = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}.$$

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, where $\alpha_1, \alpha_2, \dots, \alpha_n = 0, 1, 2, \dots$. For a function (or a distribution) $u\colon \mathbb{R}^n \to \mathbb{C}$ we denote by $D^\alpha u$ the partial derivative $D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n} u$, where $D_j = \dfrac{\partial}{\partial x_j}$.

We denote by $\mathcal{D}$ the space of all infinitely continuously differentiable functions $u\colon \mathbb{R}^n \to \mathbb{C}$, and we denote by $\mathcal{D}'$ the adjiont of $\mathcal{D}$, i. e. the space of distributions, see, e. g., [12, 11, 15] for details.

We denote by $L_2 = L_2(\mathbb{R}^n)$ the space of all measurable functions $u\colon \mathbb{R}^n \to \mathbb{C}$ with the finite norm

$$\|u\|_{L_2} = \sqrt{\int_{\mathbb{R}^n} |u(x)|^2\,dx}.$$

We denote by $L_{2\infty} = L_{2\infty}(\mathbb{R}^n)$ the space of all measurable functions $u\colon \mathbb{R}^n \to \mathbb{C}$ with the finite norm

$$\|u\|_{L_{2\infty}} = \sup_{k\in\mathbb{Z}^n} \sqrt{\int_{k+[0,1]^n} |u(x)|^2\,dx}.$$

We denote by $L_{\infty\,2} = L_{\infty\,2}(\mathbb{R}^n)$ the space of all measurable functions $u\colon \mathbb{R}^n \to \mathbb{C}$ with the finite norm

$$\|u\|_{L_{\infty\,2}} = \sqrt{\sum_{k\in\mathbb{Z}^n} \bigl(\operatorname{ess\,sup}_{x\in k+[0,1]^n} |u(x)|\bigr)^2}.$$

We note that L_2, $L_{2\infty}$, and $L_{\infty\,2}$ can be considered as subspaces of $\mathcal{D}'$, and $\mathcal{D}$ is contained in L_2, $L_{2\infty}$, and $L_{\infty\,2}$. See [4, 8] for further information about the spaces L_{pq}.

Let $m = 0, 1, 2, \dots$. We denote by $H^m = H^m(\mathbb{R}^n)$ the Sobolev space of all functions $u\colon \mathbb{R}^n \to \mathbb{C}$ such that $D^\alpha u$ (considered as a distribution), $|\alpha| \le m$, belongs to L_2, with the norm

$$\|u\| = \|u\|_{H^m} = \sqrt{\sum_{|\alpha|\le m} \int_{\mathbb{R}^n} |D^\alpha u(x)|^2\,dx}.$$

For $s \ge 0$ the space H^s is defined as the space of all functions $u \in L_2$ such that the function $\xi \mapsto (1+|\xi|)^s \hat{u}(\xi)$, where $\hat{u}$ is the Fourier transform of u, belongs to L^2, with the norm

$$\|u\| = \|u\|_{H^s} = \sqrt{\int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\hat{u}(\xi)|^2\,d\xi}.$$

Clearly, $H^0 = L_2$. It is easy to see that $\mathcal{D}$ is a dense subspace in H^s. Clearly, $H^{s+\varepsilon} \subset H^s$, $\varepsilon > 0$. See, e. g., [15, 13, 9, 14, 3] for more details about the spaces H^s. We denote by H^s_{loc} (in particular, by $L_{2,\mathrm{loc}}$) the subspace of distributions that on any open bounded set coincide with a function belonging to H^s. Let m be a non-negative integer. We say that a sequence $u_n \in H^m_{\mathrm{loc}}$ converges to $u \in H^m_{\mathrm{loc}}$ if u_n converges to u in the norm

$$\|u\|_{H^m(\Omega)} = \sqrt{\sum_{|\alpha|\le m} \int_\Omega |D^\alpha u(x)|^2\,dx}$$

for any bounded open set $\Omega \subset \mathbb{R}^n$. Clearly, $H^2_{\mathrm{loc}} \subset H^1_{\mathrm{loc}} \subset L_{2,\mathrm{loc}}$, and D_j, $j = 1, 2, \dots, n$ continuously acts from H^2_{loc} to H^1_{loc} and from H^1_{loc} to $L_{2,\mathrm{loc}}$, i. e. D_j maps convergent sequences to convergent ones.

