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ON INFINITE DIFFERENTIABILITY OF SOLUTIONS OF NONHOMOGENEOUS ALMOST HYPOELLIPTIC EQUATIONS

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Abstract. A linear differential operator $P(D)$ with constant coefficients is called *almost hypoelliptic* if all derivatives $P^{(\nu)}(\xi)$ of the characteristic polynomial $P(\xi)$ can be estimated above via $P(\xi)$. In this paper it is proved that all solutions of the equation $P(D)u = f$ where f and all its derivatives are square integrable with a certain exponential weight, which are square integrable with the same weight, are also such that all their derivatives are square integrable with this weight, if and only if the operator $P(D)$ is *almost hypoelliptic*.

1 Introduction

After in 1950's L. Hörmander introduced the concept of a hypoelliptic differential equation $P(D)u = f$ all distributional solutions u of which with an infinitely differentiable right-hand side f are infinitely differentiable (see [15], [16]), a problem arose of finding additional assumptions on solutions u of more general, non-hypoelliptic equations ensuring that these solutions are infinitely differentiable.

In [14], [8] L. Gårding, B. Malgrange and L. Ehrenprice studied the class of partially hypoelliptic equations all solutions of which with an infinitely differentiable right-hand side are infinitely differentiable under the a priori assumption that they are infinitely differentiable with respect to a certain group of the variables.

In [3] Ya.S. Bugrov constructed an example of a non-hypoelliptic equation, all solutions of which in the half-space are infinitely differentiable provided they are square integrable in the half-space together with some of their derivatives.

In [4], [6] V.I. Burenkov considered the equation $P(D)u = f$ in the cylinder $\Omega = \Omega_m \times E^{n-m}$ with $0 \leq m < n$ where Ω_m is an open set in E^m (if $m = 0$ then $\Omega = E^n$) and f and all its derivatives are m-locally square integrable on Ω, i.e. square integrable on $\Omega_m \times E^{n-m}$ for all compacts $Q_m \subset \Omega_m$ (if $m = 0$ square integrable on E^n). Necessary and sufficient conditions on P were found ensuring that all solutions u of this equation with any such f, which are m-locally square integrable on Ω together with some of their derivative are of the same class as f (in particular are infinitely differentiable).

The class of such operators is essentially wider than the class of hypoelliptic operators.

In [10], [17] the notion of an *almost hypoelliptic* polynomial was introduced and some sufficient conditions for almost hypoellipticity were found in terms of the homogeneity orders and multiplicity of the roots of certain subpolynomials.

In [9] O.R. Gabrielyan obtained necessary and sufficient conditions for almost hypoelliptipticity of two-dimensional polynomials in terms of multiplicity of the roots of the certain homogeneous subpolynomials.

We use the following standard notations: N – the set of all natural numbers, $N_0 = N \cup \{0\}$, $N_0^n = N_0 \times \ldots \times N_0$ – the set of all n-dimensional multi-indices, E^n and R^n – the n-dimensional euclidean spaces of points (vectors) $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$ respectively. For $\xi \in R^n$, $x \in E^n$ and $\alpha \in N_0^n$ we put $|\xi| = \sqrt{\xi_1^2 + \ldots + \xi_n^2}$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, where $D_j = \frac{\partial}{\partial \xi_j}$ or $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$ $(j = 1, \ldots n)$. Finally we put $C^n = R^n \times iR^n$.

For a linear differential operator with constant coefficients $P(D) = \sum_\alpha \gamma_\alpha D^\alpha$, let $P(\xi) = \sum_\alpha \gamma_\alpha \xi^\alpha$ be its characteristic polynomial (complete symbol), where the sum extends over a finite collection of multi-indices $(P) = \{\alpha \in N_0^n, \gamma_\alpha \neq 0\}$ and let $m = \operatorname{ord} P = \max\{|\alpha|, \alpha \in (P)\}$.

An operator $P(D)$ (a polynomial $P(\xi)$) is called hypoelliptic (see [15]), if all solutions $u \in D'$ $(D' = D'(E^n)$ is the set of distributions) of the equation $P(D)u = f$ are infinitely differentiable (belong to $C^\infty = C^\infty(E^n)$) for all $f \in C^\infty$.

This holds if and only if (see [16], Theorem 11.1.1) all solutions $u \in D'$ of the equation $P(D)u = 0$ are infinitely differentiable.

L. Hörmander has proved (see [16], Theorem 11.1.1 and Theorem 11.1.3), that an operator $P(D)$ is hypoelliptic if and only if any of the following equivalent conditions is satisfied.

1) $\operatorname{Sing\,Supp} u = \operatorname{Sing\,Supp} P(D)u$ for any open set $\Omega \subset E^n$ and $u \in D'$,

2) if $0 \neq \nu \in N_0^n$, then $P^{(\nu)}(\xi)/P(\xi) \equiv D^\nu P(\xi)/P(\xi) \to 0$ as $|\xi| \to \infty$,

3) $d_P(\xi) \to \infty$ as $|\xi| \to \infty$, where $d_P(\xi)$ is the distance from the point $\xi \in R^n$ to the surface $\{\zeta; \zeta \in C^n, P(\zeta) = 0\}$.

The following question naturally arises. Let the symbol $P(\xi)$ of the operator $P(D)$ satisfy a weaker condition than condition 2):

$$|P^\nu(\xi)| \, / \, [1 + |P(\xi)|] \leq C < \infty \quad \forall \xi \in R^n, \quad \forall \nu \in N_0^n, \tag{1.1}$$

or let 3) be replaced by the requirement that

$$d_P(\xi) \geq \varepsilon > 0 \tag{1.2}$$

for all sufficiently large $\xi \in R^n$. Which conditions should be imposed on a function $f \in C^\infty$ and on solutions $u \in D'(E^n)$ of the equation $P(D)u = f$ to ensure infinite differentiability of u in E^n ?

We shall use the following classes of operators and functions.

Definition 1.1. *An operator $P(D)$ (and a polynomial $P(\xi)$) is called* ***almost hypoelliptic*** *if the polynomial P satisfies one of (equivalent) conditions (1.1), (1.2).*

By Lemma 11.1.4 of [16] for any polynomial $P(\xi)$ there exists a constant $\sigma_1 = \sigma_1(P) > 0$ such that the following inequality holds:

$$\sum_{|\nu|>0} d_P^{|\nu|}(\xi)|P^{(\nu)}(\xi)| \leq \sigma_1|P(\xi)| \forall\xi \in R^n. \tag{1.3}$$

On the other hand if a polynomial $P \in I_n$, i.e $|P(\xi)| \to \infty$ as $|\xi| \to \infty$ (in this connection we note that in [11] - [12] necessary and sufficient conditions in terms of the coefficients of P ensuring that $P \in I_2$ were found), then it is **almost hypoelliptic** if and only if

$$\rho_P = \lim_{t\to\infty} \inf_{|\xi|=t} d_P(\xi) > 0 \cdot \tag{1.4}$$

For any $\delta > 0$ and $m \in N$, let $L_{2,\delta} \equiv L_{2,\delta}(E^n)$ be the set of all functions u locally integrable on E^n with finite norms

$$||u||_{L_{2,\delta}} = \left(\int_{E^n} |u(x)|^2 e^{-2\delta|x|} dx\right)^{1/2} \tag{1.5}$$

and $H_\delta^m = H_\delta^m(E^n)$ be the set of all functions $u \in L_{2,\delta}$ with finite norms

$$||u||_{H_\delta^m} = \sum_{|\alpha|\leq m} ||D^\alpha u||_{L_{2,\delta}}. \tag{1.6}$$

Finally we put

$$H_\delta^\infty = \bigcap_{m=1}^{\infty} H_\delta^m \cdot$$

It is obvious that $L_{2,\delta}$ and H_δ^m are Banach spaces and H_δ^∞ is a Fréchet space such that $H_\delta^\infty \subset C^\infty$ for any $\delta > 0$.