Proposition 1. *For any $u, v \in H^1_{\mathrm{loc}}$ and $j = 1, 2, \ldots, n$ one has*

$$D_j(uv) = D_j u \cdot v + u \cdot D_j v.$$

For any $u, v \in H^2_{\mathrm{loc}}$ and $i, j = 1, 2, \ldots, n$ one has

$$D_i D_j(uv) = D_i D_j u \cdot v + D_i u \cdot D_j v + D_j u \cdot D_i v + u \cdot D_i D_j v.$$

Here the dot means the multiplication of measurable functions.

Corollary 1. *The subspace H^m is dense in H^m_{loc}, $m = 0, 1, 2$. Consequently, the subspace $\mathcal{D}$ is dense in H^m_{loc}, $m = 0, 1, 2$, as well.*

Corollary 2. *Let $\vartheta\colon \mathbb{R}^n \to \mathbb{R}$ be an infinitely continuously differentiable function. Then the operator*

$$\bigl(\Theta u\bigr)(x) = \vartheta(x) u(x)$$

acts continuously from H^m_{loc} to H^m_{loc}, $m = 0, 1, 2$.

In this article we consider the operator

$$\bigl(Au\bigr)(x) = -\operatorname{div}\bigl(p(x) \operatorname{grad} u\bigr) + q(x)u(x), \tag{1}$$

where $p, q\colon \mathbb{R}^n \to \mathbb{R}$. We assume that the following assumption holds.

(H) The function $p\colon \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and satisfies the estimates $0 < p_0 \le p(x) \le P_0 < \infty$, $x \in \mathbb{R}^n$, and $\|\operatorname{grad} p(x)\| \le P_0$, $x \in \mathbb{R}^n$, for some p_0 and P_0. The function $q\colon \mathbb{R}^n \to \mathbb{R}$ belongs to $L_\infty(\mathbb{R}^n)$ if $n > 3$ and q belongs to $L_{2\infty}(\mathbb{R}^3)$ if $n = 1, 2, 3$.

By Proposition 1 formula (1) defines a measurable function Au for $u \in H^2_{\mathrm{loc}}$ and the representation

$$\bigl(Au\bigr)(x) = -p(x)\Delta u(x) - \bigl\langle \operatorname{grad} p(x), \operatorname{grad} u(x)\bigr\rangle + q(x)u(x), \quad u \in H^2_{\mathrm{loc}}, \tag{2}$$

holds.

3 Self-adjoint operators

In this section we recall some results of the theory of self-adjoint operators and prove that the operator (1) is self-adjoint.

Let H be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and $T\colon D(T) \subset H \to H$ be a linear operator with the *domain* $D(T)$. The operator T is called *closed* if its *graph* $\Gamma(T) = \{\, (u, Tu) \in H \times H\colon u \in D(T) \,\}$ is a closed subspace of $H \times H$.

Let the domain $D(T)$ be dense in H. In this case the domain $D(T^*)$ of the adjoint operator T^* is the set of all $v \in H$ such that the functional $u \mapsto \langle Tu, v\rangle$ is continuous in the norm of H. For such v the *adjoint operator* T^* is defined uniquely by the identity $\langle Tu, v\rangle = \langle u, T^*v\rangle$.

A linear operator $T\colon D(T) \subset H \to H$ is called *symmetric* if $\langle Tu, v\rangle = \langle u, Tv\rangle$ for all $u, v \in D(T)$. Equivalently, T is symmetric if $T \subset T^*$, i. e., T^* is an extension of T. A linear operator $T\colon D(T) \subset H \to H$ is called *self-adjoint* if $T^* = T$.

Proposition 2 ([11, Theorem 13.8]). *Let* $T\colon D(T)\subset H\to H$ *be a linear operator, and* $D(T)$ *be dense in* H. *Then the graph* $\Gamma(T^*)$ *of* T^* *is the orthogonal complement of* $V\Gamma(T)$ *in* $H\times H$, *where* $V\colon H\times H\to H\times H$ *is defined by the formula* $V(u,v)=(-v,u)$. *Consequently, the adjoint operator is closed. In particular, any self-adjoint operator is closed.*

Proposition 3 ([7, Ch. 5, § 5.2]). *The operator* $-\Delta\colon H^2\subset L_2\to L_2$ *is self-adjoint.*

Let H be a Hilbert space and $T\colon D(T)\subset H\to H$ and $B\colon D(B)\subset H\to H$ be linear operators. The operator B is called *bounded with respect to* T if $D(B)\subset D(T)$ and

$$\|Bu\|\le\varepsilon\|Tu\|+M\|u\|$$

for some $\varepsilon>0$ and $M>0$. We say that B is *completely bounded with respect to* T if for each $\varepsilon>0$ there exists $M>0$ such that this inequality holds.