Let

$$N(P,\delta) = \{u \in L_{2,\delta} : P(D)u \in H_\delta^\infty\},$$

where $P(D)u$ is understood in the distributional sense, i.e. $u \in N(P,\delta)$ means that $u \in L_{2,\delta}$ and there exists $f \in H_\delta^\infty$ such that (here and the sequel $\int$ means $\int_{E^n}$)

$$\int u\overline{P(-D)\varphi}dx = \int f\bar{\varphi}dx \quad \forall\varphi \in C_0^\infty,$$

where C_0^∞ is the set of all functions in C^∞ with compact support

The following statement is the main result of the article.

Theorem. *An operator $P \in I_n$ is almost hypoelliptic if and only if there exists a number $\delta > 0$ such that $N(P,\delta) \subset H_\delta^\infty$.*

Remark. The case $\delta = 0$ is quite different from the case $\delta > 0$. In [4,6] it is proved that $N(P, 0) \subset H_0^\infty$ if and only if there exist $c, M > 0$ such that $|P(\xi)| \geq c$ for all $\xi \in R^n$ satisfying $|\xi| \geq M\cdot$

Let $\varepsilon > 0$ and $L_{2,\delta,\varepsilon}$ be the set of all function u locally integrable on E^n with finite norms

$$||u||_{L_{2,\delta,\varepsilon}} = \left(\int\limits_{E^n} |u(x)|^2 e^{-2\delta|x|^\varepsilon} dx \right)^{1/2}.$$

Consider the space $H_{\delta,\varepsilon}^\infty$ and the sets $N(P, \delta, \varepsilon)$ obtained by replacing $L_{2,\delta}$ by $L_{2,\delta,\varepsilon}$ in the above definitions. In [7] V.I. Burenkov proved that if $\varepsilon > 1$ then $N(P, \delta, \varepsilon) \subset H_{\delta,\varepsilon}$ if and only if the operator P is hypoelliptic. An interesting question arises: what happens in the case $0 < \varepsilon < 1$?

2 Estimates for functions in H_δ^∞

Lemma 2.1. *Let $P \in I_n$ be an almost hypoelliptic polynomial, the number ρ_P be defined by formula (1.4) and σ_1 be the minimal number for which inequality (1.3) is satisfied. Then for any $\rho \in (0, \rho_P)$ there exists a number $\sigma_2 = \sigma_2(\rho, P) > 0$ such that*

$$\sum_{|\alpha|>0} \rho^{|\alpha|} \, |P^{(\alpha)}(\xi)| \leq \sigma_1 |P(\xi)| + \sigma_2 \qquad \forall \xi \in R^n. \tag{2.1}$$

Proof. Let $\rho \in (0, \rho_P)$. Then by the definition of the number ρ_P there exists a number $M = M(\rho) > 0$ such that

$$d_P(\xi) \geq \rho \quad \forall \xi \in R^n; \ |\xi| \geq M. \tag{2.2}$$

Let us put

$$\sigma_2 = \sigma_2(\rho, P) = \max_{|\xi| \leq M} \sum_{|\nu|>0} \rho^{|\nu|} |P^{(\nu)}(\xi)|,$$

then estimate (2.1) follows by estimates (1.3) and (2.2). □

For convenience instead of the weight function $e^{-\delta|x|}$ we consider the equivalent smooth weight function $g_\delta \in C^\infty$.

We assume that $g \in C^\infty$ is a fixed positive function such that for some $\kappa > 0$

$$\kappa^{-1} e^{-|x|} \leq g(x) \leq \kappa e^{-|x|} \qquad \forall x \in E^n\cdot$$

Moreover, we suppose that for any $\alpha \in N_0^n$ there exists a number $\kappa_\alpha > 0$ ($\kappa_0 = \kappa$) for which

$$|D^\alpha g(x)| \leq \kappa_\alpha 2pt g(x) \quad \forall x \in E^n\cdot$$

Note, that for example g may be constructed as the regularization (i.e. the averaging) of the function H, defined by $H(x) = e^{-|x|}$ for $|x| > 1$ and $H(x) = e^{-1}$

for $|x| \leq 1$ by means of a fixed nonnegative weight function $\varphi \in C_0^\infty$ such that $\int \varphi(x)dx = 1$.

For $\delta > 0$, we set $g_\delta(x) = g(\delta x)$. Then by the definition of g

$$\kappa^{-1} e^{-\delta|x|} \leq g_\delta(x) \leq \kappa e^{-\delta|x|} \quad \forall x \in E^n, \tag{2.3}$$

$$|D^\alpha g_\delta(x)| \leq \kappa_\alpha \delta^{|\alpha|} g_\delta(x) \quad \forall \alpha \in N_0^n, \ \forall x \in E^n. \tag{2.4}$$

We start with the statements proved in [13] (see Lemma 1.1 and Lemma 1.2 in [13])

Lemma 2.2. *Let $m \in N, a_i, b_i \geq 0$ $(i = 0, 1, ..., m)$, $d > 0$ and $t > 1$. Then*

a) if $a_0 = b_0$ and

$$a_k \leq b_k + d \sum_{j=0}^{k-1} t^j a_j \quad (k = 1, 2, ..., m), \tag{2.5}$$

then for $\sigma_3 = 2[2dt^{m-1} + 1]^m$

$$\sum_{k=0}^{m} a_k \leq \sigma_3 \sum_{k=0}^{m} b_k, \tag{2.6}$$

b) if $a_m = b_m$ and

$$t^k a_k \leq t^k b_k + d \sum_{j=k+1}^{m} t^j a_j \quad (k = 0, 1, ..., m-1), \tag{2.7}$$

then

$$\sum_{k=0}^{m} t^k a_k \leq \sum_{k=0}^{m} (1+d)^k t^k b_k. \tag{2.8}$$

Lemma 2.3. *If the set $G \subset E^n$ is contained in the ball $S_T = \{x \in E^n, |x| \leq T\}$, then for any $\delta > 0$*

$$\sup_{y \in G} g_\delta(x+y) \leq \sigma_4 g_\delta(x) \quad \forall x \in E^n, \tag{2.9}$$

$$\sup_{y \in G} |g_\delta(x+y) - g_\delta(x)| \leq \sigma_5 g_\delta(x) \quad \forall x \in E^n, \tag{2.10}$$

where the function g_δ is as defined above, $\sigma_4 = \kappa^2 e^{\delta T}$ and $\sigma_5 = \kappa^2 (\delta T) e^{\delta T} (\max_{|\alpha|=1} \kappa_\alpha) \sqrt{n}$.