Proposition 4 ([7, Ch. 4, Theorem 1.1]). *Let* H *be a Hilbert space, and* $T\colon D(T)\subset H\to H$ *and* $B\colon D(B)\subset H\to H$ *be linear operators. Let* B *be completely bounded with respect to* T. *If* T *is closed, then* $T+B$ *is also closed.*

Proposition 5. *Let* $0\le s<2$. *Then for any* $\varepsilon>0$ *there exists* M *such that for any* $u\in H^s$

$$\|u\|_{H^s}\le\varepsilon\|-\Delta u\|_{L_2}+M\|u\|_{L_2}.$$

The following Proposition is known as the Sobolev embedding theorem.

Proposition 6 ([15, Ch. 4, Proposition 1.3]). *If* $s>n/2$, *then each* $u\in H^s(\mathbb{R}^n)$ *is equivalent to a continuous bounded function* v. *Moreover* $|v(x)|\le M\|u\|_{H^s}$, $x\in\mathbb{R}^n$, *for some* M *independent of* u *and* x.

Corollary 3 ([7, Ch. 5, § 5.3]). *Let* $s>n/2$. *Then* $H^s\subset L_{\infty 2}$, *and the natural imbedding of* H^s *into* $L_{\infty 2}$ *is continuous.*

Proof. For the sake of completeness we give the proof. Let $\chi\colon\mathbb{R}^n\to[0,1]$ be an infinitely continuously differentiable function such that

$$\chi(x)=\begin{cases}1, & \text{for } x\in[0,1]^n,\\ 0, & \text{for } x\notin[-1,2]^n.\end{cases}$$

We set $\chi_k(x)=\chi(x-k)$, $k\in\mathbb{Z}^n$. For $k\in\mathbb{Z}^n$ one has

$$\begin{aligned}\operatorname{ess\,sup}_{x\in k+[0,1]^n}|u(x)|&\le\operatorname{ess\,sup}_{x\in k+[0,1]^n}|\chi_k(x)u(x)|\\&\le\operatorname{ess\,sup}_{x\in\mathbb{R}^n}|\chi_k(x)u(x)|\le M\|\chi_k u\|_{H^s}\end{aligned}$$

(the last inequality follows by Proposition 6). Therefore

$$\|u\|_{L_{\infty 2}}=\sqrt{\sum_{k\in\mathbb{Z}^n}\big(\operatorname{ess\,sup}_{x\in k+[0,1]^n}|u(x)|\big)^2}\le M\sqrt{\sum_{k\in\mathbb{Z}^n}\|\chi_k u\|^2_{H^s}}.$$

We denote by $l_2^s = L_2(\mathbb{Z}^n, H^s)$ the space of all families $v = \{\, v_k \in H^s \colon k \in \mathbb{Z}^n \,\}$ with the finite norm $\|v\| = \sqrt{\sum_{k\in\mathbb{Z}^k} \|v_k\|^2_{H^s}}$. We consider the operator $Tu = \{\chi_k u\}$. It is easy to verify that for $s = 0, 1, 2, \dots$ the operator T continuously acts from H^s to l_2^s. From the method of complex interpolation [6, p. 116], [14, Ch. 1, § 4] it follows that the operator T continuously acts from H^s to l_2^s for all $s \in \mathbb{R}$, i. e.

$$\sqrt{\sum_{k\in\mathbb{Z}^n} \|\chi_k u\|^2_{H^s}} \le N\|u\|_{H^s}$$

for some N. Combining the obtained estimates one arrives at the desirable inequality

$$\|u\|_{L_{\infty\,2}} \le MN\|u\|_{H^s}.$$

□

Proposition 7. *Let n be arbitrary and $q \in L_\infty$. Then the operator*

$$\big(Qu\big)(x) = q(x)u(x) \tag{3}$$

is completely bounded with respect to the operator $-\Delta\colon H^2 \subset L_2 \to L_2$.

Proof. Clearly for all $\varepsilon > 0$

$$\|Qu\|_{L_2} = \|qu\|_{L_2} \le \|q\|_{L_\infty} \cdot \|u\|_{L_2} \le \varepsilon\| - \Delta u\|_{L_2} + \|q\|_{L_\infty} \cdot \|u\|_{L_2}.$$

□

Proposition 8. *Let $n = 1, 2, 3$ and $q \in L_{2\infty}$. Then the operator* (3) *is completely bounded with respect to the operator $-\Delta\colon H^2 \subset L_2 \to L_2$.*

Proof. Let $u \in H^2$. Let $s = 7/4$, thus $3/2 < s < 2$. By Proposition 5

$$\|u\|_{H^s} \le \varepsilon\| - \Delta u\|_{L_2} + M\|u\|_{L_2}.$$

By Corollary 3 one has

$$\|u\|_{L_{\infty\,2}} \le N\|u\|_{H^s}$$

for some N. Finally, it is easy to see that

$$\|qu\|_{L_2} \le \|q\|_{L_{2\infty}} \cdot \|u\|_{L_{\infty\,2}}.$$

Combining all these estimates one obtains the desirable inequality

$$\|Qu\|_{L_2} \le \|q\|_{L_{2\infty}} N\big(\varepsilon\| - \Delta u\|_{L_2} + M\|u\|_{L_2}\big).$$

□

Example 1. Let $n = 3$. Then the operator

$$\big(Qu\big)(x) = \frac{u(x)}{\sqrt{x_1^2 + x_2^2 + x_3^2}},$$

where $x = (x_1, x_2, x_3)$ (which is a part of the simplest Schrödinger operator), is completely bounded with respect to the operator $-\Delta\colon H^2 \subset L_2 \to L_2$, because the coefficient $q(x) = \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}$ belongs to $L_{2\infty}$.

Proposition 9. *Let $p\colon \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function satisfying the estimate $\|\operatorname{grad} p(x)\| \le P_0$, $x \in \mathbb{R}^n$, for some P_0. Then the operator*

$$\big(Gu\big)(x) = \langle \operatorname{grad} p(x), \operatorname{grad} u(x)\rangle$$

is completely bounded with respect to the operator $-\Delta\colon H^2 \subset L_2 \to L_2$.

Proof. Let $u \in H^2$. By Proposition 5

$$\|u\|_{H^1} \le \varepsilon\| - \Delta u\|_{L_2} + M\|u\|_{L_2}.$$

It is easy to see that

$$\|Gu\|_{L_2} \le N \cdot \|u\|_{H^1}$$

for some N. Combining these estimates one obtains the desirable inequality

$$\|Gu\|_{L_2} \le N\big(\varepsilon\| - \Delta u\|_{L_2} + M\|u\|_{L_2}\big).$$

□

Proposition 10. *The operator $A\colon H^2 \to L_2$ defined by the formula* (1) *is bounded. Namely, for some M*

$$\|A\colon H^2 \to L_2\| \le M\Big(\|p\|_{L_\infty} + \sum_{j=1}^{n}\Big\|\frac{\partial p}{\partial x_j}\Big\|_{L_\infty} + \|q\|_{L_\infty}\Big)$$

or

$$\|A\colon H^2 \to L_2\| \le M\Big(\|p\|_{L_\infty} + \sum_{j=1}^{3}\Big\|\frac{\partial p}{\partial x_j}\Big\|_{L_\infty} + \|q\|_{L_{\infty 2}}\Big)$$

provided that $n = 1, 2, 3$ and $q \in L_{\infty 2}$.