In the sequel for a fixed almost hypoelliptic polynomial P of order m, fixed function g, and for any $\rho \in (0, \rho_P)$, let positive numbers κ_α, $|\alpha| \leq m$, satisfy inequalities (2.3), (2.4), numbers $\sigma_1(\rho), \sigma_2(\rho)$, satisfy inequality (2.1) and

$$h = h(g, m) = \sum_{0<|\alpha|<m} \frac{\kappa_\alpha}{\alpha!}, \tag{2.11}$$

$$\Delta_0 = \Delta_0(g, m, \rho) = \rho \min_{0<|\alpha|\leq m} \left\{ \left(\frac{\alpha!}{2\kappa_\alpha} \right)^{1/|\alpha|} [(1+h)^m \sigma_1]^{1/|\alpha|} \right\}. \tag{2.12}$$

Lemma 2.4. *Let $P(D)$ be an almost hypoelliptic operator of degree m and the number ρ_P be defined by formula (1.4), then it follows that for any $\rho \in (0, \rho_P)$, $\delta \in (0, \Delta_0(\rho)]$ and $u \in H_\delta^\infty$*

$$\sum_{|\alpha|>0} \rho^{|\alpha|} \|[P^{(\alpha)}(D)u]g_\delta\|_{L_2} \leq 2(1+h)^m[\sigma_1(\rho)\|[P(D)u]g_\delta\|_{L_2}+$$

$$+\sigma_2(\rho)\|ug_\delta\|_{L_2}]. \tag{2.13}$$

Proof. First we prove that for any $t > 0$, $\delta \in (0, t]$ and $u \in H_\delta^\infty$

$$\sum_{|\alpha|>0} t^{|\alpha|} \|[P^{(\alpha)}(D)u]g_\delta\|_{L_2} \leq (1+h)^m \sum_{|\alpha|>0} t^{|\alpha|} \|\{P^{(\alpha)}(D)[ug_\delta]\|_{L_2}. \tag{2.14}$$

Put for $j = 1, 2, ..., m$

$$a_j = \sum_{|\alpha|=j} t^{|\alpha|} \|[P^{(\alpha)}(D)u]g_\delta\|_{L_2}, \quad b_j = \sum_{|\alpha|=j} t^{|\alpha|} \|P^{(\alpha)}(D)[ug_\delta]\|_{L_2}.$$

It is obvious that $a_m = b_m$. Then by the Leibnitz formula it follows that for any $t > 0$ and $j = 1, 2, ..., m-1$

$$a_j = \sum_{|\alpha|=j} t^{|\alpha|} \|[P^{(\alpha)}(D)u\|_{L_2} = \sum_{|\alpha|=j} t^{|\alpha|} \{\|P^{(\alpha)}(D)[ug_\delta]-$$

$$-\sum_{|\beta|>0} \frac{1}{\beta!}[P^{(\alpha+\beta)}(D)u]D^\beta g_\delta\|_{L_2}\} \leq b_j+$$

$$+\sum_{|\alpha|=j} t^{|\alpha|} \sum_{|\beta|>0} \frac{1}{\beta!}[P^{(\alpha+\beta)}(D)u]D^\beta g_\delta\|_{L_2}.$$

Applying property (2.4) of the function g_δ, we get that for any $\delta \in (0, t]$

$$a_j \leq b_j + \sum_{|\alpha|=j} t^{|\alpha|} \sum_{|\beta|>0} \frac{\kappa_\beta}{\beta!} \delta^{|\beta|} \|[P^{(\alpha+\beta)}(D)u]g_\delta\|_{L_2} \leq$$

$$\leq b_j + \sum_{l=j+1}^{m} t^l \sum_{|\gamma|=l} \|[P^{(\gamma)}(D)u]g_\delta\|_{L_2} \sum_{0<\nu<\gamma} \frac{\kappa_\nu}{\nu!} =$$

$$= b_j + h \sum_{l=j+1}^{m} a_j \quad j = 1, ..., m-1\cdot$$

This means that the numbers α_j, b_j $j = 1, ..., m$ satisfy condition b) of Lemma 2.2, which proves estimate (2.14). Hence, by Lemma 2.1, inequality (2.14) and the Parseval equality we conclude that

$$\sum_{|\alpha|>0} \rho^{|\alpha|} \|[P^{(\alpha)}(D)u]g_\delta\|_{L_2} \leq (1+h)^m \sum_{|\alpha|>0} \rho^{|\alpha|} \|P^{(\alpha)}(D)[ug_\delta]\|_{L_2} =$$

$$= (1+h)^m \sum_{|\alpha|>0} \rho^{|\alpha|} \|P^{(\alpha)}(\xi) F(ug_\delta)(\xi)\|_{L_2} \leq$$

$$\leq (1+h)^m \left[\sigma_1 \|P(\xi) F(ug_\delta)\|_{L_2} + \sigma_2(\rho) \|F(ug_\delta)\|_{L_2}\right] =$$

$$= (1+h)^m \left[\sigma_1 \|P(D)(ug_\delta)\|_{L_2} + \sigma_2(\rho) \|ug_\delta\|_{L_2}\right], \tag{2.15}$$

where F is the Fourier transform.

Using the Leibnitz formula and estimates (2.4), for the first summand in the right-hand side we obtain

$$\|P(D)(ug_\delta)\|_{L_2} \leq \|[P(D)u] g_\delta\|_{L_2} + \sum_{|\alpha|>0} \frac{1}{\alpha!} \|[P^{(\alpha)}(D)u] D^\alpha g_\delta\|_{L_2} \leq$$

$$\leq \|[P(D)u]\; g_\delta\|_{L_2} + \sum_{|\alpha|>0} \frac{\kappa_\alpha}{\alpha!} \delta^{|\alpha|} \|[P^{(\alpha)}(D)u] g_\delta\|_{L_2}. \tag{2.16}$$

It follows from (2.15) - (2.16) that

$$\sum_{|\alpha|>0} \rho^{|\alpha|} \|[P^{(\alpha)}(D)u] g_\delta\|_{L_2} \leq (1+h)^m \sigma_1 [\|[P(D)u] g_\delta\|_{L_2} +$$

$$+ \sum_{|\alpha|>0} \frac{\kappa_\alpha}{\alpha!} \delta^{|\alpha|} \|[P^{(\alpha)}(D)u] g_\delta\|_{L_2}] + (1+h)^m \sigma_2(\rho) \|ug_\delta\|_{L_2}.$$