Proof. The estimate of the norm follows by representation (2). We also recall that by the proof of Proposition 8 that

$$\|qu\|_{L_2} \le \|q\|_{L_{2\infty}} \cdot \|u\|_{L_{\infty 2}}$$

and by Corollary 3 $H^2(\mathbb{R}^3) \subset L_{\infty 2}(\mathbb{R}^3)$. □

Corollary 4. *Operator* (1) *continuously maps H^2_{loc} into $L_{2,\mathrm{loc}}$, i. e. it maps convergent sequences to convergent ones.*

Proof. The proof is evident. □

Proposition 11. *The operator* $A\colon H^2 \subset L_2 \to L_2$ *defined by the formula* (1) *is symmetric.*

Proposition 12. ([11, Theorem 13.16]) *Let* T *be a symmetric operator on a Hilbert space* H. *Then* T *is a closed operator if and only if* $\operatorname{Im}(T \pm i\mathbf{1})$ *is closed, where* $\mathbf{1}$ *is the identity operator.*

Proposition 13. ([10, Theorem VIII.3]) *Let* T *be a symmetric operator on a Hilbert space* H. *Then the operator* T *is self-adjoint if and only if* $\operatorname{Im}(T \pm i\mathbf{1}) = H$.

Theorem 1. *Let assumption* (H) *hold. Then the operator* $A\colon H^2 \subset L_2 \to L_2$ *defined by formula* (1) *is self-adjoint.*

Proof. By Proposition 3 the operator $-\Delta\colon H^2 \subset L_2 \to L_2$ is self-adjoint. Hence $-\Delta\colon H^2 \subset L_2 \to L_2$ is closed. By the assumption $0 < p_0 \le p(x) \le P_0 < \infty$, $x \in \mathbb{R}^n$, it follows that the operator $p\,\Delta\colon H^2 \subset L_2 \to L_2$, where $\bigl(p\,\Delta u\bigr)(x) = p(x)\Delta u(x)$, is closed as well.

We represent the operator A in the form (2) or, more briefly, $A = -p\,\Delta + G + Q$, where

$$\bigl(Gu\bigr)(x) = \langle \operatorname{grad} p(x), \operatorname{grad} u(x)\rangle, \qquad \bigl(Qu\bigr)(x) = q(x)u(x), \qquad u \in H^2.$$

We recall that by Propositions 7, 8, and 9 the operators Q and G are completely bounded with respect to Δ. By Proposition 4 this implies that A is a closed operator for any p and q satisfying assumption (H).

Next, we consider the homotopy

$$\bigl(A\{t\}u\bigr)(x) = -\operatorname{div}\bigl((1 - t + tp(x)) \operatorname{grad} u\bigr) + tq(x)u(x), \qquad u \in H^2, t \in [0, 1].$$

Clearly, $A\{0\} = -\Delta$ and $A\{1\} = A$. By the above $A\{t\}\colon H^2 \subset L_2 \to L_2$ is a closed operator for all $t \in [0, 1]$.

By Proposition 10 it follows that $A\{t\}\colon H^2 \to L_2$ is continious for all $t \in [0, 1]$. By Proposition 12 $\operatorname{Im}(A\{t\} \pm i\mathbf{1})$ is closed for all t. At the same time by Propositions 3 and 13 $\operatorname{Im}(A\{0\} \pm i\mathbf{1}) = L_2$. By [8, Theorem 1.3.2(b) and Proposition 1.3.5] it follows that $\operatorname{Im}(A\{t\} \pm i\mathbf{1}) = L_2$ for all t. By Proposition 11 $A\{t\}\colon H^2 \subset L_2 \to L_2$ is symmetric. Finally, by Proposition 13, this implies that $A\{t\}$ is self-adjoint. In particular, $A\{1\} = A$ is self-adjoint. □

4 The transformation Ψ_ε

We denote by $\eta\colon \mathbb{R}^n \to \mathbb{R}$ a fixed infinitely continuously differentiable function satisfying the property

$$\eta(x) = \|x\| \qquad \text{for } \|x\| \ge 1. \tag{4}$$

For any $\varepsilon \in \mathbb{R}$ we consider the operator

$$\bigl(\Psi_\varepsilon u\bigr)(x) = e^{\varepsilon\eta(x)}u(x).$$

By Corollary 2 Ψ_ε acts in H^m_{loc}, $m = 0, 1, 2$. It is interesting to note that Ψ_ε forms a representation of the group $\mathbb{R}$, i. e. $\Psi_\varepsilon\Psi_\delta = \Psi_{\varepsilon+\delta}$.