Then it is clear that

$$\sum_{|\alpha|>0} \rho^{|\alpha|} \|[P^{(\alpha)}(D)u] g_\delta\|_{L_2} [1 - (1+h)^m \sigma_1 \frac{\kappa_\alpha}{\alpha!} \left(\frac{\delta}{\rho}\right)^{|\alpha|}] \leq$$

$$\leq (1+h)^m \sigma_1 \|[P(D)u] g_\delta\|_{L_2} + (1+h)^m \sigma_2 \|ug_\delta\|_{L_2}.$$

Since

$$1 - (1+h)^m \sigma_1 \frac{\kappa_\alpha}{\alpha!} \left(\frac{\delta}{\rho}\right)^{|\alpha|} \geq \frac{1}{2} \quad \forall \alpha \in N_0^n,\; 0 < |\alpha| \leq m$$

for any $\delta \in (0, \Delta_0(\rho))$, this leads to inequality (2.13). □

For any $\rho \in (0, \rho_P)$ we denote by $\Delta_1 = \Delta_1(\rho, g)$ the greatest of numbers $\delta \in (0, \Delta_0)$ for which

$$1 - \sigma_1 \left[\frac{\kappa_\alpha}{\alpha!} \left(\frac{\delta}{\rho}\right)^{|\alpha|} + \sum_{0<\gamma<\alpha} \frac{\kappa_\gamma}{\gamma!} \left(\frac{\delta}{\rho}\right)^{|\gamma|}\right] \geq \frac{1}{2} \tag{2.17}$$

for any $\alpha \in N_0^n$; $0 \neq |\,\alpha| < m$.

Lemma 2.5. *Let $P \in I_n$ be an almost hypoelliptic operator and $\rho \in (0, \rho_P)$, then for any $k \in N_0$ there exist positive numbers $A_{k,j}$, $j = 0, 1, ..., k$, and B_k such that for all $\delta \in (0, \Delta_1]$ and $u \in H_\delta^\infty$*

$$\sum_{|\beta|\le k} \sum_{|\alpha|>0} \rho^{|\alpha|} \|[D^\beta P^{(\alpha)}(D)u]g_\delta\|_{L_2} \le$$

$$\le \sum_{j=0}^{k} A_{k,j} \sum_{|\beta|=j} \|[D^\beta P(D)u]g_\delta\|_{L_2} + B_k \|ug_\delta\|_{L_2}. \tag{2.18}$$

Proof. We prove the result by induction in k. For $k = 0$ inequality (2.18) follows by (2.13) with the constants $A_{0,0} = 2(1+h)^m \sigma_1$ and $B_0 = 2(1+h)^m \sigma_2$.

Assuming that inequalities (2.18) hold for $k \le r$, let us prove that they hold for $k = r + 1$.

By the inductive assumption it follows that for any $\delta \in (0, \Delta_1)$ and $u \in H_\delta^\infty$ the following inequality holds:

$$\sum_{|\beta|\le r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} \left\|[D^\beta P^{(\alpha)}(D)u]g_\delta\right\|_{L_2} \le$$

$$\le \sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} \left\|[D^\beta P^{(\alpha)}(D)u]g_\delta\right\|_{L_2} +$$

$$+ \sum_{j=0}^{r} A_{r,j} \sum_{|\beta|=j} \left\|[D^\beta P(D)u]g_\delta\right\|_{L_2} + B_r \left\|ug_\delta\right\|_{L_2}. \tag{2.19}$$

If $\beta, \nu \in N_0^n$ and $0 \ne \nu \le \beta$, then $|\beta - \nu| \le r$. Hence, by the Leibnitz formula and estimates (2.4) we conclude that for some positive constants $C_1 = C_1(g, \rho)$, $C_2 = C_2(g, \rho)$ and for any $\delta \in (0, \rho]$ the following inequalities hold

$$\sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} \left\|[D^\beta P^{(\alpha)}(D)u]g_\delta\right\|_{L_2} = \sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} \left[\left\| D^\beta P^{(\alpha)}(D)[ug_\delta] - \right.\right.$$

$$\left.\left. - \sum_{\gamma+\nu\ne 0, \nu\le\beta} \frac{C_\beta^\nu}{\gamma!} [D^{\beta-\nu} P^{(\alpha+\gamma)}(D)u] D^{\gamma+\nu} g_\delta \right\|_{L_2} \right] \le$$

$$\le \sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} \left\|D^\beta P^{(\alpha)}(D)[ug_\delta]\right\|_{L_2} +$$

$$+ \sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} \sum_{\gamma\ne 0} \frac{1}{\gamma!} \left\|[D^\beta P^{(\alpha+\gamma)}(D)u] D^\gamma g_\delta]\right\|_{L_2} +$$

$$+ \sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} \sum_{0\ne\nu\le\beta} \sum_{\gamma} \frac{C_\beta^\nu}{\gamma!} \left\|[D^{\beta-\nu} P^{(\alpha+\gamma)}(D)u] D^{\gamma+\nu} g_\delta]\right\|_{L_2} \le$$

$$\leq \sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} \left\| D^\beta P^{(\alpha)}(D)[u g_\delta] \right\|_{L_2} +$$

$$+ \sum_{|\beta|=r+1} \sum_{|\alpha|\geq 2} \rho^{|\alpha|} \left\| [D^\beta P^{(\alpha)}(D) u] g_\delta] \right\|_{L_2} \left[\sum_{0\neq\nu<\alpha} \left(\frac{\delta}{\rho}\right)^{|\gamma|} \frac{\kappa_\gamma}{\gamma!} \right] +$$

$$+C_1 \sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} \sum_{0\neq\nu\leq\beta} \sum_{\gamma\neq 0} \frac{C_\beta^\nu}{\gamma!} \kappa_{\nu+\gamma} \delta^{|\nu+\gamma|} \left\| [D^{\beta-\nu} P^{(\alpha+\gamma)}(D) u] D^{\gamma+\nu} g_\delta] \right\|_{L_2} \leq$$

$$\leq \sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} \left\| D^\beta P^{(\alpha)}(D)[u g_\delta] \right\|_{L_2} +$$

$$+ \sum_{|\beta|=r+1} \sum_{|\alpha|\geq 2} \rho^{|\alpha|} \left\| [D^\beta P^{(\alpha)}(D) u] g_\delta] \right\|_{L_2} \left[\sum_{0\neq\nu<\alpha} \left(\frac{\delta}{\rho}\right)^{|\gamma|} \frac{\kappa_\gamma}{\gamma!} \right] +$$

$$+C_2 \sum_{|\beta|\leq r} \sum_{|\alpha|\geq 1} \rho^{|\alpha|} \left\| [D^\beta P^{(\alpha)}(D) u] g_\delta] \right\|_{L_2}.$$