For any $\varepsilon \in \mathbb{R}$ and $s \geq 0$ we denote by $H^s_\varepsilon = H^s_\varepsilon(\mathbb{R}^n)$ the space of all functions of the form $\Psi_\varepsilon u$, where $u \in H^s$. In particular, $(L_2)_\varepsilon = (L_2)_\varepsilon(\mathbb{R}^n)$ is the space of all functions of the form $\Psi_\varepsilon u$, where $u \in L_2$. It is easy to see that the definition of H^2_ε does not depend on the choice of a function η with the property (4). Evidently, H^2_0 coincides with H^2 and $(L_2)_0$ coincides with L_2. Obviously, $H^2_\varepsilon \subset H^2_{\mathrm{loc}}$ for all $\varepsilon \in \mathbb{R}$.

We set

$$A[\varepsilon] = \Psi_\varepsilon A \Psi_{-\varepsilon}, \qquad \varepsilon \in \mathbb{R}.$$

By Corollary 4 the operator $A[\varepsilon]$ acts from H^2_{loc} to $L_{2,\mathrm{loc}}$.

Proposition 14. *Let assumption* (H) *hold. Then for* $u \in H^2_{\mathrm{loc}}$ *one has*

$$\begin{aligned}\big(A[\varepsilon]u\big)(x) = &-\operatorname{div}\big(p(x)\operatorname{grad} u(x)\big) + q(x)u(x)\\ &+ \varepsilon\big\langle u(x)\operatorname{grad} p(x) + 2p(x)\operatorname{grad} u(x),\ \operatorname{grad}\eta(x)\big\rangle\\ &+ \varepsilon p(x)u(x)\Delta\eta(x) - \varepsilon^2 p(x)u(x)\|\operatorname{grad}\eta(x)\|.\end{aligned} \tag{5}$$

Proof. For $u \in \mathcal{D}$ one has

$$\big(\operatorname{grad}\Psi_{-\varepsilon}u\big)(x) = \operatorname{grad}\big(e^{-\varepsilon\eta(x)}u(x)\big) = u(x)\operatorname{grad} e^{-\varepsilon\eta(x)} + e^{-\varepsilon\eta(x)}\operatorname{grad} u(x)$$

and

$$\begin{aligned}\operatorname{div}\big(p(x)\operatorname{grad}\Psi_{-\varepsilon}u\big)(x) &= \operatorname{div}\big(p(x)\big(u(x)\operatorname{grad} e^{-\varepsilon\eta(x)} + e^{-\varepsilon\eta(x)}\operatorname{grad} u(x)\big)\big)\\ &= \operatorname{div}\big(p(x)u(x)\operatorname{grad} e^{-\varepsilon\eta(x)} + e^{-\varepsilon\eta(x)}p(x)\operatorname{grad} u(x)\big)\\ &= u(x)\big\langle\operatorname{grad} p(x), \operatorname{grad} e^{-\varepsilon\eta(x)}\big\rangle + p(x)\big\langle\operatorname{grad} u(x), \operatorname{grad} e^{-\varepsilon\eta(x)}\big\rangle\\ &+ p(x)u(x)\operatorname{div}\operatorname{grad} e^{-\varepsilon\eta(x)} + p(x)\big\langle\operatorname{grad} e^{-\varepsilon\eta(x)}, \operatorname{grad} u(x)\big\rangle\\ &+ e^{-\varepsilon\eta(x)}\operatorname{div}\big(p(x)\operatorname{grad} u(x)\big)\\ &= \big\langle u(x)\operatorname{grad} p(x) + 2p(x)\operatorname{grad} u(x), \operatorname{grad} e^{-\varepsilon\eta(x)}\big\rangle\\ &+ p(x)u(x)\operatorname{div}\operatorname{grad} e^{-\varepsilon\eta(x)} + e^{-\varepsilon\eta(x)}\operatorname{div}\big(p(x)\operatorname{grad} u(x)\big)\\ &= -\varepsilon e^{-\varepsilon\eta(x)}\big\langle u(x)\operatorname{grad} p(x) + 2p(x)\operatorname{grad} u(x), \operatorname{grad}\eta(x)\big\rangle\\ &- \varepsilon p(x)u(x)\operatorname{div}\big(e^{-\varepsilon\eta(x)}\operatorname{grad}\eta(x)\big) + e^{-\varepsilon\eta(x)}\operatorname{div}\big(p(x)\operatorname{grad} u(x)\big)\\ &= -\varepsilon e^{-\varepsilon\eta(x)}\big\langle u(x)\operatorname{grad} p(x) + 2p(x)\operatorname{grad} u(x), \operatorname{grad}\eta(x)\big\rangle\\ &- \varepsilon p(x)u(x)\big\langle\operatorname{grad} e^{-\varepsilon\eta(x)}, \operatorname{grad}\eta(x)\big\rangle - \varepsilon e^{-\varepsilon\eta(x)}p(x)u(x)\Delta\eta(x)\\ &+ e^{-\varepsilon\eta(x)}\operatorname{div}\big(p(x)\operatorname{grad} u(x)\big)\\ &= -\varepsilon e^{-\varepsilon\eta(x)}\big\langle u(x)\operatorname{grad} p(x) + 2p(x)\operatorname{grad} u(x), \operatorname{grad}\eta(x)\big\rangle\\ &+ \varepsilon^2 e^{-\varepsilon\eta(x)}p(x)u(x)\|\operatorname{grad}\eta(x)\| - \varepsilon e^{-\varepsilon\eta(x)}p(x)u(x)\Delta\eta(x)\\ &+ e^{-\varepsilon\eta(x)}\operatorname{div}\big(p(x)\operatorname{grad} u(x)\big).\end{aligned}$$