From this and by the inductive assumption we get that for any $\delta \in (0, \rho]$ and $u \in H_\delta^\infty$

$$\sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} \left\| [D^\beta P^{(\alpha)}(D) u] g_\delta \right\|_{L_2} \leq$$

$$\leq \sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} \left\| D^\beta P^{(\alpha)}(D)[u g_\delta] \right\|_{L_2} +$$

$$+ \sum_{|\beta|=r+1} \sum_{|\alpha|\geq 2} \rho^{|\alpha|} \left\| [D^\beta P^{(\alpha)}(D) u] g_\delta] \right\|_{L_2} \left[\sum_{0\neq\nu<\alpha} \left(\frac{\delta}{\rho}\right)^{|\gamma|} \frac{\kappa_\gamma}{\gamma!} \right] +$$

$$+ \sum_{j=0}^{r} C_1 A_{r,j} \sum_{|\beta|=j} \left\| [D^\beta P^{(\alpha)}(D) u] g_\delta] \right\|_{L_2} + C_1 B_r \left\| u g_\delta \right\|_{L_2}. \tag{2.20}$$

Let $\rho \in (0, \rho_P)$ and let $M = M(\rho)$ be the minimal of numbers satisfying the inequality (2.2) and

$$\sigma_6 = \max_{|\xi|\leq M} \sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} |\xi^\beta P^{(\alpha)}(\xi)|.$$

To evaluate the first summand in the right-hand side of (2.20) we use the Parseval equality, estimates (2.4) and Lemma 2.1. We conclude that for any $\rho \in (0, \rho_P)$

$$\sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} \left\| D^\beta P^{(\alpha)}(D)[u g_\delta] \right\|_{L_2} =$$

$$= \sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} \left\| \xi^\beta P^{(\alpha)}(\xi) F(ug_\delta) \right\|_{L_2} \le$$

$$\le \sigma_1 \sum_{|\beta|=r+1} \left\| \xi^\beta P(\xi) F(ug_\delta) \right\|_{L_2} + \sigma_6 \left\| F(ug_\delta) \right\|_{L_2} =$$

$$= \sigma_1 \sum_{|\beta|=r+1} \left\| D^\beta P(D)[ug_\delta] \right\|_{L_2} + \sigma_6 \left\| u\, g_\delta \right\|_{L_2} . \tag{2.21}$$

To evaluate the first summand in the right-hand side of (2.21) we use the Leibnitz formula and estimates (2.3). Then by inductive assumption we conclude that for some positive constant $C_3 = C_3(g, \rho)$, and for any $\delta \in (0, \rho]$ the folowing inequalities hold

$$\sum_{|\beta|=r+1} || \, D^\beta P(D)[\, u\, g_\delta \,] \, ||_{L_2} \le \sum_{|\beta|=r+1} || \, [\, D^\beta P(D) \; u \,] \, g_\delta \, ||_{L_2} +$$

$$+ \sum_{|\beta|=r+1} \sum_{\gamma+\nu\neq 0;\; \nu\le\beta} \frac{C_\beta^\nu}{\gamma!} || \, [D^{\beta-\nu} P^{(\gamma)}(D) u \,] \, D^{\gamma+\nu} \, g_\delta \, ||_{L_2} \le$$

$$\le \sum_{|\beta|=r+1} || \, [\, D^\beta P(D) \; u \,] \, g_\delta \, ||_{L_2} +$$

$$+ \sum_{|\beta|=r+1} \sum_{|\gamma|>0} \frac{\kappa_\gamma}{\gamma!} \, \delta^{|\gamma|} \, || \, [D^\beta P^{(\gamma)}(D) u \,] \, g_\delta \, ||_{L_2} +$$

$$+ \sum_{|\beta|=r+1} \sum_{0<\nu<\beta} (\delta^{|\nu|} \, \kappa_\nu) \cdot C_\beta^\nu \; || \, [D^{\beta-\nu} P(D) u] \, g_\delta \, ||_{L_2} +$$

$$+ \sum_{|\beta|=r+1} \sum_{|\gamma|>0} \sum_{|\nu|>0} \frac{C_\beta^\nu}{\gamma!} \, \frac{\kappa_{\gamma+\nu}}{\gamma!} \, \delta^{|\gamma+\nu|} \; || \, [D^{\beta-\nu} P^{(\gamma)}(D) u] \, g_\delta \, ||_{L_2} \le$$

$$\le \sum_{|\beta|=r+1} || \, [\, D^\beta P(D) \; u \,] \, g_\delta \, ||_{L_2} +$$

$$+ \sum_{|\beta|=r+1} \sum_{|\gamma|>0} \frac{\kappa_\gamma}{\gamma!} \left(\frac{\delta}{\rho} \right)^{|\gamma|} \rho^{|\gamma|} \; || \, [D^\beta P^{(\gamma)}(D) u \,] \, g_\delta \, ||_{L_2} +$$

$$+ C_3 \left[\sum_{j=0}^{r} A_{j,r} \sum_{|\beta|=j} || \, [D^\beta P(D) u \,] \, g_\delta \, ||_{L_2} + B_r ||u g_\delta||_{L_2} \right] . \tag{2.22}$$

Applying estimates (2.20) – (2.22) we obtain that for any $\delta \in (0, \rho]$ and $u \in H_\delta^\infty$

$$\sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} || D^\beta P^{(\alpha)}(D)[\, u\, g_\delta \,] \, ||_{L_2} \le$$

$$\leq \sigma_1 \sum_{|\beta|=r+1} || \, [\, D^\beta P(D) \, u \,] \, g_\delta \, ||_{L_2} +$$

$$+\sigma_1 \sum_{|\beta|=r+1} \sum_{|\gamma|>0} \frac{\kappa_\gamma}{\gamma!} \left(\frac{\delta}{\rho}\right)^{|\gamma|} \rho^{|\gamma|} \, || \, [D^\beta P^{(\gamma)}(D) u \,] \, g_\delta \, ||_{L_2} +$$

$$+ \sum_{|\beta|=r+1} \sum_{|\alpha|\geq 2} \rho^{|\alpha|} \, || \, [\, D^\beta P^{(\alpha)}(D) \, u \,] \, g_\delta \, ||_{L_2} \sum_{0<\gamma<\alpha} \frac{\kappa_\gamma}{\gamma!} \left(\frac{\delta}{\rho}\right)^{|\gamma|} +$$

$$+[\sigma_1 \, (C_2 + C_3) + C_1] \sum_{j=0}^{r} A_{j,r} \sum_{|\beta|=j} || \, [D^\beta P(D) u \,] \, g_\delta \, ||_{L_2} +$$

$$+\{[\sigma_1 \, (C_2 + C_3) + C_1] \; B_r + \sigma_6\} \, ||u g_\delta||_{L_2}. \tag{2.23}$$

Applying estimates (2.19) and (2.23) we obtain that for any $\delta \in (0, \rho]$ and $u \in H_\delta^\infty$

$$\sum_{|\beta|=r+1} \sum_{|\alpha|>0} \rho^{|\alpha|} || D^\beta P^{(\alpha)}(D) [\, u \, g_\delta \,] \, ||_{L_2} \leq \sigma_1 \sum_{|\beta|=r+1} || \, [\, D^\beta P(D) \, u \,] \, g_\delta \, ||_{L_2} +$$

$$+[\sigma_1 \, (C_2 + C_3) + C_1 + 1] \sum_{j=0}^{r} A_{j,r} \sum_{|\beta|=j} || \, [D^\beta P(D) u \,] \, g_\delta \, ||_{L_2} +$$

$$+[\sigma_1 \, (C_2 + C_3) + C_1 + 1] \, B_r + \sigma_6\} \, ||u \, g_\delta||_{L_2}] +$$

$$+\sigma_1 \sum_{|\beta|=r+1} \sum_{|\gamma|>0} \frac{\kappa_\gamma}{\gamma!} \left(\frac{\delta}{\rho}\right)^{|\gamma|} \rho^{|\gamma|} \, || \, [D^\beta P^{(\gamma)}(D) u \,] \, g_\delta \, ||_{L_2} +$$

$$+\sigma_1 \sum_{|\beta|=r+1} \sum_{|\alpha|\geq 2} \rho^{|\alpha|} \, || \, [\, D^\beta P^{(\alpha)}(D) \, u \,] \, g_\delta \, ||_{L_2} \sum_{0<\gamma<\alpha} \frac{\kappa_\gamma}{\gamma!} \left(\frac{\delta}{\rho}\right)^{|\gamma|}.$$

Estimate (2.18) immediately follows by this inequality and by inequality (2.17) in the definition of $\Delta_1(\rho, g)$ with the constants

$$A_{j,r+1} = 2[\sigma_1 \, (C_2 + C_3) + C_1 + 1] \; A_{j,r} \quad j = 0, 1, ..., r; \quad A_{r+1,r+1} = 2\sigma_1;$$

$$B_{r+1} = 2 \, \{[\sigma_1 \, (C_2 + C_3) \, C_1 + 1] \; B_r + \sigma_6\},$$

which in turn completes the proof of the lemma. □

3 Density of smooth functions in weighted Sobolev spaces

In this section we consider almost hypoelliptic operators in weighted Sobolev function spaces $H_\delta^m = H_\delta^m(E^n)$ and $H_\delta^\infty = H_\delta^\infty(E^n)$. We begin with a general result on linear differential operators with constant coefficients.

Lemma 3.1. *For any linear differential operator $P(D)$ with constant coefficients and for any $\delta > 0$ the set H_δ^∞ is dense in $N(P, \delta) = \{u \in L_{2,\delta}; P(D)u \in H_\delta^\infty\}$ with respect to the topology, induced by the seminorms*

$$\|u\|_{P,k,\delta} = \|ug_\delta\|_{L_2} + \sum_{|\alpha|\le k} \|[D^\alpha(P(D)u)]g_\delta\|_{L_2}, \quad k = 0, 1, \ldots .$$

Proof. Assuming that $S_1 = \{x \in E^n : |x| < 1\}$, $\varphi \in C_0^\infty(S_1)$, $\int \varphi(x)dx = 1$, $u \in L_{2,\delta}$ and $\varepsilon > 0$, we denote $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, and set

$$u_\varepsilon(x) = u * \varphi_\varepsilon = \int u(x-y)\varphi_\varepsilon(y)dy = \varepsilon^{-n}\int u(x-y)\varphi(y/\varepsilon)dy\cdot$$

The function u_ε is called a regularization (or mollification) of u (for the properties of u_ε see for example [5], Chapter 1, or [1], Chapter 2, Section 17).

First we prove that $u_\varepsilon \in H_\delta^\infty\cdot$ We observe that the following takes place for any $k \in N_0$, using property (2.10) of the function g_δ and Young's inequality

$$\sum_{|\alpha|\le k} ||(D^\alpha u_\varepsilon)\, g_\delta||_{L_2} = \sum_{|\alpha|\le k} \left\| \int u(x-y)\, D^\alpha\varphi_\varepsilon(y)\, g_\delta(x)dy \right\|_{L_2} \le$$

$$\le \sum_{|\alpha|\le k} \left[||\,(ug_\delta) * D^\alpha\varphi_\varepsilon\,||_{L_2} + \left\| \int u(x-y)\,[\,g_\delta(x-y) - g_\delta(x)\,]D^\alpha\varphi_\varepsilon(y)dy \right\|_{L_2} \right] \le$$

$$\le \sum_{|\alpha|\le k} [\,||\,(u\,g_\delta) * D^\alpha\varphi_\varepsilon\,||_{L_2} + \sigma_5(\varepsilon)||\,|u\,g_\delta| * |D^\alpha\varphi_\varepsilon|\,||_{L_2}\,] \le$$

$$\le (1+\sigma_5(\varepsilon)) \sum_{|\alpha|\le k} ||\,ug_\delta\,||_{L_2}\,||D^\alpha\varphi_\varepsilon\,||_{L_1} =$$

$$= (1+\sigma_5(\varepsilon))\; ||\,ug_\delta\,||_{L_2} \sum_{|\alpha|\le k} \varepsilon^{-|\alpha|}||D^\alpha\varphi\,||_{L_1} < \infty\cdot$$

Since $k \in N_0$ is arbitrary, it follows that $u_\varepsilon \in H_\delta^\infty\cdot$

To complete the proof it remains to show that as $\varepsilon \to 0$

$$||\,u_\varepsilon - u\,||_{P,\,k,\,\delta} \to 0\cdot \tag{3.1}$$

Let a function $v \in L_{2,\,loc}(E^n)$ and a linear differential operator $Q(D)$ satisfy the following condition : $Q(D)v \in L_{2,\,loc}(E^n)\cdot$ Then (see [2], 6.3 (2)) $Q(D)v_\varepsilon(x) = [Q(D)v]_\varepsilon(x)$ for all $x \in E^n$ and by the continuity in the mean of functions $u \in L_2$

$$||\,v_\varepsilon - v\,||_{L_2} \to 0$$

as $\varepsilon \to 0$.

Therefore, using property (2.10) of the weight function g and Young's inequality the following holds for any $k \in N_0$

$$|| u_\varepsilon - u ||_{P,k,\delta} = || [u_\varepsilon - u] g_\delta ||_{L_2} + \sum_{|\alpha| \le k} || [D^\alpha(P(D)u_\varepsilon) - D^\alpha(P(D)u)] g_\delta ||_{L_2} =$$

$$= || [u_\varepsilon - u] g_\delta ||_{L_2} + \sum_{|\alpha| \le k} || \{ [D^\alpha(P(D)u)]_\varepsilon - D^\alpha(P(D)u) \} g_\delta ||_{L_2} \le$$

$$\le ||(u\, g_\delta)_\varepsilon - (u\, g_\delta)||_{L_2} + ||(u\, g_\delta)_\varepsilon - (u_\varepsilon\, g_\delta)||_{L_2} +$$

$$+ \sum_{|\alpha| \le k} || [(D^\alpha P(D)u)\, g_\delta]_\varepsilon - (D^\alpha P(D)u)\, g_\delta ||_{L_2} +$$