Consequently,

$$\begin{aligned}\big(A[\varepsilon]u\big)(x) &= -\Psi_\varepsilon \operatorname{div}\big(p(x)\operatorname{grad}\Psi_{-\varepsilon}u\big)(x) + q(x)u(x)\\ &= -\operatorname{div}\big(p(x)\operatorname{grad}u(x)\big) + q(x)u(x)\\ &+ \varepsilon\big\langle u(x)\operatorname{grad}p(x) + 2p(x)\operatorname{grad}u(x), \operatorname{grad}\eta(x)\big\rangle\\ &+ \varepsilon p(x)u(x)\Delta\eta(x) - \varepsilon^2 p(x)u(x)\|\operatorname{grad}\eta(x)\|.\end{aligned}$$

Clearly (cf. Proposition 10), the right-hand side of the last formula defines a continuous operator, acting from H^2_{loc} to $L_{2,\mathrm{loc}}$. By Corollary 1 it coincides with $A[\varepsilon]\colon H^2_{\mathrm{loc}} \to L_{2,\mathrm{loc}}$. □

Proposition 15. *The operator $A[\varepsilon]$ continuously acts from H^2 to L_2 for all $\varepsilon \in \mathbb{R}$. The operator $A[\varepsilon]\colon H^2 \to L_2$ continuously depends on the parameter ε.*

Proof. The proof is similar to that of Proposition 10. □

Corollary 5. *The operator A maps H^2_ε to $(L_2)_\varepsilon$ for all $\varepsilon \in \mathbb{R}$.*

Proof. The proof is evident. □

5 Exponential decay of eigenfunctions

Proposition 16. *Let X and Y be Banach spaces and $A, B\colon X \to Y$ be bounded linear operators. If the operator A is invertible and*

$$\|B\| \cdot \|A^{-1}\| < 1$$

then the operator $A - B$ is also invertible. Moreover

$$\|(A-B)^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|B\| \cdot \|A^{-1}\|}.$$

Proof. The proof follows by the representation

$$(A-B)^{-1} = A^{-1} + A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} + \dots.$$

□

Theorem 2. *Let assumption* (H) *hold. Let λ_0 be an isolated eigenvalue of the operator A defined by formula* (1)*, and the eigenspace E_{λ_0} associated with λ_0 be finite dimensional. Then $E_{\lambda_0} \subset H^2_\varepsilon$ for some $\varepsilon < 0$.*

Proof. Let Γ be a circumference with the centre λ_0 oriented anticlockwise. We assume that the radius of the circumference Γ is sufficiently small, so it does not surround points of the spectrum of A different from λ_0. We consider the resolvent $R(\lambda, A) = (\lambda\mathbf{1} - A)^{-1}\colon L_2 \to H^2$, where $\mathbf{1}$ is the identity operator. By Proposition 16 the resolvent $R(\lambda, A)$ continuously depends on $\lambda \in \Gamma$. Hence the maximum of $\|R(\lambda, A)\colon L_2 \to H^2\|$ over $\lambda \in \Gamma$ is finite.