$$+ \sum_{|\alpha| \le k} || [(D^\alpha P(D)u)\, g_\delta]_\varepsilon - (D^\alpha P(D)u)_\varepsilon\, g_\delta ||_{L_2} \le$$

$$\le ||(u\, g_\delta)_\varepsilon - (u\, g_\delta)||_{L_2} + \sum_{|\alpha| \le k} || [(D^\alpha(P(D)u)\, g_\delta]_\varepsilon - [D^\alpha(P(D)u)]\, g_\delta ||_{L_2} +$$

$$+ \left\| \int u(x-y)\, [g_\delta(x-y) - g_\delta(x)] \varphi_\varepsilon(y) dy \right\|_{L_2} +$$

$$+ \sum_{|\alpha| \le k} \left\| \int [D^\alpha P(D)u](x-y)\, [g_\delta(x-y) - g_\delta(x)] \varphi_\varepsilon(y) dy \right\|_{L_2} \le$$

$$\le ||(u\, g_\delta)_\varepsilon - (u\, g_\delta)||_{L_2} + \sum_{|\alpha| \le k} || [(D^\alpha(P(D)u)\, g_\delta]_\varepsilon - [D^\alpha(P(D)u)]\, g_\delta ||_{L_2} +$$

$$+ \sigma_5(\varepsilon)\, ||\, |u g_\delta| * |\varphi_\varepsilon|\, ||_{L_2} + \sigma_5(\varepsilon) \sum_{|\alpha| \le k} || [\, |D^\alpha(P(D)u| * |\varphi_\varepsilon|\, ||_{L_2} \le$$

$$\le ||(u\, g_\delta)_\varepsilon - (u\, g_\delta)||_{L_2} + \sum_{|\alpha| \le k} || [(D^\alpha(P(D)u)\, g_\delta]_\varepsilon - [D^\alpha(P(D)u)]\, g_\delta ||_{L_2} +$$

$$+ \sigma_5(\varepsilon)\, \{ ||u g_\delta ||_{L_2} + \sum_{|\alpha| \le k} || [D^\alpha(P(D)u)]\, g_\delta ||_{L_2} \}.$$

Since $\sigma_5(\varepsilon) \to 0$ as $\varepsilon \to 0$, the right-hand side tends to zero as $\varepsilon \to 0$. Thus, (3.1) is true and the proof is complete. □

4 Proof of the main result

Theorem 4.1. *Let $P \in I_n$ be an almost hypoelliptic operator, $\rho \in (0, \rho_P)$ and $\delta \in (0, \Delta_1(\rho))$, then $N(P, \delta) \subset H_\delta^\infty$.*

Proof. Let $u \in N(P,\delta)$, $\varphi \in C_0^\infty(S_1)$, $\varphi \geq 0$, $\int \varphi(x)dx = 1$, $\varepsilon > 0$, $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\frac{x}{\varepsilon})$ and let $u_\varepsilon = u * \varphi_\varepsilon$ be a regularization of u. Then (see the proof of Lemma 3.1) $u_\varepsilon \in H_\delta^\infty$ and by the Lemma 2.5 for any $\delta \in (0, \Delta_1(\rho))$ and $k \in N_0$

$$\sum_{|\beta|\leq k} \sum_{|\alpha|>0} \rho^{|\alpha|} \left\|[D^\alpha P^{(\alpha)}(D)u_\varepsilon]g_\delta\right\|_{L_2} \leq$$

$$\leq \sum_{j=0}^{k} A_{k,j} \sum_{|\beta|=j} \left\|[D^\beta P(D)u_\varepsilon]g_\delta\right\|_{L_2} + B_k \left\|u_\varepsilon g_\delta\right\|_{L_2}.$$

Note that by Lemma 3.1

$$\sum_{j=0}^{k} A_{k,j} \sum_{|\beta|=j} ||\ [\ D^\beta P(D)u_\varepsilon\ ]\ g_\delta\ ||_{L_2} + B_k||u_\varepsilon\ g_\delta\ ||_{L_2} \to$$

$$\to \sum_{j=0}^{k} A_{k,j} \sum_{|\beta|=j} ||\ [\ D^\beta P(D)u\ ]\ g_\delta\ ||_{L_2} + B_k||u\ g_\delta\ ||_{L_2}$$

as $\varepsilon \to 0$. Hence, there exist numbers $\varepsilon_0 > 0$ and $C = C(\varepsilon_0) > 0$ such that

$$\sum_{|\beta|\leq k} \sum_{|\alpha|>0} \rho^{|\alpha|}||\ [\ D^\alpha P^{(\alpha)}(D)u_\varepsilon\ ]\ g_\delta\ ||_{L_2} \leq C \quad \forall\varepsilon \in (0, \varepsilon_0).$$

On the other hand, since $P^{(\alpha^0)}(\xi) = const \neq 0$ for some multi-index $\alpha^0 \neq 0$, by this inequality it follows that for some constant $C_1 > 0$

$$\sum_{|\beta|\leq k} ||\ (D^\beta u_\varepsilon)\ g_\delta\ ||_{L_2} \leq C_1 \quad \forall\varepsilon \in (0, \varepsilon_0). \tag{4.1}$$

This means that the set $\{u_\varepsilon;\ \varepsilon \in (0, \varepsilon_0)\}$ is uniformly bounded in H_δ^∞. Therefore, using Lemmas 2.5 and 2.6 we get that for some constant $C_2 > 0$ and any $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0)$

$$C_2 \sum_{|\beta|\leq k} ||\ D^\beta(u_{\varepsilon_1} - u_{\varepsilon_2})\ g_\delta\ ||_{L_2} \leq$$

$$\leq \sum_{|\beta|\leq k} \sum_{|\alpha|>0} \rho^{|\alpha|}||\ D^\beta P^{(\alpha)}(D)(u_{\varepsilon_1} - u_{\varepsilon_2})\ g_\delta\ ||_{L_2} \leq$$

$$\leq \sum_{j=0}^{k} A_{k,j} \sum_{|\beta|=j} ||\ [\ D^\beta P(D)u_{\varepsilon_1} - D^\beta P(D)u_{\varepsilon_2}\ ].g_\delta\ ||_{L_2} +$$

$$+B_k||(u_{\varepsilon_1}-u_{\varepsilon_2})\ g_\delta\ ||_{L_2}\le$$

$$\le\sum_{|\beta|\le k}\ \sum_{|\alpha|>0}\rho^{|\alpha|}||\ [\ D^\beta P^{(\alpha)}(D)u_{\varepsilon_1}-D^\beta P^{(\alpha)}(D)u)\ ]\ g_\delta\ ||_{L_2}+$$

$$+\sum_{|\beta|\le k}\ \sum_{|\alpha|>0}\rho^{|\alpha|}\ ||\ [\ D^\beta P^{(\alpha)}(D)u_{\varepsilon_2}-D^\beta P^{(\alpha)}(D)u)\ ]\ g_\delta\ ||_{L_2}+$$

$$+B_k||(u_{\varepsilon_1}-u)\ g_\delta\ ||_{L_2}+B_k||(u_{\varepsilon_2}-u)\ g_\delta\ ||_{L_2}\cdot$$