By Propositions 15 and 16 the operator $\lambda\mathbf{1} - A[\varepsilon]$ is invertible for $\lambda \in \Gamma$ and $|\varepsilon|$ small enough and the inverse operator $R(\lambda, A[\varepsilon]) = (\lambda\mathbf{1} - A[\varepsilon])^{-1}\colon L_2 \to H^2$ continuously depends on ε.

For sufficiently small $|\varepsilon|$ we consider the Riesz projector

$$P[\varepsilon] = \frac{1}{2\pi i}\int_\Gamma (\lambda\mathbf{1} - A[\varepsilon])^{-1}\,d\lambda.$$

It acts from L_2 to H^2 and continuously depends on ε. The image of $P[\varepsilon]$ is the eigenspace associated with the part of the spectrum of $A[\varepsilon]$ surrounded by Γ. In particular, the image of $P[0]$ is E_{λ_0}. Since E_{λ_0} is finite-dimensional and $P[\varepsilon]$ continuously depends on ε, the dimension of the image of $P[\varepsilon]$ does not depend on ε.

We consider the operator

$$A\{\varepsilon\} = \Psi_{-\varepsilon}A[\varepsilon]\Psi_\varepsilon.$$

It acts from H^2_ε to $(L_2)_\varepsilon$ (cf. Corollary 5). Clearly, it coincides with the restriction of the operator A to H^2_ε. We notice that the operator $\lambda\mathbf{1} - A\{\varepsilon\}\colon H^2_\varepsilon \to (L_2)_\varepsilon$ coincides with $\Psi_{-\varepsilon}\big(\lambda\mathbf{1} - A[\varepsilon]\big)\Psi_\varepsilon$. Therefore the spectrum of $A\{\varepsilon\}\colon H^2_\varepsilon \subset (L_2)_\varepsilon \to (L_2)_\varepsilon$ coincides with the spectrum of $A[\varepsilon]\colon H^2 \subset L_2 \to L_2$. Next we consider the operator

$$\begin{aligned} P\{\varepsilon\} = \Psi_{-\varepsilon}P[\varepsilon]\Psi_\varepsilon &= \frac{1}{2\pi i}\int_\Gamma \Psi_{-\varepsilon}(\lambda\mathbf{1} - A[\varepsilon])^{-1}\Psi_\varepsilon\,d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma \big(\Psi_{-\varepsilon}(\lambda\mathbf{1} - A[\varepsilon])\Psi_\varepsilon\big)^{-1}\,d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma (\lambda\mathbf{1} - A\{\varepsilon\})^{-1}\,d\lambda. \end{aligned}$$

By this representation it follows that $P\{\varepsilon\}$ is the spectral projector associated with the part of the spectrum of $A\{\varepsilon\}$ surrounded by Γ. Since $P[\varepsilon]$ and $P\{\varepsilon\}$ are similar projectors (namely, $P\{\varepsilon\} = \Psi_{-\varepsilon}P[\varepsilon]\Psi_\varepsilon$), their images are isomorphic spaces and the dimensions of their images are the same.

Let ε be less than zero. In this case the operator $A\{\varepsilon\}\colon H^2_\varepsilon \to (L_2)_\varepsilon$ is a restriction of the operator $A\colon H^2 \to L_2$ to the subspace $H^2_\varepsilon \subset H^2$. Consequently, the operator $(\lambda\mathbf{1} - A\{\varepsilon\})^{-1}\colon (L_2)_\varepsilon \to H^2_\varepsilon$ is the restriction of the operator $(\lambda\mathbf{1} - A)^{-1}\colon L_2 \to H^2$. Therefore by the representation

$$P\{\varepsilon\} = \frac{1}{2\pi i}\int_\Gamma (\lambda\mathbf{1} - A\{\varepsilon\})^{-1}\,d\lambda$$

it follows that the operator $P\{\varepsilon\}\colon (L_2)_\varepsilon \to H^2_\varepsilon$ is the restriction of the operator $P\{0\} = P[0]\colon L_2 \to H^2$. The operators $P\{\varepsilon\}$ and $P[0]$ are projectors and dimensions of their images coincide. This implies that the images of $P\{\varepsilon\}$ and $P[0]$ coincide. But the image of $P\{\varepsilon\}$ is contained in H^2_ε. Therefore the image of $P[0]$ is also a subspace of H^2_ε with $\varepsilon < 0$. □

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