Hence

$$\sum_{|\beta|\le k}\ ||\ D^\beta(u_{\varepsilon_1}-u_{\varepsilon_2})\ g_\delta\ ||_{L_2}\to 0 \text{ as } \varepsilon_1\,,\varepsilon_2\to 0^+\cdot$$

From this and (3.2) we get that for any bounded set $G\subset E^n$ and any $k\in N_0$ there exists a number $C_3=C_3(G,k)>0$ such that

$$\sum_{|\beta|\le k}\ ||\ (D^\beta u_\varepsilon)\ ||_{L_2(G)}\le C_3\quad\forall\varepsilon\in(0,\varepsilon_0),$$

and

$$\sum_{|\beta|\le k}\ ||\ D^\beta(u_{\varepsilon_1}-u_{\varepsilon_2})\ ||_{L_2(G)}\to 0\ \ as\ \ \varepsilon_1+\varepsilon_2\to 0\ \cdot$$

Since the Sobolev space $H^k\equiv W_2^k$ is complete and $||u_\varepsilon-u\ ||_{L_2(G)}\to 0$ as $\varepsilon\to 0$ (see the proof of Lemma 3.1) it follows that $u\in H^k\cdot$ Moreover, since $G\subset E^n$ and $k\in N_0$ are arbitrary, we have $u\in H^\infty_{loc}(E^n)\ \cdot$

Passing in (4.1) to the limit as $\varepsilon\to 0$ we conclude that $u\in H^k_\delta$ for any $k\in N_0$. Hence $u\in H^\infty_\delta(E^n)\cdot$ □

Remark 4.1 It follows by estimate (4.1) that for any $\delta\in(0,\Delta_1(\rho))$ the set $N(P,\delta)$ is continuously embedded in H^∞_δ in the topology of H^∞_δ.

Since $[D^\beta(P(D)u)].\ g_\delta\in L_2$ for any $\beta\in N_0^n$ and $u\in N(P,\delta)$, the following corollary is an immediate consequence of the above Theorem 4.1 :

Corollary 4.1. *Let $P(D)$ be an almost hypoelliptic operator, $\rho\in(0,\rho_P)$, $\delta\in(0,\Delta_1(\rho))$, $f\in H^\infty_\delta$ and let $u\in L_{2,\delta}$ be a solution of the equation $P(D)u=f$. Then $u\in H^\infty_\delta$.*

Using Theorem 3.1 and noting that

$$H^\infty_\delta(E^n)\subset H^\infty_{loc}(E^n)\subset C^\infty(E^n),$$

we can prove

Corollary 4.2. *Let the assumptions of Corollary 4.1 hold. Then $u\in C^\infty(E^n)$.*

Theorem 4.2. *Let $P(D)$ be a linear differential operator with constant coefficients, such that $N(P,\delta) \subset H_\delta^\infty$ for a $\delta > 0$. Then $\rho_P \geq \delta$ and operator $P(D)$ is almost hypoelliptic.*

Proof. First note that by the closed graph theorem and the condition $N(P,\delta) \subset H_\delta^\infty$ there exist numbers $k \in N$ and $C_1 > 0$ such that

$$\sum_{j=1}^{n} \|(D_j u) g_\delta\|_{L_2} \leq C_1 \|u\|_{P,k,\delta} \quad \forall u \in N(P,\delta). \tag{4.2}$$

To prove the theorem it is suffices to show that $\rho_P \geq \delta$.

Suppose to the contrary that $\rho_P < \delta$. Then by the definition of the number ρ_P it follows that there exists a sequence $\{\xi^s\}$ of points in R^n such that $|\xi^s| \to \infty$ as $s \to \infty$ and

$$d_P(\xi^s) \leq \frac{\rho_P + \delta}{2} \quad s = 1, 2, \ldots. \tag{4.3}$$

Let $\zeta^s \in D(P)$ be such that

$$d_P(\xi^s) = |\xi^s - \zeta^s| \quad s = 1, 2, \ldots.$$

Then by (4.3)

$$|\mathrm{Im}\,\zeta^s| \leq d_P(\xi^s) \leq \frac{\rho_P + \delta}{2} < \delta \quad s = 1, 2, \ldots. \tag{4.4}$$

We set $u_s(x) = e^{i(x,\,\zeta^s)}$ $(s \in N)$. Then $u_s \in N(P,\delta)$ and by (4.2) we obtain that

$$\sum_{j=1}^{n} \|(D_j u_s) g_\delta\|_{L_2} \leq C_1 \|u_s\|_{P,k,\delta} \quad s = 1, 2, \ldots. \tag{4.5}$$

By estimates (2.3), (4.5) and by the definition of the points $\{\zeta^s\}$ we obtain

$$\|u_s\|_{P,k,\delta} = \|u_s \cdot g_\delta\|_{L_2} + \sum_{|\beta| \leq k} \left\|[D^\beta P(D) u_s] g_\delta\right\|_{L_2} =$$

$$= \|u_s \cdot g_\delta\|_{L_2} \leq \kappa \left\| e^{\frac{\rho_P+\delta}{2}|x|} e^{-\delta|x|} \right\|_{L_2} \equiv C_2, \quad s = 1, 2, \ldots. \tag{4.6}$$

On the other hand, by estimates (2.3) and (4.4)

$$\sum_{j=1}^{n} \|(D_j u_s) g_\delta\|_{L_2} = \sum_{j=1}^{n} |\zeta_j^s| \left\| e^{i(x,\zeta^s)} g_\delta \right\|_{L_2} \geq$$

$$\geq \sum_{j=1}^{n} |\zeta_j^s| \left\| e^{-|\,Im\zeta^s| \cdot |x|} g_\delta \right\|_{L_2} \geq \kappa^{-1} \sum_{j=1}^{n} |\zeta_j^s| \left\| e^{-(|Im\zeta^s|+\delta)\cdot|x|} \right\|_{L_2} \geq$$

$$\geq \kappa^{-1} \sum_{j=1}^{n} |\,\zeta_j^s| \left\| e^{-\frac{\rho_P+\delta}{2}|x|} e^{-\delta|x|} \right\|_{L_2} \equiv C_3 \sum_{j=1}^{n} |\zeta_j^s|. \tag{4.7}$$

From (4.5) – (4.7) it follows that

$$\sum_{j=1}^{n} |\,\zeta_j^s| \leq C_4, \quad s = 1, 2, \ldots,$$

where $C_4 = C_1 C_2 / C_3$.

From this and (4.3) we get

$$|\xi^s| \leq |\zeta^s - \xi^s| + |\zeta^s| = d_P(\xi^s) + |\zeta^s| \leq \frac{\rho_P + \delta}{2} + C_4.$$

Therefore, the sequence $\{\xi^s\}$ is bounded, which contradicts the assumption. This contradiction completes the proof. □

Using the statements of Theorems 4.1 and 4.2, we arrive at the main result stated in Section 1.

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